Problem 2(a)

Chapter 2.

$$D_N(\theta) = \frac{\sin((N+\frac{1}{2})\theta)}{\sin\frac{\theta}{2}}$$

$$\boxed{\text{Dirichlet kernel is not a good kernel because the property (iii) fails.}}$$

Claim $|D_N(\theta)| \geq c \dfrac{\sin(N+\frac{1}{2})\theta}{|\theta|}$

Since $\theta \in [-\pi, \pi]$, $\frac{\theta}{2} \in [-\frac{\pi}{2}, \frac{\pi}{2}]$ $|\sin\frac{\theta}{2}| \leq |\frac{\theta}{2}|$

$$\frac{1}{|\sin\frac{\theta}{2}|} \geq \frac{1}{|\frac{\theta}{2}|} = 2\frac{1}{|\theta|} \Rightarrow c = 2.$$

Now
$$L_N = \frac{1}{2\pi}\int_{-\pi}^{\pi} c\,|D_N(\theta)|\,d\theta$$
$$\geq \frac{c}{2\pi}\int_{-\pi}^{\pi} \frac{|\sin(N+\frac{1}{2})\theta|}{|\theta|}\,d\theta$$

change variable: $\varphi = (N+\frac{1}{2})\theta$ $d\varphi = (N+\frac{1}{2})d\theta$

Since $\dfrac{|\sin(N+\frac{1}{2})\theta|}{|\theta|}$ is an even function $\displaystyle\int_{-\pi}^{\pi}\frac{|\sin(N+\frac{1}{2})\theta|}{|\theta|}d\theta = 2\int_0^{\pi}\frac{|\sin(N+\frac{1}{2})\theta|}{|\theta|}d\theta.$

$$\int_0^{\pi}\frac{|\sin(N+\frac{1}{2})\theta|}{|\theta|}d\theta = \int_0^{(N+\frac{1}{2})\pi}\frac{|\sin\varphi|}{|\varphi|}d\varphi.$$

$$\geq \int_{\pi}^{N\pi}\frac{|\sin\varphi|}{|\varphi|}d\varphi \quad \text{缩小上下限}$$

Write
$$\int_{\pi}^{N\pi}\frac{|\sin\varphi|}{|\varphi|} = \sum_{k=1}^{N-1}\int_{k\pi}^{(k+1)\pi}\frac{|\sin\varphi|}{|\varphi|}d\varphi$$
$$\geq \sum_{k=1}^{N-1}\int_{k\pi}^{(k+1)\pi}\frac{1}{(k+1)\pi}|\sin\varphi|\,d\varphi$$
$$= \sum_{k=1}^{N-1}\frac{1}{(k+1)\pi}\int_{k\pi}^{(k+1)\pi}|\sin\varphi|\,d\varphi$$
$$= \sum_{k=1}^{N-1}\frac{1}{(k+1)\pi} = \frac{1}{\pi}\sum_{k=1}^{N-1}\frac{1}{(k+1)\pi}$$

Using the fact $\displaystyle\sum_{k=1}^{N}\frac{1}{k} \geq \log(N+1)$, we have
$$\geq \frac{1}{\pi}\log N$$
$$L_N \geq \frac{c}{2\pi}\cdot 2\cdot\frac{1}{\pi}\log N = \frac{c}{\pi^2}\log N$$

Problem 2(b). From (a) we get

$$L_N = \frac{1}{2\pi}\int_{-\pi}^{\pi} |D_N(\theta)|\, d\theta \geq c \log N \quad \text{for some constant } c.$$

Define $g_n = \begin{cases} 1 & \text{when } D_n \text{ is positive} \\ -1 & \text{when } D_n \text{ is negative} \end{cases}$

$$S_n(g_n)(x) = g_n * D_n(x)$$

$$= \frac{1}{2\pi}\int_{-\pi}^{\pi} g_n(y) D_n(x-y)\, dy$$

$$S_n(g_n)(0) = \frac{1}{2\pi}\int_{-\pi}^{\pi} g_n(y) D_n(-y)\, dy \quad (\text{Note that } D_N(y) \text{ is even})$$

$$= \frac{1}{2\pi}\int_{-\pi}^{\pi} g_n(y) D_n(y)\, dy$$

$$= \frac{1}{2\pi}\int_{-\pi}^{\pi} |D_n(y)|\, dy$$

$$\geq c \log n.$$

g_n is not continuous

By Lemma 3.2 There exists a sequence $\{g_{n_k}\}$ of continuous functions $\int_{-\pi}^{\pi} |g_{n_k}(x) - g_n(x)|\, dx \to 0$ as $k \to \infty$ and, moreover

$$g_{n_k}(x) \leq g_n(x)$$

$$|S_n(g_{n_k})(0) - S_n(g_n)(0)| = \left|\frac{1}{2\pi}\int_{-\pi}^{\pi} (g_{n_k} - g_n) D_n(y)\, dy\right|$$

$$\leq \frac{1}{2\pi}\int_{-\pi}^{\pi} |g_{n_k} - g_n| |D_n(y)|\, dy$$

Take an upper bound of $D_n(y)$ as M

$$\leq \frac{1}{2\pi} \cdot M \int_{-\pi}^{\pi} |g_{n_k} - g_n|\, dy \to 0 \quad \text{as } k \to \infty$$

Then for any $\varepsilon > 0$. there exists a k such that

$$S_n(g_k)(0) \geq S_n(g_n)(0) - \varepsilon \quad \text{take } \varepsilon = \frac{1}{2} c \log n.$$

$$S_n(g_k)(0) \geq \frac{1}{2} c \log n$$ and let $c' = c/2$ we are done

Exercise 15

Prove that the Fejér kernel is given by

$$\boxed{F_N(x) = \frac{1}{N} \frac{\sin^2(Nx/2)}{\sin^2(x/2)}}$$

Proof: $F_N(x) = \dfrac{D_0(x) + \cdots + D_{N-1}(x)}{N}$

$$NF_N(x) = D_0(x) + \cdots + D_{N-1}(x)$$

Recall that $D_n(x) = \dfrac{\omega^{-n} - \omega^{n+1}}{1-\omega}$ where $\omega = e^{ix}$

$$NF_N(x) = \sum_{n=0}^{N-1} \frac{\omega^{-n} - \omega^{n+1}}{1-\omega}$$

$$= \frac{1}{1-\omega}\left(\sum_{n=0}^{N-1} \omega^{-n} - \sum_{n=0}^{N-1} \omega^{n+1}\right)$$

$$= \frac{1}{1-\omega}\left(\frac{1-\omega^{-N}}{1-\omega^{-1}} - \frac{\omega(1-\omega^N)}{1-\omega}\right)$$

$$= \frac{1}{1-\omega}\left(\frac{\omega - \omega^{-N+1}}{\omega - 1} - \frac{\omega - \omega^{N+1}}{1-\omega}\right)$$

$$= \frac{1}{1-\omega}\left(\frac{\omega^{-N+1} - \omega - \omega + \omega^{N+1}}{1-\omega}\right)$$

$$= \frac{\omega^{N+1} - 2\omega + \omega^{-N+1}}{(1-\omega)^2}$$

$$= \frac{\omega^N - 2 + \omega^{-N}}{(\omega^{-\frac{1}{2}} - \omega^{\frac{1}{2}})^2} = \frac{(\omega^{\frac{N}{2}} - \omega^{-\frac{N}{2}})^2}{(\omega^{\frac{1}{2}} - \omega^{-\frac{1}{2}})^2}$$

$$= \frac{\sin^2(Nx/2)}{\sin^2(x/2)}$$

Therefore, $F_N(x) = \dfrac{1}{N} \dfrac{\sin^2(Nx/2)}{\sin^2(x/2)}$

The Fejér kernel

$$F_N(x) = \frac{1}{N}(D_0(x) + \cdots + D_{N-1}(x)) = \frac{1}{N}\frac{\sin^2(Nx/2)}{\sin^2(x/2)}$$

is a good kernel.

Proof: ① For all $N \geq 1$

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} F_N(x)dx = \frac{1}{2\pi}\int_{-\pi}^{\pi}\frac{1}{N}(D_0(x) + \cdots + D_{N-1}(x))dx$$

$$= \frac{1}{2\pi N}\left(\int_{-\pi}^{\pi} D_0(x)dx + \cdots + \int_{-\pi}^{\pi} D_{N-1}(x)dx\right)$$

For $k \geq 0$

$$\int_{-\pi}^{\pi} D_k(x)dx = \sum_{n=-k}^{k}\int_{-\pi}^{\pi} e^{inx}dx$$

$$= \int_{-\pi}^{\pi} 1\,dx = 2\pi.$$

and $\frac{1}{2\pi}\int_{-\pi}^{\pi} F_N(x)dx = \frac{1}{2\pi N}\cdot N\cdot 2\pi = 1.$

② From ① $\int_{-\pi}^{\pi}|F_N(x)|dx = \int_{-\pi}^{\pi} F_N(x)dx = 2\pi.$

③. For every $\delta > 0$, there exists a $C_\delta > 0$ such that $\sin^2(\frac{x}{2}) \geq C_\delta > 0$ if $\delta \leq |x| \leq \pi$. hence

$$\int_{\delta\leq|x|\leq\pi}|F_N(x)|dx \leq \int_{-\pi}^{\pi}\frac{1}{N}\frac{\sin^2(Nx/2)}{C_\delta}dx$$

Since $\sin^2(Nx/2) \leq 1$

$$\leq \int_{-\pi}^{\pi}\frac{1}{NC_\delta}dx$$

$$= \frac{2\pi}{NC_\delta}$$

Therefore $\int_{\delta\leq|x|\leq\pi}|F_N(x)|dx \longrightarrow 0$ as $N\to\infty$.

Exercise 3 Chapter 2.

Construct a sequence of integrable function $\{f_k\}$ on $[0, 2\pi]$ such that

$$\lim_{k\to\infty} \frac{1}{2\pi}\int_0^{2\pi} |f_k(\theta)|^2 \, d\theta = 0$$

but $\lim_{k\to\infty} f_k(\theta)$ fails to exist for any θ.

Construction. We first construct the sequence $\{g_k\}$ on $[0,1]$

Let $I_1 = [0,1]$

$I_2 = [0, \frac{1}{2}]$, $I_3 = [\frac{1}{2}, 1]$

$I_4 = [0, \frac{1}{3}]$, $I_5 = [\frac{1}{3}, \frac{2}{3}]$, $I_6 = [\frac{2}{3}, 1]$

... preceed with this rule

Then eachpoint $\theta \in [0,1]$ preceed infinitely many of $\{I_k\}_{k=1}^{\infty}$

Let $g_k = \chi_{I_k}$, where $|I_k| \to 0$ as $k \to \infty$, then

$\lim_{k\to\infty} g_k$ fails to exists for any $\theta \in [0,1]$, but

$$\lim_{k\to\infty} \int_0^1 |g_k(\theta)|^2 \, d\theta = \lim_{k\to\infty} |I_k| = 0.$$

For functions $\{f_k\}$ defined on $[0, 2\pi]$, consider

$f_k(\theta) = g_k(\frac{1}{2\pi}\theta)$ and then we are done.

Mean square convergence does not imply pointwise convergence.

Exercise 5 Chapter 3.

Let $f(\theta) = \begin{cases} 0 & \text{for } \theta = 0 \\ \log(1/\theta) & \text{for } 0 < \theta \leq 2\pi \end{cases}$

and define the sequence of functions in R by

$$f_n(\theta) = \begin{cases} 0 & \text{for } 0 < \theta < \frac{1}{n} \\ f(\theta) & \text{for } \frac{1}{n} < \theta \leq 2\pi \end{cases}$$

Prove that $\{f_n\}_{n=1}^{\infty}$ is a Cauchy sequence in R. However, f does not belong to R.

Proof: We first explain that f does not belong to R. This is because $f(\theta)$ is unbounded in $[0, 2\pi]$ ($\lim_{\theta \to 0^+} f(\theta) = \lim_{\theta \to 0^+} \log(\frac{1}{\theta}) = \infty$)

Second we prove that $f_n(\theta) = \begin{cases} 0 & 0 < \theta < \frac{1}{n} \\ \log(1/\theta) & \frac{1}{n} < \theta \leq 2\pi \end{cases}$ is a Cauchy sequence in R.

Write $\|f_n - f_m\| = \left(\frac{1}{2\pi}\int_0^{2\pi} |f_n(\theta) - f_m(\theta)|^2 d\theta\right)^{\frac{1}{2}}$, $n > m$, $m, n \in \mathbb{N}^*$.

$$\int_0^{2\pi} |f_n(\theta) - f_m(\theta)|^2 d\theta = \int_{\frac{1}{n}}^{\frac{1}{m}} \left|\log\left(\frac{1}{\theta}\right)\right|^2 d\theta = \int_{\frac{1}{n}}^{\frac{1}{m}} (\log\theta)^2 d\theta.$$

We show that $\int_a^b (\log\theta)^2 d\theta \to 0$ if $0 < a < b$ and $b \to 0$

$$\int_a^b (\log\theta)^2 d\theta = \theta(\log\theta)^2 - 2\theta\log\theta + 2\theta \Big|_a^b$$

$$= b(\log b)^2 - 2b\log b + 2b - (a(\log a)^2 - 2a\log a + 2a)$$

Note that $\lim_{b \to 0} b(\log b)^2 \overset{b = e^t}{=} \lim_{t \to -\infty} e^t t^2 = \lim_{t \to +\infty} \frac{t^2}{e^t} = 0.$

Similarly $\lim_{b \to 0} b\log b = 0$. Since $0 < a < b$, then $a \to 0$

Therefore $\int_a^b (\log\theta)^2 d\theta \to 0$ as $b \to 0$. and $0<a<b$

By the argument above, Since

$$\|f_n - f_m\| = \int_{\frac{1}{n}}^{\frac{1}{m}} (\log\theta)^2 d\theta, \quad \text{there exists an } N \text{ such}$$

that

$$\|f_n - f_m\| < \varepsilon \text{ whenever } n > m \geq N.$$

Therefore $\{f_n\}_{n=1}^{\infty}$ is a Cauchy sequence.

It is obvious that

$$\|f_n - f\| = \int_0^{\frac{1}{n}} (\log\theta)^2 d\theta = \frac{1}{n}\left(\log\frac{1}{n}\right)^2 - 2\frac{1}{n}\log\frac{1}{n} + 2\frac{1}{n}$$

$$\to 0 \quad \text{as} \quad n \to \infty.$$

which implies that $\mathcal{R}$ is not a complete vector space.

$$\boxed{\text{The space of Riemann integrable functions, } \mathcal{R}, \text{ is not a complete vector space.}}$$

Exercise 6 Chapter 3

Consider the sequence $\{a_k\}_{k=-\infty}^{\infty}$ defined by

$$a_k = \begin{cases} 1/k & \text{if } k \geq 1 \\ 0 & \text{if } k \leq 0 \end{cases}$$

Note that $\{a_k\} \in \ell^2(\mathbb{Z})$, but that no Riemann integrable function has k-th Fourier coefficient equals to a_k for all k.

Proof: Suppose there exists a Riemann integrable function $\tilde{f}$ defined on $[0, 2\pi]$ has k-th Fourier coefficient equal to a_k for all k. Similar as the argument in Section 2.2.

$$f(\theta) \sim \sum_{n\geq 1}^{\infty} \frac{e^{in\theta}}{n},$$ where in particular $\tilde{f}$ is bounded

Using Abel means, we have

$$|A_r(\tilde{f})(0)| = \sum_{n=1}^{\infty} \frac{r^n}{n}$$

which tends to infinity as $r \to 1^-$, because $\sum 1/n$ diverges.

This gives the desired contradiction since

$$|A_r \tilde{f}(0)| \leq \frac{1}{2\pi}\int_0^{2\pi} |\tilde{f}(\theta)| P_r(\theta)\, d\theta \leq \sup_{\theta} |f(\theta)|$$

where $P_r(\theta)$ is the Poisson kernel.

There exist sequences $\{a_n\}_{n\in\mathbb{Z}}$ such that $\sum_{n\in\mathbb{Z}} |a_n| < \infty$ yet no Riemann integrable function F has n-th Fourier coefficient equal to a_n for all n.

Exercise 17 Chapter 2

Let f be an integrable function on the circle.

(a) Prove that if f has a jump discontinuity at θ, then

$$\lim_{r\to 1} A_r(f)(\theta) = \frac{f(\theta^+)+f(\theta^-)}{2} \quad \text{with } 0\le r<1.$$

(b) Using the similar argument, show that if f has a jump discontinuity at θ, the Fourier series of f at θ is Cesàro summable to $\frac{f(\theta^+)+f(\theta^-)}{2}$

Proof (a) Since $P_r(\theta)$ is an even function, we have

$$\frac{1}{2\pi}\int_{-\pi}^{0} P_r(\theta)\,d\theta = \frac{1}{2\pi}\int_{0}^{\pi} P_r(\theta)\,d\theta = \frac{1}{2}$$

Let $\varepsilon>0$ be given. Choose $\delta>0$ so that $0<h<\delta$ implies $|f(\theta-h)-f(\theta^-)|<\varepsilon/2$ and $|f(\theta+h)-f(\theta^+)|<\varepsilon/2$

Let M be such that $|f(y)|\le M$ for all y.

Then $\left|(f*P_r)(\theta) - \frac{f(\theta^+)+f(\theta^-)}{2}\right|$

$$= \left|\frac{1}{2\pi}\int_{-\pi}^{\pi} f(\theta-y)P_r(y)\,dy - \frac{f(\theta^+)+f(\theta^-)}{2}\right|$$

$$\le \frac{1}{2\pi}\int_{-\pi}^{0}|P_r(y)||f(\theta-y)-f(\theta^+)|\,dy + \frac{1}{2\pi}\int_{0}^{\pi}|P_r(y)||f(\theta-y)-f(\theta^-)|\,dy$$

$$\le \frac{1}{2\pi}\int_{-\delta<y<0}|P_r(y)||f(\theta-y)-f(\theta^+)\,dy|$$

$$+ \frac{1}{2\pi}\int_{0<y<\delta}|P_r(y)||f(\theta-y)-f(\theta^-)\,dy|$$

$$+ \frac{1}{2\pi}\int_{\delta\le|y|\le\pi} 2M|P_r(y)|\,dy$$

$$\leq \frac{\varepsilon}{2} + \frac{\varepsilon}{2} + \frac{M}{\pi}\int_{\delta \leq |y| \leq \pi} |P_r(y)|\, dy$$

Letting $r \to 1^-$

$$\left|(f * P_r)(\theta) - \frac{f(\theta^+) + f(\theta^-)}{2}\right| \leq 2\varepsilon \quad \text{when } |r-1| \text{ is}$$

sufficiently small.

Therefore $\lim_{r\to 1} A_r(f)(\theta) = \frac{f(\theta^+) + f(\theta^-)}{2}$

(2) Since the Fejér kernel $F_n(\theta)$ is even and positive, we also have $\frac{1}{2\pi}\int_{-\pi}^{0} F_n(\theta)\, d\theta = \frac{1}{2\pi}\int_{0}^{\pi} F_n(\theta)\, d\theta = \frac{1}{2}$ for all n. Repeat the argument above and we get the same result

Exercise 7 Chapter 3

Show that the trigonometric series

$$\sum_{n\geq 2} \frac{1}{\log n} \sin nx$$

converges for every x, yet it is not the Fourier series of a Riemann integrable function.

Proof: Let $b_n = \frac{1}{\log n}$ $a_n = \sin nx$. If $x \neq 2k\pi$ for some $k \in \mathbb{Z}$,

note that $\left|\sum_{n=1}^{N} a_n\right| = \left|\sum_{n=1}^{N} \sin nx\right| = \left|\frac{\sin(N+\frac{1}{2})x - \sin\frac{x}{2}}{2\sin(x/2)}\right|$

$$\leq \frac{1}{|\sin\frac{x}{2}|}$$

By Dirichlet's test, the series $\sum_{n\geq 2} \frac{1}{\log n} \sin nx$ converges

If $x = 2k\pi$, $\sin nx = 0$ for all x. The series is obviously convergent.

Next we prove that it is not the Fourier series of a Riemann integrable function.

Suppose it were a Fourier series of an integrable function $f(\theta)$. The Parseval's identity implies that

$$\sum_{n=2}^{\infty} \left|\frac{1}{\log n}\right|^2 = \sum_{n=2}^{\infty} \left(\frac{1}{\log n}\right)^2 = \frac{1}{2\pi}\int_0^{2\pi} |f(\theta)|^2 d\theta < \infty$$

which leads to a contradiction since $\log n < \frac{1}{n}$ for all $n \geq 2$ and $\sum_{n=1}^{\infty} \frac{1}{n}$ diverges

Remark: The same is true for $\sum \frac{\sin nx}{n^\alpha}$ for $0<\alpha<1$. We can check directly by Dirichlet's test and Parsevals identity that the same conclusion holds for $0<\alpha\leq\frac{1}{2}$, but the case $\frac{1}{2}<\alpha<1$ is more difficult.

Exercise 8

Chapter 3

(a) Let f be the function defined on $[-\pi,\pi]$ by $f(\theta)=|\theta|$. Use Parseval's identity to find the sums of the following two series: $\sum_{n=0}^{\infty}\frac{1}{(2n+1)^4}$ and $\sum_{n=1}^{\infty}\frac{1}{n^4}$

(b) Consider the 2π-periodic odd function defined on $[0,\pi]$ by $f(\theta)=\theta(\pi-\theta)$. Show that $\sum_{n=0}^{\infty}\frac{1}{(2n+1)^6}=\frac{\pi^6}{960}$ and $\sum_{n=1}^{\infty}\frac{1}{n^6}=\frac{\pi^6}{945}$

Proof: (a) By the results given in Exercise 6 of Chapter 2, the Fourier coefficients of $f(\theta)=|\theta|$ are $\hat{f}(n)=\begin{cases}\frac{\pi}{2}, & \text{if } n=0\\ \frac{-1+(-1)^n}{\pi n^2} & \text{if } n\neq 0\end{cases}$

Apply the Parseval's identity, $\sum_{n=-\infty}^{\infty}|\hat{f}(n)|^2=\frac{1}{2\pi}\int_{-\pi}^{\pi}|f(\theta)|^2\,d\theta$

we get $$\frac{\pi^2}{4}+\sum_{n=-\infty}^{-1}\left|\frac{-1+(-1)^n}{\pi n^2}\right|^2+\sum_{n=1}^{\infty}\left|\frac{-1+(-1)^n}{\pi n^2}\right|^2=\frac{2}{3}\pi^3\cdot\frac{1}{2\pi}$$

$$\frac{1}{4}\pi^2+\sum_{n\text{ odd}\leq -1}\frac{2}{\pi n^4}+\sum_{n\text{ odd}\geq 1}\frac{2}{\pi^2 n^4}=\frac{1}{3}\pi^2$$

$$2\sum_{n\text{ odd}\geq 1}\left(\frac{2}{\pi n^4}\right)^2=\frac{\pi^2}{12}.$$

$$\frac{8}{\pi^2}\sum_{n\text{ odd}\geq 1}\frac{1}{n^4}=\frac{\pi^2}{12}$$

Finally we obtain the result

$$\sum_{n \text{ odd} \geq 1} \frac{1}{n^4} = \frac{\pi^4}{96} \quad \text{or} \quad \sum_{n=0}^{\infty} \frac{1}{(2n+1)^4} = \frac{\pi^4}{96}$$

Let $S = \sum_{n=1}^{\infty} \frac{1}{n^4}$, then $\sum_{n=1}^{\infty} \frac{1}{(2n)^4} = \frac{1}{16} \sum_{n=1}^{\infty} \frac{1}{n^4} = \frac{1}{16} S$

Then $S - \frac{1}{16} S = \sum_{n \text{ odd} \geq 1} \frac{1}{n^4} = \frac{\pi^4}{96}$. and $\frac{15}{16} S = \frac{\pi^4}{96}$

and finally
$$S = \sum_{n=1}^{\infty} \frac{1}{n^4} = \frac{\pi^4}{90}$$

(b) Write $f(\theta) = \begin{cases} \theta(\pi - \theta), & \theta \in [0, \pi] \\ & \theta \in [\pi, 0] \end{cases}$