Fourier Transform on $\mathbb{R}$

A function f defined on $\mathbb{R}$ is said to be moderate decrease if f is continuous and there exists a constant $A > 0$ so that $|f(x)| \leq \frac{A}{1+x^2}$ for all $x \in \mathbb{R}$.

Example ① $f(x) = \frac{1}{1+|x|^n}$ ② $f(x) = e^{-a|x|}$ $(a > 0)$

Denote by $M(\mathbb{R})$ the set of functions of moderate decrease on $\mathbb{R}$.

$$M(\mathbb{R}) = \{ f \in C(\mathbb{R}) \mid \exists A > 0, \text{so that } |f(x)| \leq \frac{A}{1+x^2}, \forall x \in \mathbb{R} \}$$

Proposition. $M(\mathbb{R})$ forms a vector space over $\mathbb{C}$.

We define: $\int_{-\infty}^{\infty} f(x)\,dx = \lim_{N \to \infty} \int_{-N}^{N} f(x)\,dx$ whenever f belongs to $M(\mathbb{R})$.

Proposition. If $f \in M(\mathbb{R})$, then for each $N \in \mathbb{N}^*$, let $I_N = \int_{-N}^{N} f(x)\,dx$. Then $\{I_N\}_{N=1}^{\infty}$ is a Cauchy sequence.

Remark: We may replace the exponent 2 in the definition of moderate increase by $1+\varepsilon$ where $\varepsilon > 0$.

Proposition The integral of a function f of moderate decrease satifies the following properties:

(i) Linearity: if $f, g \in M(\mathbb{R})$, and $a, b \in \mathbb{C}$, then

$$\int_{-\infty}^{\infty} (af(x) + bg(x))\,dx = a\int_{-\infty}^{\infty} f(x)\,dx + b\int_{-\infty}^{\infty} g(x)\,dx$$

(ii) Translation invariance: for every $h \in \mathbb{R}$ we have

$$\int_{-\infty}^{\infty} f(x-h)\,dx = \int_{-\infty}^{\infty} f(x)\,dx.$$

(iii) Scaling under dilations: if $\delta > 0$, then

$$\delta \int_{-\infty}^{\infty} f(\delta x)\,dx = \int_{-\infty}^{\infty} f(x)\,dx$$

(iv) Continuity: if $f \in \mathcal{M}(\mathbb{R})$, then

$$\int_{-\infty}^{\infty} |f(x-h) - f(x)|\,dx \to 0 \quad \text{as} \quad h \to 0.$$

Proof of (ii): It suffices to see that

$$\int_{-N}^{N} f(x-h)\,dx - \int_{-N}^{N} f(x)\,dx \to 0 \quad \text{as} \quad N \to \infty$$

By change of variable formula

$$\int_{-N}^{N} f(x-h)\,dx = \int_{-N-h}^{N-h} f(x)\,dx$$

The above difference is majorized by

$$\left|\int_{-N-h}^{-N} f(x)\,dx\right| + \left|\int_{N-h}^{N} f(x)\,dx\right| = O\left(\frac{1}{1+N^2}\right)$$

for some constant A and large N, which tends to 0 as N tends to infinity.

Proof of (iv) It suffices to take $|h| \le 1$. For a preassigned $\varepsilon > 0$, we first choose N so large that

$$\int_{|x| \ge N} |f(x)|\,dx < \frac{\varepsilon}{4} \quad \text{and} \quad \int_{|x| \ge N} |f(x-h)|\,dx < \frac{\varepsilon}{4}.$$

Now with N fixed, we use ~~the~~ fact that since f is continuous, it is uniformly continuous in the interval $[-N-1, N+1]$. Hence $\sup_{|x| \le N} |f(x-h) - f(x)| \to 0$ as $h \to 0$.

So we can take h so small that this supremum is less than $\varepsilon/4N$. Altogether, then

$$\int_{-\infty}^{\infty} |f(x-h) - f(x)|\, dx \leq \int_{-N}^{N} |f(x-h) - f(x)|\, dx$$

$$+ \int_{|x| \geq N} |f(x-h)|\, dx + \int_{|x| \geq N} |f(x)|\, dx$$

$$\leq \varepsilon/2 + \varepsilon/4 + \varepsilon/4 = \varepsilon.$$

and thus conclusion (iv) holds.

Definition: If $f \in M(\mathbb{R})$, we define its Fourier transform for $\xi \in \mathbb{R}$ by

$$\hat{f}(\xi) = \int_{-\infty}^{\infty} f(x)\, e^{-2\pi i x \xi}\, dx$$

The Fourier transform for $f \in M(\mathbb{R})$ is well-defined because $|e^{-2\pi i x \xi}| = 1$. Then the integrand is of moderate decrease.

Observation: $\hat{f}$ is bounded, and $\hat{f}$ is continuous and tends to 0 as $|\xi| \to \infty$. (Exercise 5).

The Schwartz space on $\mathbb{R}$ consists of the set of all indefinitely differentiable functions f so that f and all its derivatives $f', f'', \dots, f^{(l)}, \dots$ are rapidly decreasing, in the sense that $\sup_{x \in \mathbb{R}} |x|^k |f^{(l)}(x)| < \infty$, for every $k, l \geq 0$.

We denote this space by $S(\mathbb{R})$.

$$S(\mathbb{R}) = \{ f \in C^\infty \mid \sup_{x \in \mathbb{R}} |x|^k |f^{(l)}(x)| < \infty, \ \forall k, l \geq 0 \}$$

Observation 1 $S(\mathbb{R}) \subseteq M(\mathbb{R})$

This is because if $f \in S(\mathbb{R})$, then f is obviously continuous. Next, take $k = 1$ and $k = 2$ respectively with $l = 0$. There exists an A and a B such that

$$|f(x)| \leq A$$

and $x^2 |f(x)| \leq B$.

Then $(x^2+1)|f(x)| \leq A + B$. and thus $|f(x)| \leq \frac{A+B}{x^2+1}$.

Therefore $f \in M(\mathbb{R})$

Observation 2. $S(\mathbb{R})$ is a vector space over $\mathbb{C}$.

Observation 3. If $f(x) \in S(\mathbb{R})$, then $\frac{df}{dx} \in S(\mathbb{R})$ and

$$x f(x) \in S(\mathbb{R})$$

They are easy to verify. The Schwartz space is closed under differentiation and multiplication.

Example. ① $f(x) = e^{-x^2} \in S(\mathbb{R})$

$$e^{-ax^2} \in S(\mathbb{R}) \quad \text{whenever } a > 0.$$

(v) We must show that $\hat{f}$ is differentiable and find its derivative.

Let $\varepsilon > 0$ and consider

$$\frac{\hat{f}(\xi+h) - \hat{f}(\xi)}{h} - (\widehat{-2\pi i x f})(\xi)$$

$$= \frac{1}{h} \int_{-\infty}^{\infty} f(x)[e^{-2\pi i x(\xi+h)} - e^{-2\pi i x \xi}]\, dx - \int_{-\infty}^{\infty} -2\pi i x f(x) e^{-2\pi i x \xi} dx$$

$$= \int_{-\infty}^{\infty} f(x) e^{-2\pi i x \xi} \left[\frac{e^{-2\pi i x h} - 1}{h} + 2\pi i x \right] dx$$

Since $f(x)$ and $xf(x)$ are of rapid decrease, there exists an integer N such that $\int_{|x| \geq N} |f(x)| dx \leq \varepsilon$ and $\int_{|x| \geq N} |x||f(x)| dx \leq \varepsilon$.

Moreover, for $|x| \leq N$, there exists h_0 so that $|h| < h_0$ implies $\left| \frac{e^{-2\pi i x h} - 1}{h} + 1 \right| \leq \frac{\varepsilon}{N}$. $\left(\text{since } \lim_{t \to 0} \frac{e^{-t} - 1}{t} = 1\right)$

Hence for $|h| < h_0$ we have

$$\left| \frac{f(\xi+h) - f(\xi)}{h} - \widehat{2\pi i x f}(\xi) \right|$$

$$\leq \int_{-N}^{N} \left| f(x) e^{-2\pi i x \xi} \left[\frac{e^{-2\pi i x h} - 1}{h} + 2\pi i x \right] \right| dx$$

$$+ \int_{|x| \geq N} \left| f(x) e^{-2\pi i x \xi} \left[\frac{e^{-2\pi i x h} - 1}{h} + 2\pi i x \right] \right| dx$$

By the argument above,

$$\left| \frac{f(\xi+h) - f(\xi)}{h} - \widehat{2\pi i x f}(\xi) \right| \leq 2\varepsilon + C\varepsilon = C'\varepsilon \text{ for some } C' \geq 0$$

Therefore $-2\pi i x f(x) \longrightarrow \frac{d}{d\xi} \hat{f}(\xi)$

Theorem If $f \in S(\mathbb{R})$, then $\hat{f} \in S(\mathbb{R})$

Proof: If $f \in S(\mathbb{R})$, then

$$|\hat{f}(\xi)| = \left|\int_{-\infty}^{\infty} f(x) e^{-2\pi i x \xi} d\xi\right|$$

$$\leq \int_{-\infty}^{\infty} |f(x)||e^{-2\pi i x \xi}| d\xi$$

$$= \int_{-\infty}^{\infty} |f(x)| d\xi$$

Since $|x|^2 |f(x)|$ is bounded by M for some constant M,

then $|\hat{f}(\xi)| \leq \int_{-\infty}^{\infty} \frac{M}{x^2} dx < \infty$, which means $\hat{f}(\xi)$ is bounded.

From (v) of the above propositions, we conclude that

$$(-2\pi i)^n x^n f(x) \longrightarrow \left(\frac{d}{d\xi}\right)^n \hat{f}(\xi) \quad \text{for every } n \geq 0$$

Thus $\hat{f}$ is infinitely differentrable.

Consider the expression $\xi^k \left(\frac{d}{d\xi}\right)^\ell \hat{f}(\xi)$. By the propositions (iv) and (v) above, it is the Fourier transform of

$$\frac{1}{(2\pi i)^k} \left(\frac{d}{dx}\right)^k [(-2\pi i x)^\ell f(x)]$$

where the expression above belongs to $S(\mathbb{R})$

Proposition. The normalization of Gaussian integral

$$\int_{-\infty}^{\infty} e^{-\pi x^2} dx = 1$$

Theorem (Fundamental property of Gaussian).

$e^{-\pi x^2}$ equals its Fourier transform. That is to say.

If $f(x) = e^{-\pi x^2}$, then $\hat{f}(\xi) = f(\xi)$

Proof: Define $F(\xi) = \hat{f}(\xi) = \int_{\infty}^{\infty} e^{-\pi x^2} e^{-2\pi i x \xi} dx$ and observe that $F(0) = 1$. By property (v) in Proposition 1.2, and the fact that $f'(x) = 2\pi x f(x)$, we obtain

$$F'(\xi) = \int_{-\infty}^{\infty} f(x)(-2\pi i x) e^{-2\pi i x \xi} dx = i \int_{-\infty}^{\infty} f'(x) e^{-2\pi i x \xi} dx$$

By (iv) of the same proposition. we find that

$$F'(\xi) = i(2\pi i \xi) \hat{f}(\xi) = -2\pi i \xi F(\xi)$$

If we define $G(\xi) = F(\xi) e^{\pi \xi^2}$, then from what we have seen above, it follows that $G'(\xi) = 0$, hence G is constant.

Since $F(0) = 1$, we conclude that G is identically to 1, therefore $F(\xi) = e^{-\pi \xi^2}$, as was to be shown.

Corollary If $\delta > 0$. and $K_\delta(x) = \delta^{-\frac{1}{2}} e^{-\pi x^2/\delta}$, then $\hat{K}_\delta(\xi) = e^{-\pi \delta \xi^2}$

This is directly from the previous theorem and (iii) in Property 1.2. (with δ replaced by $\delta^{\frac{1}{2}}$)

Proposition If $f, g \in S(\mathbb{R})$, then

$$\int_{-\infty}^{\infty} f(x)\hat{g}(x)\,dx = \int_{-\infty}^{\infty} \hat{f}(y)g(y)\,dy$$

Proof: Let $F(x,y) = f(x)\,g(y)\,e^{-2\pi i x y}$. The aim is to show that

$$\int_{-\infty}^{\infty} f(x)\left(\int_{-\infty}^{\infty} g(y)e^{-2\pi i x y}dy\right)dx = \int_{-\infty}^{\infty}\left(\int_{-\infty}^{\infty} f(x)e^{-2\pi i x y}dx\right)g(y)\,dy$$

where the left hand side is $\int_{-\infty}^{\infty}\left(\int_{-\infty}^{\infty} f(x)\,g(y)\,e^{-2\pi i x y}dy\right)dx$

and the right hand side is $\int_{-\infty}^{\infty}\left(\int_{-\infty}^{\infty} f(x)\,g(y)\,e^{-2\pi i x y}\,dx\right)dy$

Now recall the concepts of the double (improper) integrals.

Consider the closed square centered at origin

$$Q_N = \{\, x = (x_1, x_2) : |x_1| < N/2,\ |x_2| < N/2 \,\}$$

Let $f(x,y)$ be a continuous function on $\mathbb{R}^2$. If the limit

$$\lim_{N\to\infty} \int_{Q_N} f(x)\,dx \qquad \text{exists,}$$

we denote it by

$$\int_{\mathbb{R}^2} f(x)\,dx$$

Theorem (Fubini theorem for double integrals)

Let f be a continuous function defined on a closed rectangle $R \subset \mathbb{R}^2$. Suppose $R = R_1 \times R_2$ where $R_1 \subset \mathbb{R}$ and $R_2 \subset \mathbb{R}$. If we write $x = (x_1, x_2)$ with $x_1, x_2 \in \mathbb{R}$. then $F(x_1) = \int_{\mathbb{R}_2} f(x_1, x_2)\,dx_2$ is continuous on R_1, and we have

$$\int_R f(x)\,dx = \int_{R_1}\left(\int_{R_2} f(x_1,x_2)\,dx_2\right)dx_1$$

We continue to prove the proposition

First we show that the improper integral $\int_{\mathbb{R}^2} F(x,y)\,dx\,dy$ exists

This is obvious since $|F(x,y)| = |f(x)||g(y)| \leq \frac{A}{(1+x^2)(1+y^2)}$ for some $A>0$

and $\int_{\mathbb{R}^2} F(x,y)\,dx\,dy \leq \int_{\mathbb{R}^2} |F(x,y)|\,dx\,dy$

$$\leq A \int_{I_N \times I_N} \frac{1}{(1+x^2)(1+y^2)}\,dx\,dy,$$

where $I_N = [-N, N]$. By Fubini's theorem for double integrals,

$$\int_{\mathbb{R}^2} |F(x,y)|\,dx\,dy \leq A \int_{I_N \times I_N} \frac{1}{(1+x^2)(1+y^2)}\,dx\,dy$$

$$= A \int_{-N}^{N} \left(\int_{-N}^{N} \frac{1}{1+x^2} \frac{1}{1+y^2}\,dx \right) dy$$

$$= A \left(\int_{-N}^{N} \frac{1}{1+x^2}\,dx \right) \left(\int_{-N}^{N} \frac{1}{1+y^2}\,dy \right)$$

$$\to A \cdot \pi^2. \quad \text{a finite number.}$$

Therefore $\int_{\mathbb{R}^2} F(x,y)\,dx\,dy < \infty$

Next we show that $\int_{\mathbb{R}^2} F(x,y)\,dx\,dy = \int_{\mathbb{R}} \left(\int_{\mathbb{R}} F(x,y)\,dx \right) dy$

Given $\varepsilon > 0$, choose N so large that

$$\left| \int_{\mathbb{R}^2} F(x,y)\,dx\,dy - \int_{I_N \times I_N} F(x,y)\,dx\,dy \right| < \varepsilon.$$

where by Fubini's theorem,

$$\int_{I_N \times I_N} F(x,y)\,dx\,dy = \int_{I_N} \left(\int_{I_N} F(x,y)\,dy \right) dx$$

$$= \int_{\mathbb{R}} \int_{\mathbb{R}} - \int_{I_N^c} \int_{\mathbb{R}} - \int_{I_N} \int_{I_N^c}$$

Let us first investigate $\int_{I_N^c} \left(\int_{\mathbb{R}} F(x,y)\, dy \right) dx$

$$\left| \int_{I_N^c} \int_{\mathbb{R}} F(x,y)\, dy\, dx \right| \le \int_{I_N^c} \left(\int_{\mathbb{R}} |F(x,y)|\, dy \right) dx$$

$$\le A \int_{I_N^c} \left(\int_{\mathbb{R}} \frac{1}{1+x^2} \frac{1}{1+y^2}\, dy \right) dx$$

$$= A \int_{I_N^c} \frac{1}{1+x^2} \left(\int_{\mathbb{R}} \frac{1}{1+y^2}\, dy \right) dx$$

Here $\int_{-\infty}^{\infty} \frac{1}{1+y^2}\, dy = \pi$ $\quad = A\pi \int_{I_N^c} \frac{1}{1+x^2}\, dx$

and N may be again so large such that $\int_{I_N^c} \frac{1}{1+x^2}\, dx < \frac{\varepsilon}{2\pi \cdot A}$

and therefore $\left| \int_{I_N^c} \int_{\mathbb{R}} \right| < \frac{\varepsilon}{2}$

Then we investigate $\int_{I_N} \int_{I_N^c} F(x,y)\, dy\, dx$

$$\left| \int_{I_N} \int_{I_N^c} F(x,y)\, dy\, dx \right| \le \int_{I_N} \int_{I_N^c} |F(x,y)|\, dy\, dx$$

$$\le A \int_{I_N} \int_{I_N^c} \frac{1}{1+x^2} \frac{1}{1+y^2}\, dy\, dx$$

$$\le A \int_{\mathbb{R}} \left(\int_{I_N^c} \frac{1}{1+x^2} \frac{1}{1+y^2}\, dy \right) dx$$

$$= A\pi \int_{I_N^c} \frac{1}{1+y^2}\, dy$$

$$< \frac{\varepsilon}{2}$$

Therefore $\left| \int_{\mathbb{R}^2} F(x,y)\, dx\, dy - \int_{\mathbb{R}} \left(\int_{\mathbb{R}} F(x,y)\, dy \right) dx \right|$

$$\le \left| \int_{\mathbb{R}^2} - \int_{I_N \times I_N} \right| + \left| \int_{I_N \times I_N} - \int_{\mathbb{R}} \int_{\mathbb{R}} \right| < \varepsilon + \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = 2\varepsilon.$$

and then $\int_{\mathbb{R}^2} F(x,y)\,dx\,dy = \int_{\mathbb{R}}\left(\int_{\mathbb{R}} F(x,y)\,dy\right)dx = \int_{\mathbb{R}}\left(\int_{\mathbb{R}} F(x,y)\,dx\right)dy$

by the symmetry of x and y.

The proposition is immediate since

$$\int_{-\infty}^{\infty} f(x)\,\hat{g}(x)\,dx = \int_{-\infty}^{\infty} f(x)\left(\int_{-\infty}^{\infty} g(y)\,e^{-2\pi i x y}\,dy\right)dx = \int_{-\infty}^{\infty} g(y)\left(\int_{-\infty}^{\infty} f(x)\,e^{-2\pi i x y}\,dx\right)dy$$

$$= \int_{-\infty}^{\infty} \hat{f}(y)\,g(y)\,dy. \qquad \square$$

Theorem (Fourier inversion). If $f \in S(\mathbb{R})$, then

$$f(x) = \int_{-\infty}^{\infty} \hat{f}(\xi)\,e^{2\pi i x \xi}\,d\xi$$

Proof: We first claim that

$$f(0) = \int_{-\infty}^{\infty} \hat{f}(\xi)\,d\xi$$

Let $G_\delta(x) = e^{-\pi\delta x^2}$, so that $\widehat{G_\delta}(\xi) = K_\delta(\xi)$.

By the multiplication formula we get

$$\int_{-\infty}^{\infty} f(x)\,K_\delta(x)\,dx = \int_{-\infty}^{\infty} \hat{f}(\xi)\,G_\delta(\xi)\,d\xi$$

Since K_δ is a good kernel, the first integral goes to $f(0)$ as δ tends to 0. (A rigorous explanation will be given later). Since the second integral clearly converges to $\int_{-\infty}^{\infty} \hat{f}(\xi)\,d\xi$ as δ tends to 0, the claim is proved.

An explanation is as follows:

Because $\hat{f}(\xi) \in S(\mathbb{R})$, $I = \int_{-\infty}^{\infty} |\hat{f}(\xi)|\,d\xi < \infty$, so given $\varepsilon > 0$, we find M so that $\int_{|\xi|>M} |\hat{f}(\xi)|\,d\xi < \frac{\varepsilon}{2}$.

Now find a $\delta > 0$ so that

$$\sup_{|\xi| \le M} (1 - G_\delta(\xi)) < \frac{\varepsilon}{2I}$$

Then $\left| \int_{|\xi| \le M} \hat{f}(\xi)(1 - G_\delta(\xi))\, d\xi \right| < \frac{\varepsilon}{2}$.

Combining with the result above, noting that $0 < G_\delta \le 1$, and we obtain the result that $\int_{-\infty}^{\infty} \hat{f}(\xi) G_\delta(\xi)\, d\xi \to \int_{-\infty}^{\infty} \hat{f}(\xi)\, d\xi$ as $\delta \to 0$.

In general, let $F(y) = f(y + x)$ so that

$$f(x) = F(0) = \int_{-\infty}^{\infty} \hat{F}(\xi)\, d\xi = \int_{-\infty}^{\infty} \hat{f}(\xi) e^{2\pi i x \xi}\, d\xi$$

and then we are done. $\square$

We may define two mappings $\mathcal{F} : S(\mathbb{R}) \to S(\mathbb{R})$ and $\mathcal{F}^* : S(\mathbb{R}) \to S(\mathbb{R})$ by

$$\mathcal{F}(f)(\xi) = \int_{-\infty}^{\infty} f(\xi) e^{-2\pi i x \xi}\, dx \quad \text{and} \quad \mathcal{F}^*(g)(x) = \int_{-\infty}^{\infty} g(\xi) e^{2\pi i x \xi}\, d\xi.$$

Thus $\mathcal{F}$ is the Fourier transform and the Fourier inversion theorem guarantees that $\mathcal{F}^* \circ \mathcal{F} = I$, where I is the identity mapping. Moreover we see that $\mathcal{F}(f)(y) = \mathcal{F}^*(f)(-y)$ so we also have $\mathcal{F} \circ \mathcal{F}^* = I$.

Corollary The Fourier transform is a bijective mapping on the Schwartz space.