A continuous function with diverging Fourier series

We construct a continuous periodic function whose Fourier series diverges at one point. Principle: "symmetry-breaking"

Step 1. The sawtooth function f defined by

$$f(\theta)=\begin{cases} i(-\pi-\theta) & 0<\theta<\pi \\ i(\pi-\theta) & -\pi<\theta<0 \\ 0 & \theta=0. \end{cases}$$

has Fourier series $f(\theta)\sim \sum_{n\neq 0}\frac{e^{in\theta}}{n}$

Break the symmetry, the resulting series is

$$\sum_{n=-\infty}^{-1}\frac{e^{in\theta}}{n}, \qquad a_n=\frac{1}{n}$$

Claim: the series above is no longer the Fourier series of a Riemann integrable function.

Proof: Suppose it were the Fourier series of an integrable function $\tilde{f}$. $\tilde{f}$ is bounded. Using the Abel means, recall that

$$A_r(\tilde{f})(\theta)=\sum_{n=-\infty}^{\infty} r^{|n|}a_n e^{in\theta}$$

$$|A_r(\tilde{f})(0)|=\left|\sum_{n=-\infty}^{-1} r^{|n|}\cdot\frac{1}{n}\right|=\sum_{n=1}^{\infty}\frac{r^n}{n}$$

which tends to infinity as r tends to 1, because $\sum_{n=1}^{\infty}\frac{1}{n}$ diverges. (Note that $\sum_{n=1}^{\infty}\frac{r^n}{n}=-\log(1-r)$, $0\leq r<1$ by Taylor expansion).

This gives the desired contradiction since

$$|A_r(\tilde{f})(0)|\leq\frac{1}{2\pi}\int_{-\pi}^{\pi}|\tilde{f}(\theta)|P_r(\theta)\,d\theta \leq \sup_{\theta}|\tilde{f}(\theta)|$$

where $P_r(\theta)$ denotes the Poisson kernel.

(Note that the Poisson kernel is a non-negative even function, and $\frac{1}{2\pi}\int_{-\pi}^{\pi} P_r(\theta)\,d\theta = 1$.

$$|A_r(\tilde{f})(0)| = \left|\frac{1}{2\pi}\int_{-\pi}^{\pi} \tilde{f}(\theta)\, P_r(-\theta)\,d\theta\right| = \left|\frac{1}{2\pi}\int_{-\pi}^{\pi} \tilde{f}(\theta)\, P_r(\theta)\,d\theta\right|$$

$$\leq \frac{1}{2\pi}\int_{-\pi}^{\pi} |\tilde{f}(\theta)|\, P_r(\theta)\,d\theta$$

$$\leq \frac{1}{2\pi}\int_{-\pi}^{\pi} \left(\sup_\theta |\tilde{f}(\theta)|\right) P_r(\theta)\,d\theta$$

$$= \sup_\theta |\tilde{f}(\theta)| \left(\frac{1}{2\pi}\int_{-\pi}^{\pi} P_r(\theta)\,d\theta\right)$$

$$= \sup |\tilde{f}(\theta)| \quad \text{finite since } \tilde{f} \text{ is bounded.}$$

The equality holds and contradict with the unboundeness of $A_r(\tilde{f})(0)$ as $r \to 1^-$)

Step 2. For each $N \geq 1$ we define two functions on $[-\pi, \pi]$.

$$f_N(\theta) = \sum_{1 \leq |n| \leq N} \frac{e^{in\theta}}{n} \quad \text{and} \quad \tilde{f}_N(\theta) = \sum_{-N \leq n \leq -1} \frac{e^{in\theta}}{n}$$

Claim (i) $|\tilde{f}_N(0)| \geq c \log N$.

(ii) $f_N(\theta)$ is uniformly bounded in N and θ

Proof of (i) follows from the inequality:

$$\sum_{n=1}^{N} \frac{1}{n} \geq \sum_{n=1}^{N-1} \int_n^{n+1} \frac{dx}{x} = \int_1^N \frac{dx}{x} = \log N$$

So we see that if M satisfies both $|A_r| \leq M$ and $n|C_n| \leq M$, then $|S_N| \leq 3M$.

We apply the lemma to the series $\sum_{n \neq 0} \frac{e^{in\theta}}{n}$, where the Fourier coefficients $C_n = \frac{e^{in\theta}}{n} + \frac{e^{-in\theta}}{-n}$ for $n \neq 0$, so clearly $C_n = O(1/|n|)$. Finally $A_r(f)(\theta) = (f * P_r)(\theta)$. But f is bounded and P_r is a good kernel, so $S_N(f)(\theta)$ is uniformly bounded in N and θ, as was to be shown.

(f is bounded, $(f * P_r)(\theta) \xrightarrow[r \to 1^-]{} f(\theta)$ choose an m such that $|f(\theta)| \leq m$ $|A_r| \leq m$ and $n|C_n| \leq m$, then $|S_N(f)(\theta)| \leq 3m$, uniformly bounded)

Step 3. $f_N(\theta) = \sum_{1 \leq |n| \leq N} \frac{e^{in\theta}}{n}$, $\tilde{f}_N(\theta) = \sum_{-N \leq n \leq 1} \frac{e^{in\theta}}{n}$ are trigonometric polynomials of degree N.

Define $P_N(\theta) = e^{i(2N)\theta} f_N(\theta)$, $\tilde{P}_N(\theta) = e^{i(2N)\theta} \tilde{f}_N(\theta)$ the degree of which are $3N$ and $2N-1$.

Now consider the partial sums of $P_N(\theta)$, which is denoted by $S_M(P_N)$.

Lemma
$$S_M(P_N) = \begin{cases} P_N, & \text{if } M \geq 3N \\ \tilde{P}_N & \text{if } M = 2N \\ 0 & \text{if } M < N. \end{cases}$$

This is easy to verify. The effect is that when $M = 2N$, the operator S_M breaks the symmetry of P_N.

To prove (ii) we introduce a lemma, whose proof is similar as the Tauber's theorem. (Exercise 14 Chapter 2)

Lemma: Suppose that the Abel means $A_r = \sum_{n=1}^{\infty} r^n c_n$ of the series $\sum_{n=1}^{\infty} c_n$ are bounded as r tends to 1 (with $r < 1$). If $c_n = O(1/n)$, then the partial sums $S_N = \sum_{n=1}^{N} c_n$ are bounded.

Proof: Let $r = 1 - 1/N$ and choose an M such that $n|c_n| \leq M$.

We estimate the difference

$$S_N - A_r = \sum_{n=1}^{N} (c_n - r^n c_n) - \sum_{n=N+1}^{\infty} r^n c_n$$

as follows:

$$|S_N - A_r| \leq \sum_{n=1}^{N} |c_n|(1-r^n) + \sum_{n=N+1}^{\infty} r^n |c_n|$$

$$\leq M \sum_{n=1}^{N} (1-r) + \frac{M}{N} \sum_{n=N+1}^{\infty} r^n$$

$$\leq MN(1-r) + \frac{M}{N} \frac{1}{1-r} \quad \text{recall that } r = 1 - 1/N$$

$$= 2M.$$

where the second inequality follows from the simple observation

$$1 - r^n = (1-r)(1 + r + \cdots + r^{n-1}) \leq n(1-r).$$

and the third equality follows from the simpler inequality

$$\sum_{n=N+1}^{\infty} r^n \leq \sum_{n=1}^{\infty} r^n = \frac{1}{1-r}$$

Step 4. Find a convergent series of positive terms $\sum \alpha_k$ and a sequence of integers $\{N_k\}$, which increases rapidly enough so that

(i) $N_{k+1} > 3N_k$ (ii) $\alpha_k \log N_k \to \infty$ as $k \to \infty$.

Choose (for example) $\alpha_k = \frac{1}{k^2}$, $N_k = 3^{2^k}$, which are easily seen to satisfy the criteria above.

Step 5 $\quad f(\theta) = \sum_{k=1}^{\infty} \alpha_k P_{N_k}(\theta)$ is the desired function.

Because $|P_N(\theta)| = |e^{i(2N)\theta} f_N(\theta)| = |f_N(\theta)|$. the series converges uniformly to a continuous periodic function.

How ever by our lemma we get.

$$|S_{2N_m}(f)(0)| \geq C\alpha_m \log N_m + O(1) \to \infty \quad \text{as } m \to \infty$$

Indeed, first we have

$$S_M(f)(\theta) = \sum_{k=1}^{\infty} \alpha_k S_M(P_{N_k})(\theta).$$

where $S_M(P_{N_k})(\theta) = \begin{cases} P_{N_k} & M \geq 3N_k \\ P_{N_k} & M = 2N_k \\ 0 & M < N_k \end{cases}$

$$S_{2N_m} f(\theta) = \sum_{k=1}^{m} \alpha_k S_{2N_m}(P_{N_k})(\theta) + \sum_{k=m+1}^{\infty} \alpha_k S_{2N_m}(P_{N_k})(\theta) + \alpha_m S_{2N_m}(P_{N_m})(\theta$$

when $k < m$, $|S_{2N_m}(P_{N_k})(\theta)| = O(1)$ since $2N_m > N_m > N_k$ and

$S_{2N_m}(P_{N_k})(\theta) = P_{N_k}(\theta) = e^{i(2N)\theta} f_{N_k}(\theta)$ is uniformly bounded by the claim in Step 2. Hence $\sum_{k=1}^{m} \alpha_k S_{2N_m}(P_{N_k})(\theta)$ is uniformly bounde when $k > m$,

$$S_{2N_m}(P_{N_k})(\theta) = 0 \quad \text{since } 2N_m < N_k$$

Now we substitute $\theta = 0$ to the formula.

$$\alpha_m S_{2N_m}(P_{N_m})(\theta) = \alpha_m \tilde{P}_{N_m}(\theta) = \alpha_m e^{i(2N_m)\theta} \tilde{f}_{N_m}(\theta)$$

Take the absolute value,

$$|\alpha_m S_{2N_m}(P_N)(\theta)| = \alpha_m |\tilde{f}_{N_m}(\theta)| \geq c\alpha_m \log N_m \to \infty$$

as $m \to \infty$

Therefore

$$|S_{2N_m}(f)(0)| \geq c\alpha_m \log N_m + O(1) \to \infty \quad \text{as} \quad m \to \infty.$$

We are done since this proves the divergence of the Fourier series of f at $\theta = 0$.

To produce a function whose series diverges at any other preassigned $\theta = \theta_0$, it suffices to consider the function $f(\theta - \theta_0)$.