Exercise 13

(a) Lemma: (Abel summation by parts, Exercise 7).

$$\sum_{k=1}^{n} a_k b_k = \sum_{k=1}^{n-1} S_k(b_k - b_{k+1}) + S_n b_n$$

where $S_k = \sum_{l=1}^{k} a_l$.

Using summation by parts we find that

$$\sum_{n=1}^{N} c_n r^n = (1-r)\sum_{n=1}^{N} S_n r^n + S_N r^{N+1}$$

Without loss of generality we may assume $\sum_{n=1}^{N} c_n = S_N \to S = 0$.

or we replace S_N by $S_N - S$.

Letting $N \to \infty$. $\sum_{n=1}^{N} c_n r^n = (1-r)\sum_{n=1}^{N} S_n r^n + S_N r^{N+1}$

$$\to (1-r)\sum_{n=1}^{\infty} S_n r^n$$

Since $S_n \to 0$, there exises a k such that when $n \geq k+1$, $S_n \leq 1-r$

$$(1-r)\sum_{n=1}^{\infty} S_n r^n \leq (1-r)\left(\sum_{n=1}^{k} S_n r^n + \sum_{n=k+1}^{\infty} S_n r^n\right)$$
$$= (1-r)(A + r^{k+1})$$

Leteing $r \to 1$, $(1-r)\sum_{n=1}^{\infty} S_n r^n \to 0$.

(b) There exist series which are Abel summable, but that do not converge.

Let $c_n = (-1)^n$ $\quad \sum_{n=1}^{N} c_n = \begin{cases} -1 & N \text{ is odd} \\ 0 & N \text{ is even} \end{cases}$

which does not converge.

The Abel means $A(r) = \sum_{n=1}^{\infty} c_n r^n$ converges because

$$A(r) = \sum_{n=1}^{\infty} c_n r^n = \lim_{N\to\infty} \sum_{n=1}^{N} (-1)^n r^n = \lim_{N\to\infty} \frac{r((-r)^N - 1)}{r+1} = -\frac{r}{r+1}$$

when $0 \leq r < 1$

Letting $r \to 1$, $\lim_{r\to 1} A(r) = -\frac{1}{2}$

Therefore the series $\{(-1)^n\}$ is Abel summable but that does not converge

(c) There exists a series that is Abel summable but not Cesàro summable. We note that if $\sum c_n$ is Cesàro summable, then $c_n/n \to 0$ as $n \to \infty$. In fact, let $S_N = \sum_{n=1}^{N} c_n$

$$\sigma_n = \frac{S_1 + S_2 + \cdots + S_n}{n}$$

$$\sigma_{n-1} = \frac{S_1 + \cdots + S_{n-1}}{n-1} \quad \Rightarrow \quad \sigma_n - \sigma_{n-1} = (S_1 + \cdots + S_{n-1})\left(\frac{1}{n} - \frac{1}{n-1}\right) + \frac{S_n}{n}$$

$$= -\frac{S_1 + \cdots + S_{n-1}}{n(n-1)} + \frac{S_n}{n}.$$

$\to 0$ as $n \to \infty$

This is because $\sigma_n, \sigma_{n-1} \to s$ for some s and thus $\sigma_n - \sigma_{n-1} \to 0$

Now $-\frac{S_1 + \cdots + S_n}{n(n-1)} \to 0$ follows from the fact that $\frac{S_1 + \cdots + S_{n-1}}{n-1} \to s$

and $\frac{1}{n} \to 0$. Then we get $\frac{S_n}{n} \to 0$. $\frac{S_n}{n} = \sum_{k=1}^{n} \frac{c_k}{n} = \sum_{k=1}^{n-1} \frac{c_k}{n} + \frac{c_n}{n}$.

$\frac{S_n}{n} = \frac{S_{n-1}}{n} + \frac{c_n}{n}$. Since $S_n/n \to 0$ $(n \to \infty)$, $\frac{S_{n-1}}{n-1} \to 0$, $S_{n-1} = o(n-1) = o(n)$

$\Rightarrow \frac{c_n}{n} = \frac{S_n}{n} - \frac{S_{n-1}}{n} \to 0$. $(n \to \infty)$

Now let $c_n = (-1)^{n-1} n$. $\left\{\frac{c_n}{n}\right\}$ fails to tend to zero because $\{(-1)^{n-1}\}$ is oscillating, therefore not Cesáro summable.

Consider the Abel sums $\sum_{n=1}^{\infty} c_n r^n = \sum_{n=1}^{\infty} (-1)^{n-1} n r^n$, the $\sum_{n=1}^{\infty} c_n$ is Abel summable when $0 \leq r < 1$, with $A(r) = \frac{r}{(r+1)^2}$ $(0 \leq r < 1)$

and $\lim_{r \to 0} A(r) = 0$.

Therefore $\{(-1)^{n-1} n\}$ is Abel summable but not Cesàro summable.

(C). We prove that

$$\sum_{n=1}^{\infty} c_n r^n = (1-r)^2 \sum_{n=1}^{\infty} n \sigma_n r^n.$$

For the finite case N. $c_1 = s_1$, $c_n = s_n - s_{n-1}$ $(n \geq 2)$

$$\sigma_n = \frac{s_1 + \cdots + s_n}{n} \qquad s_1 = \sigma_1 \quad s_n = n\sigma_n - (n-1)\sigma_{n-1} \ (n \geq 2)$$

$$\sum_{n=1}^{N} c_n r^n = \sum_{n=3}^{N} c_n r^n + c_1 r + c_2 r^2 = \sum_{n=3}^{N} c_n r^n + \sigma_1 r + (2\sigma_2 - \sigma_1) r^2$$

$$\sum_{n=1}^{N} c_n r^n = \sum_{n=3}^{N} [n\sigma_n - 2(n-1)\sigma_{n-1} + (n-2)\sigma_{n-2}] r^n + \sigma_1 r + (2\sigma_2 - \sigma_1) r^2$$

$$= \sum_{n=1}^{N} n\sigma_n r^n + \sum_{n=2}^{N} -2(n-1)\sigma_{n-1} r^n + \sum_{n=3}^{N} (n-2)\sigma_{n-2} r^n$$

$$= \sum_{n=1}^{N} n\sigma_n r^n - 2r \sum_{n=1}^{N-1} n\sigma_n r^n + r^2 \sum_{n=1}^{N-2} n\sigma_n r^n$$

$$= \sum_{n=1}^{N} n\sigma_n r^n - \left(2r \sum_{n=1}^{N} n\sigma_n r^n\right) + 2r(N\sigma_N r^N)$$
$$+ \left(r^2 \sum_{n=1}^{N} n\sigma_n r^n\right) - r^2\left((N-1)\sigma_{N-1} r^{N-1} + N\sigma_N r^N\right)$$

$$= (1-r)^2 \sum_{n=1}^{N} n\sigma_n r^n + \left[(2r - r^2) N\sigma_N r^N - r^2 (N-1)\sigma_{N-1} r^{N-1}\right]$$

Letting $N \to \infty$ $\quad N\sigma_N r^N \to 0$ we find that.

$$\sum_{n=1}^{\infty} c_n r^n = (1-r)^2 \sum_{n=1}^{\infty} n\sigma_n r^n$$

Suppose $\sum_{n=1} c_n$ is Cesàro summable to σ.

Without loss of generality we may assume $\sigma = 0$

Consider the finite sum $(1-r)^2 \sum_{n=1}^{N} n\sigma_n r^n$. Since $\sigma_n \to \sigma = 0$,

there exists an integer k such that $\sigma_n \le (1-r)^2$ when $n \ge k+1$

$$\sum_{n=1}^{N} n\sigma_n r^n \le \sum_{n=1}^{k} n\sigma_n r^n + \sum_{n=k+1}^{N} n(1-r)^2 r^n$$

Let $A = \sum_{n=1}^{k} n\sigma_n r^n$ denotes the partial sum

$$\sum_{n=1}^{N} n\sigma_n r^n \le A + (1-r)^2 \sum_{n=k+1}^{N} n r^n$$

Letting $N \to \infty$,

$$\sum_{n=1}^{\infty} n\sigma_n r^n \le A + (1-r)^2 \sum_{n=k+1}^{N} n r^n$$

$$= A + r^{k+1}(k+1-kr)$$

Letting $r \to 1$

$$(1-r) \sum_{n=1}^{\infty} n\sigma_n r^n = (1-r)\left[A + r^{k+1}(k+1-kr)\right]$$

$$\to 0.$$

Therefore $\sum_{n=1}^{\infty} c_n$ is Abel summable.

$\boxed{\text{convergent} \Rightarrow \text{Cesàro summable} \Rightarrow \text{Abel summable and none of the arrows can be reversed}}$

Exercise 14 Chapter 2

(Theorem of Tauber) Under an additional condition on the coefficients c_n, the arrows in Exercise 13 can be reversed

(a) If $\sum c_n$ is Cesàro summable to σ and $c_n = o(\frac{1}{n})$ (that is $nc_n \to 0$), then $\sum c_n$ converges to σ.

(b) The above statement holds if we replace Cesàro summable by Abel summable.

Proof: (a) Let $S_n = \sum_{k=1}^{n} c_k$ be the n-th partial sum of $\sum c_n$, and

$\sigma_n = \frac{1}{n} \sum_{k=1}^{n} S_k$ be the n-th Cesàro sum of $\sum c_n$ and $\lim_{n\to\infty} \sigma_n = \sigma$ (Cesàro summable to σ)

Then
$$S_n - \sigma_n = c_1 + \cdots + c_n + \frac{nc_1 + \cdots + c_n}{n}$$
$$= \frac{1}{n} [(n-1)c_n + (n-2)c_{n-1} + \cdots + c_2]$$

We assert that $S_n - \sigma_n = \frac{1}{n}[(n-1)c_n + (n-2)c_{n-1} + \cdots + c_2] \to 0$ as $n \to \infty$

Since $nc_n \to 0$ as $n \to \infty$, then $\lim_{n\to\infty} (n-1)c_n = \lim_{n\to\infty} nc_n \frac{n-1}{n} = 0$

Thus there exists an $N \in \mathbb{N}^*$, such that $|(k-1)c_k| < \varepsilon/2$ whenever $k > N$.

$$|S_n - \sigma_n| = \left| \frac{1}{n}[(n-1)c_n + \cdots + (N+1)c_{N+2}] + \frac{1}{n}[(N+1)c_{N+1} + \cdots + c_2] \right|$$
$$\le \frac{1}{n}\left(|(n-1)c_n| + \cdots + |Nc_{N+1}|\right) + \left|\frac{1}{n}[(N-1)c_N + \cdots + c_2]\right|$$
$$\le \frac{1}{n}(n-N)\,\varepsilon/2 + \frac{M}{n}, \quad \text{where } M = |(N-1)c_N + \cdots + c_2| \ge 0$$

Now $\lim_{n\to\infty} (n-N)\cdot\frac{1}{n} = 1$ and $\lim_{n\to\infty} \frac{M}{n} = 0$. There exists a sufficiently large N, $\frac{M}{n} < \frac{\varepsilon}{2}$ whenever $n > N$. Thus

$|S_n - \sigma_n| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$ whenever $n > N$ and $S_n - \sigma_n = 0$.

Therefore $\lim_{n\to\infty} S_n = \lim_{n\to\infty} \sigma_n = \sigma$, i.e. $\sum c_n$ converges to σ.

(b) Suppose C_n is Abel summable. Given $\varepsilon > 0$, there exists an $N' = N'(\varepsilon)$ such that $n|C_n| < \varepsilon$ whenever $|n| > N'$

Consider $r_n = 1 - \frac{1}{n} \to 1^-$

$$|S_n - Ar_n| = \left|\sum_{j=1}^{n} c_j - \sum_{j=1}^{n} c_j r_n^j\right|$$

$$= \sum_{j=1}^{n} |C_j|(1 - r_n^j) \leq \sum_{j=1}^{n} |C_j|(1 - r_n^j)$$

We require N' large enough such that $1 - r_n^j < r_n^j$ whenever $n > N'$.

$$|S_n - Ar_n| \leq \sum_{j=1}^{N'} |C_j|(1 - r_n^j) + \sum_{j=N+1}^{\infty} r_n^j |C_j|$$

$$\leq \sum_{j=1}^{N'} |C_j|(1 - r_n^j) + \left(\sum_{j=N+1}^{\infty} \frac{r_n^j}{j}\right) \varepsilon.$$

For the first term of the finite sum, observe that $\{n|C_n|\}$ bounded, let $j|c_j| \leq M$ for a constant $M \geq 0$.

$$\sum_{j=1}^{N'} j|C_j| \cdot \frac{1 - r_n^j}{j} \leq \sum_{j=1}^{N'} M \frac{(1 - r_n^j)}{j} = M \sum_{j=1}^{N} \frac{1 - r_n^j}{j}$$

$$\leq M \sum_{j=1}^{N} 1 - r_n^j$$

Since $\lim_{n\to\infty} 1 - r_n^j = \lim_{n\to\infty} [1 - (1 - \frac{1}{n})^j] = 0$

then $\lim_{n\to\infty} \sum_{j=1}^{N} 1 - r_n^j = 0$. By definition, there exists an N'' such that $\left|\sum_{j=1}^{N} |C_j|(1 - r_n^j)\right| < \varepsilon$ whenever $n \geq N''$

For the second term, by the ratio test, $\sum_{j=1}^{\infty} \frac{r_n^j}{j}$ converges, denote by K the infinite sum. Now take $N = \max\{N', N''\}$

Then $|S_n - Ar_n| \leq (M + K)\varepsilon$. Therefore, S_n converges to the same limit of Ar_n.

The Weierstrass approximation theorem:
Let f be a continuous function on a closed and bounded interval $[a,b] \subset \mathbb{R}$. Then for any $\varepsilon > 0$, there exists a polynomial P such that $\sup_{x \in [a,b]} |f(x) - P(x)| < \varepsilon$

Proof. By Corollary 5.4 it is obvious that if we modify the condition $[-\pi, \pi]$ to any bounded interval $[x_1, x_2]$ where $g(x_1) = g(x_2)$, the conclusion holds for the trigonometric polynomial $P(x) = \sum_{n=-N}^{N} c_n e^{2\pi i n x/L}$ where $L = x_2 - x_1$

That is, given any $\varepsilon > 0$, there exists a trigonometric polynomial $Q(x) = \sum_{n=-N}^{N} c_n e^{2\pi i n x/L}$ such that

$$\left| g(x) - \sum_{n=-N}^{N} c_n e^{2\pi i n x/L} \right| < \varepsilon/2 \quad \text{for all } x_1 \le x \le x_2$$

Now observed the fact that e^{ix} can be approximated by polynomials uniformly on any interval. Since

$$e^{ix} = \sum_{k=0}^{\infty} \frac{(ix)^k}{k} \qquad \text{for } x \in [x_1, x_2]$$

and $c_n e^{\frac{2\pi i n x}{L}} = c_n e^{ix \cdot \frac{2\pi n}{L}} = c_n \sum_{k=0}^{\infty} \frac{\left(ix \cdot \frac{2\pi n}{L}\right)^k}{k} = c_n \sum_{k=0}^{\infty} \frac{\left(i\frac{2\pi n}{L}\right)^k}{k} x^k$

Therefore given any $\varepsilon > 0$ there exists a polynomial $Q_n(x)$ such that $\left| c_n e^{\frac{2\pi i n x}{L}} - Q_n(x) \right| < \frac{\varepsilon}{2N+1}$

Therefore $\left| g(x) - \sum_{n=-N}^{N} P_n(x) \right| \le \left| g(x) - \sum_{n=-N}^{N} c_n e^{2\pi i n x/L} \right| + \left| \sum_{n=-N}^{N} \left(c_n e^{2\pi i n x/L} - Q_n(x)\right) \right|$

(Note that $\sum_{n=-N}^{N} Q_n(x)$ is a polynomial)

$$\leq \left| g(x) - \sum_{n=-N}^{N} c_n e^{2\pi i n x/L} \right| + \sum_{n=-N}^{N} \left| c_n e^{2\pi i n x/L} - Q_n(x) \right|$$

$$< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon \qquad \text{for all} \quad x \in [x_1, x_2]$$

For any closed and bounded interval $[a,b] \subseteq \mathbb{R}$,

choose $[x_1, x_2] \supseteq [a,b]$ by taking $x_1 \leq a < b \leq x_2$.

Then taking $f = g|_{[a,b]}$ and $P_n = Q_n|_{[a,b]}$

Thus the theorem is proved.