Inner product

An inner product on a vector space V over $\mathbb{C}$ associates to any pair X, Y of elements in V a complex number, denoted by $\langle X, Y\rangle$, satisfying the following conditions.

(i) Positive definiteness: $\langle X, X\rangle \geq 0$ for all $X \in V$.

(ii) Linearity in the first slot $\langle \alpha X+\beta Y, Z\rangle = \alpha\langle X, Z\rangle + \beta\langle Y, Z\rangle$

(iii) Conjugate symmetry $\langle X, Y\rangle = \overline{\langle Y, X\rangle}$

Given an inner product $\langle\cdot,\cdot\rangle$ we may define the norm of X by $\|X\| = \langle X, X\rangle^{\frac{1}{2}}$

If in addition $\|X\|=0$ implies $X=0$, we say the inner product is positive-definite.

Two elements X and Y are orthogonal if $\langle X, Y\rangle = 0$, and we write $X \perp Y$.

Properties: (i) The Pythagorean theorem: if X and Y are orthogonal, then $\|X+Y\|^2 = \|X\|^2+\|Y\|^2$.

(ii) The Cauchy-Schwarz inequality: for any $X, Y \in V$, we have $\|\langle X, Y\rangle\| \leq \|X\|\cdot\|Y\|$.

(iii) The triangle inequality: for any $X, Y \in V$
$$\|X+Y\| \leq \|X\| + \|Y\|$$

Proof: (i) $\|X+Y\|^2 = \langle X+Y, X+Y\rangle = \langle X, X\rangle + \langle X, Y\rangle + \langle Y, X\rangle + \langle Y, Y\rangle$

Since $\langle X, Y\rangle = 0$, $\|X+Y\|^2 = \|X\|^2+\|Y\|^2$.

(ii) If $\|Y\|=0$, we show that $\langle X, Y\rangle = 0$. Indeed, for all real t,
$$\|X+tY\|^2 = \langle X+tY, X+tY\rangle = \|X\|^2 + \langle X, tY\rangle + \langle tY, X\rangle + \|Y\|^2$$
$$= \|X\|^2 + 2t\,\mathrm{Re}\langle X, Y\rangle$$

If $\mathrm{Re}\langle X,Y\rangle \neq 0$, we take t to be large positive or large negative such that $\|X\|^2 + 2t\,\mathrm{Re}\langle X,Y\rangle < 0$, which leads to a contradiction. Similarly, by considering $\|X + itY\|$, we find that $\mathrm{Im}\langle X,Y\rangle = 0$. Thus $\langle X,Y\rangle = 0$.

If $\|Y\| \neq 0$, set $c = \langle X,Y\rangle / \|Y\|^2$ then $X - cY$ is orthogonal to Y, and therefore also cY. Write $X = X - cY + cY$ and apply Pythagorean theorem,

$$\|X\|^2 = \|X - cY\|^2 + \|cY\|^2 \geq |c|^2 \|Y\|^2.$$

Taking square roots on both sides,

$$\|X\| \geq \langle X,Y\rangle / \|Y\|^2 \cdot \|Y\|.$$

and

$$|\langle X,Y\rangle| \leq \|X\|\,\|Y\|.$$

Note that we have equality in the above precisely when $X = cY$

(iii) $$\|X+Y\|^2 = \langle X,X\rangle + \langle X,Y\rangle + \langle Y,X\rangle + \langle Y,Y\rangle$$

$$= \|X\|^2 + \|Y\|^2 + \langle X,Y\rangle + \langle Y,X\rangle$$

By Cauchy-Schwarz Inequality

$$|\langle X,Y\rangle + \langle Y,X\rangle| \leq |\langle X,Y\rangle| + |\langle Y,X\rangle| \leq 2\|X\|\|Y\|$$

Then $\|X+Y\|^2 \leq \|X\|^2 + 2\|X\|\|Y\| + \|Y\|^2 = (\|X\| + \|Y\|)^2$.

Hilbert space: An inner product space with the following two properties is called a Hilbert space:

(i) The inner product is strictly positive definite, that is $\|X\| = 0$ implies $X = 0$

(ii) The vector space is complete, which by definition means that every Cauchy sequence in the norm converges to a limit in the vector space. If either of the conditions fail, the space is called a pre-Hilbert space.

Example for a pre-Hilbert space that (i) and (ii) fail.

Let R denote the set of complex-valued Riemann integrable functions on $[0, 2\pi]$. This is a vector space over $\mathbb{C}$.

Define the inner product by

$$\langle f, g\rangle = \frac{1}{2\pi}\int_0^{2\pi} f(\theta)\overline{g(\theta)}\,d\theta$$

Check Cauchy-Schwarz inequality: $|\langle f, g\rangle| \leq \|f\|\,\|g\|$.

Set $A = \lambda^{\frac{1}{2}}|f(\theta)|$, $B = \lambda^{-\frac{1}{2}}|g(\theta)|$. By the inequality

$2AB \leq A^2 + B^2$, for any two real numbers A and B

$$2|f(\theta)|\,|g(\theta)| \leq \lambda|f(\theta)|^2 + \lambda^{-1}|g(\theta)|^2$$

$$|\langle f, g\rangle| = \left|\frac{1}{2\pi}\int_0^{2\pi} f(\theta)\overline{g(\theta)}\,d\theta\right| \leq \frac{1}{2\pi}\int_0^{2\pi}|f(\theta)|\,|\overline{g(\theta)}|\,d\theta$$

$$\leq \frac{1}{2}\left(\lambda\|f\|^2 + \lambda^{-1}\|g\|^2\right)$$

Put $\lambda = \dfrac{\|g\|}{\|f\|}$

$$|\langle f, g\rangle| \leq \frac{1}{2}\left(\frac{\|g\|}{\|f\|}\|f\|^2 + \frac{\|f\|}{\|g\|}\|g\|^2\right) = \|f\|\cdot\|g\|$$

(i) $\|f\| = 0$ implies only f vanishes at its point of continuity. Therefore if we modify the value of f on a set of "measure zero" on $[0, 2\pi]$, the f can be not identically zero

(ii) R is not complete. The function $f(\theta) = \begin{cases} 0 & \theta = 0 \\ \log\frac{1}{\theta} & 0 < \theta \leq 2\pi \end{cases}$

f is not bounded and thus f does not belong to R.

The sequence of truncations f_n defined by

$$f_n(\theta) = \begin{cases} 0 & 0 < \theta < \frac{1}{n} \\ f(\theta) & \frac{1}{n} < \theta < 2\pi \end{cases}$$

$f_n(\theta)$ is a Cauchy sequence in $\mathcal{R}$. However the limit of $\{f_n(\theta)\}_{n=1}^{\infty}$, if existed, would have to be f. which is not belong to $\mathcal{R}$ (Exercise 5)

Theorem (Mean-square Convergence)

Suppose f is integrable on the circle. Then

$$\frac{1}{2\pi}\int_0^{2\pi} |f(\theta) - S_N(f)(\theta)|^2 d\theta \to 0 \quad \text{as } N \to \infty.$$

Sketch of proof: Define $\langle f, g\rangle = \frac{1}{2\pi}\int_0^{2\pi} f(\theta)\overline{g(\theta)}\, d\theta$.
which is an inner product. $e_n(\theta) := e^{in\theta}$, $\{e_n\}_{n\in\mathbb{Z}}$ is an orthonormal basis.

$$a_n = \frac{1}{2\pi}\int_0^{2\pi} f(\theta) e^{-in\theta} d\theta = \langle f, e_n\rangle$$

$$S_N(f) = \sum_{n=-N}^{N} a_n e_n = \sum_{n=-N}^{N} \langle f, e_n\rangle e_n$$

Observation: $(f - S_N(f)) \perp e_m$, $m = -N, \cdots, -1, 0, 1, \cdots N$

Because

$$\begin{aligned}
&\langle f - S_N(f), e_m\rangle \\
&= \langle f - \sum_{n=-N}^{N}\langle f, e_n\rangle e_n, e_m\rangle \\
&= \langle f, e_m\rangle - \langle \sum_{n=-N}^{N}\langle f, e_n\rangle e_n, e_m\rangle \\
&= \langle f, e_m\rangle - \langle\langle f, e_m\rangle e_m, e_m\rangle \\
&= \langle f, e_m\rangle - \langle f, e_m\rangle = 0.
\end{aligned}$$

By linearity, for any $c_n \in \mathbb{C}$, $n = -N, \cdots, -1, 0, 1, \cdots, N$,

$$(f - S_N(f)) \perp \sum_{n=-N}^{N} c_n e_n$$

We use the best approximation lemma.

Lemma (Best Approximation) If f is integrable on the circle with Fourier coefficients a_n, then

$$\|f - S_N(f)\| \leq \|f - \sum_{|n|\leq N} c_n e_n\|$$

for any complex numbers c_n. Moreover, equality holds precisely when $c_n = a_n$ for all $|n| \leq N$.

Write

$$f - \sum_{n=-N}^{N} c_n e_n = f - S_N(f) + \sum_{n=-N}^{N} b_n e_n$$

$b_n = a_n - c_n$.

Take the norm and applying Pythagorean theorem

$$\| f - \sum_{n=-N}^{N} c_n e_n \| = \|f - S_N(f)\| + \| \sum_{n=-N}^{N} b_n e_n \|$$
$$\geq \|f - S_N(f)\|$$

Suppose f is continuous on the circle, by Corollary 5.4 in Chapter 2, there exists a trigonometric polynomial P of degree M such that $|f(\theta) - P(\theta)| < \varepsilon$. Then

$$\| f - P \| = \left(\frac{1}{2\pi} \int_0^{2\pi} |f(\theta) - P(\theta)|^2 \right)^{\frac{1}{2}} = (\varepsilon^2)^{\frac{1}{2}} < \varepsilon.$$

by the best approximation lemma, $\|f - S_N(f)\| < \varepsilon$ whenever $N \geq M$.

If f is merely integrable, using Lemma 3.2 in Chapter 2, we choose a continuous function g such that

$$\sup_{\theta \in [0, 2\pi]} |g(\theta)| \leq \sup_{\theta \in [0, 2\pi]} |f(\theta)| := B.$$

and

$$\int_0^{2\pi} |f(\theta) - g(\theta)| < \varepsilon^2.$$

Then

$$\|f - g\|^2 = \frac{1}{2\pi} \int_0^{2\pi} |f(\theta) - g(\theta)|^2 d\theta$$
$$= \frac{1}{2\pi} \int_0^{2\pi} |f(\theta) - g(\theta)| \, |f(\theta) - g(\theta)| \, d\theta$$
$$\leq \frac{2B}{2\pi} \int_0^{2\pi} |f(\theta) - g(\theta)| \, d\theta$$

$\leq C\varepsilon^2$ where we take $C = B/\pi$.

Now we approximate g by a trigonometric polynomial P so that $\|g-P\| < \varepsilon$. Then $\|f-P\| < C'\varepsilon$ and we may again conclude by applying the best approximation lemma.

This completes the proof of the partial sums of f converg to f in the mean square sense.

Parseval's Identity

Let $a_n = \hat{f}(n)$ be the n-th Fourier coefficient of an integrable function f, then the series $\sum_{n=-\infty}^{\infty} |a_n|^2$ converges and

$$\sum_{n=-\infty}^{\infty} |a_n| = \|f\|^2.$$

Proof: Write

$$f = f - S_N(f) + S_N(f), \quad S_N(f) = \sum_{n=-N}^{N} a_n e_n$$

By the obervation above

$$(f - S_N(f)) \perp S_N(f)$$

Then by Pythagorean theorem

$$\|f\| = \|f - S_N(f)\| + \|S_N(f)\|$$

and $\|S_N(f)\| = \sum_{n=-N}^{N} |a_n|$ again by the orthogonality of $\{e_n\}_{n=-\infty}^{\infty}$

Now letting $N \to \infty$, by the mean square convergence theorem

$$\lim_{N\to\infty} \|f - S_N(f)\| = 0 \quad \text{and} \quad \lim_{n\to\infty} \|S_N(f)\| = \sum_{n=-\infty}^{\infty} |a_n|^2$$

Then we obtain the identity

$$\|f\| = \sum_{n=-\infty}^{\infty} |a_n|^2$$

Remark 1. If $\{e_n\}$ is any orthonormal family of functions on the circle, and $a_n = \langle f, e_n \rangle$, then we have the Bessel's inequality

$$\sum_{n=-\infty}^{\infty} |a_n|^2 \leq \|f\|^2.$$

This is obtained by the relation

$$\|f\|^2 = \|f - S_N(f)\|^2 + \sum_{n=-N}^{N} |a_n|^2.$$

and the equality holds (Parseval's identity) when

$$\|f - S_N(f)\| \to 0 \quad \text{as } N \to \infty,$$

in this sense we say that $\{e_n\}_{n=-\infty}^{\infty}$ is also a "basis".

Remark 2 There exists sequences $\{a_n\}_{n\in\mathbb{Z}}$ such that $\sum_{n\in\mathbb{Z}} |a_n| < \infty$, yet no Riemann-integrable function F has n-th Fourier coefficient equal to a_n for all n. (Exercise 6)

Theorem (Riemann-Lebesgue Lemma)

If f is integrable on the circle, then $\hat{f}(n) \to 0$ as $|n| \to \infty$.

This is immediately from the Parseval's identity.

Since $\sum_{n=-\infty}^{\infty} |a_n|^2 = \|f\|^2$ converges, then $|a_n|^2 \to 0$ as $|n| \to \infty$ and hence $\hat{f}(n) = a_n \to 0$.

By writing $a_n = \frac{1}{2\pi}\int_0^{2\pi} f(\theta) e^{-in\theta}\, d\theta$

$$= \frac{1}{2\pi}\int_0^{2\pi} f(\theta)(\cos n\theta - i\sin n\theta)\, d\theta$$

$$= \frac{1}{2\pi}\left[\left(\int_0^{2\pi} f(\theta)\cos n\theta\, d\theta\right) + i\left(\int_0^{2\pi} f(\theta)\sin n\theta\, d\theta\right)\right]$$

Then we obtain an equivalent description.

$$\int_0^{2\pi} f(\theta)\cos(n\theta)\,d\theta \to 0 \qquad \text{and}$$

$$\int_0^{2\pi} f(\theta)\sin(n\theta)\,d\theta \to 0 \qquad \text{as} \quad n\to\infty$$

Generalization of Parseval's identity

Lemma. Suppose F and G are integrable on the circle with

$$F \sim \sum a_n e^{in\theta} \quad \text{and} \quad G \sim \sum b_n e^{in\theta}$$

Then
$$\frac{1}{2\pi}\int_0^{2\pi} F(\theta)\overline{G(\theta)}\,d\theta = \sum_{n=-\infty}^{\infty} a_n \overline{b_n}$$

Proof: We claim the identity that

$$\langle F, G\rangle = \frac{1}{4}\left[\|F+G\|^2 - \|F-G\|^2 + i\left(\|F+iG\|^2 - \|F-iG\|^2\right)\right]$$

By Parseval's identity and the claim,

$$\frac{1}{2\pi}\int_0^{2\pi} F(\theta)\overline{G(\theta)}\,d\theta = \frac{1}{4}\sum_{n=-\infty}^{\infty}\left[|a_n+b_n|^2 - |a_n-b_n|^2 + i\left(|a_n+ib_n|^2 - |a_n-ib_n|^2\right)\right]$$

$$|a_n+b_n|^2 - |a_n-b_n|^2 + i\left(|a_n+ib_n|^2 - |a_n-ib_n|^2\right)$$

$$= (a_n+b_n)(\bar{a}_n+\bar{b}_n) - (a_n-b_n)(\bar{a}_n-\bar{b}_n) + i\big((a_n+ib_n)(\bar{a}_n+i\bar{b}_n) - (a_n-ib_n)(\bar{a}_n-i\bar{b}_n)\big)$$

$$= |a_n|^2 + a_n\bar{b}_n + \bar{a}_n b_n + |b_n|^2 - \left(|a_n|^2 - a_n\bar{b}_n - \bar{a}_n b_n + |b_n|^2\right)$$
$$+\, i\left(|a_n|^2 + a_n\overline{ib_n} + \bar{a}_n i b_n + |ib_n|^2 - \left(|a_n|^2 - a_n\overline{ib_n} - \bar{a}_n i b_n + |ib_n|^2\right)\right)$$

$$= 2a_n\bar{b}_n + 2\bar{a}_n b_n + i\left(-a_n i\bar{b}_n + \bar{a}_n i b_n - a_n i\bar{b}_n + \bar{a}_n i b_n\right)$$

$$= 2a_n\bar{b}_n + \cancel{2\bar{a}_n b_n} + a_n\bar{b}_n - \cancel{\bar{a}_n b_n} + a_n\bar{b}_n - \cancel{\bar{a}_n b_n}$$

$$= 4a_n\bar{b}_n$$

Therefore
$$\frac{1}{2\pi}\int_0^{2\pi} F(\theta)\overline{G(\theta)}\,d\theta = \sum_{n=-\infty}^{\infty} a_n\bar{b}_n$$

The claim is verified directly

$$\text{RHS} = \frac{1}{4}\Big[\langle F+G, F+G\rangle - \langle F-G, F-G\rangle + i\big(\langle F+iG, F+iG\rangle - \langle F-iG, F-iG\rangle\big)\Big]$$

$$= \frac{1}{4}\Big[\langle F,F\rangle + \langle F,G\rangle + \langle G,F\rangle + \langle G,G\rangle - \big(\langle F,F\rangle - \langle F,G\rangle - \langle G,F\rangle + \langle G,G\rangle\big) + i\Big(\big(\langle F,F\rangle - i\langle F,G\rangle + i\langle G,F\rangle + \langle G,G\rangle\big) - \big(\langle F,F\rangle + i\langle F,G\rangle - i\langle G,F\rangle + \langle G,G\rangle\big)\Big)\Big]$$

$$= \frac{1}{4}\Big[\langle F,G\rangle + \langle G,F\rangle + \langle F,G\rangle + \langle G,F\rangle + i\big(-i\langle F,G\rangle + i\langle G,F\rangle - i\langle F,G\rangle + i\langle G,F\rangle\big)\Big]$$

$$= \frac{1}{4}\Big[\langle F,G\rangle + \langle G,F\rangle + \langle F,G\rangle + \langle G,F\rangle + \langle F,G\rangle - \langle G,F\rangle + \langle F,G\rangle - \langle G,F\rangle\Big]$$

$$= \frac{1}{4}\cdot 4\langle F,G\rangle$$

$$= \langle F,G\rangle = \text{LHS}.$$

Then the proof is completed.

Remark: $\displaystyle\sum_{n=-\infty}^{\infty} |a_n \bar{b}_n| = \sum_{n=-\infty}^{\infty} |a_n||b_n| \le \sum_{n=-\infty}^{\infty} \frac{1}{2}\left(|a_n|^2 + |b_n|^2\right)$

$$= \frac{1}{2}\left(\sum_{n=-\infty}^{\infty} |a_n|^2 + \sum_{n=-\infty}^{\infty} |b_n|^2\right).$$

Since $\displaystyle\sum_{n=-\infty}^{\infty} |a_n|^2$ and $\displaystyle\sum_{n=-\infty}^{\infty} |b_n|^2$ converge absolutely by Parseval's identity, we conclude that $\displaystyle\sum_{n=-\infty}^{\infty} a_n \bar{b}_n$ converges also absolutely.