Lecture 1, GOADyn
September 7, 2021

Let A be a (unital) C^*-algebra. We define

$$A_{\mathrm{sa}} = \{a \in A \,|\, a = a^*\}.$$

Note that $A = A_{\mathrm{sa}} + iA_{\mathrm{sa}}$, since for all $a \in A$ we have

$$a = \underbrace{\left(\frac{a+a^*}{2}\right)}_{a_R} + i\underbrace{\left(\frac{a-a^*}{2i}\right)}_{a_I}, \quad a_R, a_I \in A_{\mathrm{sa}}.$$

We define

$$\begin{aligned} A_+ &= \{a \in A \,:\, a = a^*,\ \sigma(a) \subset [0, +\infty)\} \\ &= \{a \in A \,:\, a = b^*b \text{ for some } b \in A\} \\ &= \{a \in A \,:\, a = b^2 \text{ for some } b \in A_{\mathrm{sa}}\}. \end{aligned}$$

Further, for all $a \in A_{\mathrm{sa}}$, $a = a_+ - a_-$, where $a_+, a_- \in A_+$ with $a_+a_- = 0$. Hence every element in A is a linear combination of 4 positive elements.

<u>Exercises:</u> *(Facts about A_+ and A_{sa})*

(1) For all $a \in A_{\mathrm{sa}}$, $-1_A\|a\| \leq a \leq \|a\|1_A$.
(2) A_+ is a cone, i.e.,
  - $a \in A_+$ and $\lambda > 0$ implies $\lambda a \in A_+$,
  - $a, b \in A_+$ implies $a + b \in A_+$.

(3) $0 \leq a \leq b \in A$ implies $0 \leq c^*ac \leq c^*bc$, for all $c \in A$.
(4) $0 \leq a \leq b \in A$ implies $\|a\| \leq \|b\|$.
(5) For $a \in A_{\mathrm{sa}}$, $\|a\| \leq 1$ if and only if $-1_A \leq a \leq 1_A$.

Let A, B be C^*-algebras. Then $M_n(A), M_n(B)$ are C^*-algebras for all $n \geq 1$. Indeed, if $A \subset B(H)$, then

$$M_n(A) \subset B(\underbrace{H \oplus \cdots \oplus H}_{n}).$$

Let $\varphi\colon A \to B$ be a linear map. For $n \geq 1$, consider $\varphi_n\colon M_n(A) \to M_n(B)$ given by

$$\varphi_n([a_{ij}]) := [\varphi(a_{ij})], \quad [a_{ij}] \in M_n(A).$$

Sometimes we use the notation $\varphi_n = \varphi \otimes \mathrm{Id}_{M_n(\mathbb{C})}$.

Definition 1.1. The map φ is called:

- *positive* if $\varphi(A_+) \subset B_+$.
- *n-positive* if φ_n is positive.
- *completely positive (c.p.)* if φ_n is positive for all $n \geq 1$.

Remark 1.2. In order to talk about positivity, we do not really need C^*-algebras. Given a C^*-algebra A, consider a closed linear subspace $E \subset A$ such that $\{e^* \,:\, e \in E\} = E^* = E$ and $1_A \in E$. E is called an *operator (sub)system.*

Note that some books, e.g., Paulsen: "Completely bounded maps and Operator Algebras", do not require E to be closed. Here we follow the convention from Brown-Ozawa [BO].

Examples of operator systems.

- Unital C^*-algebras are operator systems.
- $\left\{ \begin{bmatrix} \lambda I & x \\ y^* & \mu I \end{bmatrix} : x, y \in B(H) \right\} \subset M_2(B(H))$ is an operator (sub)system.
- $\left\{ \begin{bmatrix} \alpha & \beta \\ 0 & 0 \end{bmatrix} : \alpha, \beta \in \mathbb{C} \right\} \subset M_2(\mathbb{C})$ is <u>not</u> an operator system.

If $E \subset A$ is an operator subsystem, then we define

$$\begin{aligned} E_{\text{sa}} &= \{a \in E \,|\, a^* = a\} \subset A_{\text{sa}} = A_+ - A_+ \\ E_+ &= E_{\text{sa}} \cap A_+ \\ M_n(E)_+ &= (M_n(E))_{\text{sa}} \cap M_n(A)_+ \end{aligned}$$

(so $M_n(E)$ inherits the order structure from $M_n(A)$). One can then consider $\varphi\colon E \to B$ ($E \subset A$) and define positive, n-positive and c.p. as above.

Note that there are <u>positive</u> maps which are <u>not</u> c.p. Let $\varphi\colon M_2(\mathbb{C}) \to M_2(\mathbb{C})$ be given by $[a_{ij}] \mapsto [a_{ij}]^T$. Then

- **φ is positive:** Let $a \in M_2(\mathbb{C})_+$. Then $a = b^*b$ for some $b \in M_2(\mathbb{C})_+$ and

$$\varphi(a) = \varphi(b^*b) = (b^*b)^T = b^T(b^*)^T = b^T(b^T)^* = \varphi(b)\varphi(b)^* \geq 0.$$

- **φ is <u>not</u> 2-positive:** Let $\begin{bmatrix} e_{11} & e_{12} \\ e_{21} & e_{22} \end{bmatrix} \in M_2(M_2(\mathbb{C}))_+$ (see the proof of Proposition 1.5.12). However,

$$\varphi_2\left(\begin{bmatrix} e_{11} & e_{12} \\ e_{21} & e_{22} \end{bmatrix}\right) = \begin{bmatrix} e_{11}^T & e_{12}^T \\ e_{21}^T & e_{22}^T \end{bmatrix} = \begin{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} & \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} \\ \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} & \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \boxed{0} & \boxed{1} & 0 \\ 0 & \boxed{1} & \boxed{0} & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

where the middle matrix is not positive, since it has determinant -1.

Examples of c.p. maps. ① Let $E \subset A$ be an operator subsystem. If $\varphi\colon E \to C(\Omega)$ (where Ω is a compact Hausdorff topological space) is positive, then φ must be c.p.

Proof. Let $n \geq 1$ and let $[a_{ij}] \in M_n(E)_+$. Then

$$\varphi_n([a_{ij}]) = [\varphi(a_{ij})] \in M_n(C(\Omega)) \cong C(\Omega, M_n(\mathbb{C})).$$

We must show that $\forall \omega \in \Omega$, $[\varphi(a_{ij})(\omega)] \geq 0$ or, equivalently, $\alpha^*[\phi(a_{ij})(\omega)]\alpha \geq 0$, $\forall \alpha = \begin{bmatrix} \alpha_1 \\ \vdots \\ \alpha_n \end{bmatrix} \in M_{n,1}(\mathbb{C})$.

We have $\alpha^*[\phi(a_{ij})(\omega)]\alpha = \sum_{i,j} \overline{\alpha_i}\varphi(a_{ij})(\omega)\alpha_j = \varphi\left(\sum_{i,j} \overline{\alpha_i} a_{ij} \alpha_j\right)(\omega) = \varphi(\alpha^*[a_{ij}]\alpha)(\omega) \geq 0$, by positivity of φ. ☐

② If $\pi\colon A \to B$ is a *-homomorphism, then π is positive. In fact, π is c.p. (since $\pi_n\colon M_n(A) \to M_n(B)$ is a *-homomorphism for all $n \geq 1$).

Definition 1.3. Let $\varphi\colon A \to B$ be linear and bounded. We say that φ is *completely bounded (c.b.)* if

$$\|\varphi\|_{\mathrm{cb}} = \sup_n \|\varphi_n\| < \infty.$$

If $\|\varphi\|_{\mathrm{cb}} \leq 1$, we say that φ is *completely contractive (c.c.)*. If all φ_n are isometries, we say that φ is a *complete isometry*.

Remark 1.4. Let $\varphi\colon A \to B$ be a positive, linear map. Then:

(1) $\varphi(A_{\mathrm{sa}}) \subset B_{\mathrm{sa}}$.
(2) $\varphi(a^*) = \varphi(a)^*$ for all $a \in A$.

Proof. (1) Follows from $a \in A_{\mathrm{sa}} \Rightarrow a = a_+ - a_-$ with $a_+, a_- \in A_+$.

(2) Let $a \in A$. Then $a = a_R + ia_I$, where

$$a_R = \frac{a + a^*}{2}, a_I = \frac{a - a^*}{2i} \in A_{\mathrm{sa}}.$$

By using (1), $\varphi(a)^* = (\varphi(a_R + ia_I))^* = (\varphi(a_R) + i\varphi(a_I))^* = \varphi(a_R)^* - i\varphi(a_I)^* = \varphi(a_R - ia_I) = \varphi(a^*)$. □

On the connection between order and norms

Recall that if $\varphi\colon A \to \mathbb{C}$ is positive and linear, where A is unital, then φ is bounded with $\|\varphi\| = \varphi(1)$. The state space of A is given by

$$S(A) = \{\varphi\colon A \to \mathbb{C} \text{ positive and linear} : \|\varphi\| = 1\}.$$

Proposition 1.5. *Let A, B be C^*-algebras. If $\varphi\colon A \to B$ is positive and linear, then φ is bounded. Suppose further that A is unital. If, moreover, φ is 2-positive, then*

$$\|\varphi\| = \|\varphi(1)\|.$$

In particular, if φ is c.p., then φ is c.b. with

$$\|\varphi\| = \|\varphi\|_{\mathrm{cb}} = \|\varphi(1)\|.$$

(Hence, if φ is unital and c.p. (u.c.p.)*, then φ is completely contractive.)*

Proof. We first show that $\varphi\colon A \to B$ positive and linear implies that φ is bounded. Consider the family $\{f \circ \varphi \,|\, f \in S(B)\}$. Then for all $f \in S(B)$,

$$|(f \circ \varphi)(a)| \leq \|\varphi(a)\|, \quad a \in A.$$

By the Uniform boundedness principle, there exists $K > 0$ such that

$$|(f \circ \varphi)(a)| \leq K\|a\|, \quad f \in S(B), \ a \in A.$$

This implies that

$$\|\varphi(a)\| \leq K\|a\|, \quad a \in A_{\mathrm{sa}}.$$

(Use the fact that for any self-adjoint element in a C^*-algebra, there exists a (pure) state on the C^*-algebra, whose absolute value on the given self-adjoint element is equal to the norm of that element.) From here we deduce that $\|\varphi(a)\| \leq 2K\|a\|$ for all $a \in A$. Hence φ is bounded.

Now, suppose that A is unital and assume that φ is 2-positive. For all $a \in A_{\text{sa}}$,

$$-\|a\|1_A \leq a \leq \|a\|1_A \;\Rightarrow\; -\|a\|\varphi(1) \leq \varphi(a) \leq \|a\|\varphi(1) \;\Rightarrow\; \|\varphi(a)\| \leq \|a\|\|\varphi(1)\|.$$

To pass from A_{sa} to A, we use the following 2×2 matrix trick: Given $a \in A$, set

$$\tilde{a} = \begin{bmatrix} 0 & a^* \\ a & 0 \end{bmatrix} \in M_2(A).$$

Then $\tilde{a}^* = \tilde{a}$, $\|\tilde{a}\|_{M_2(A)} = \|a\|_A$. Note that

$$\varphi_2(\tilde{a}) = \begin{bmatrix} 0 & \varphi(a)^* \\ \varphi(a) & 0 \end{bmatrix}, \; \|\varphi_2(\tilde{a})\| = \|\varphi(a)\|.$$

φ_2 is positive, $\tilde{a}$ is self-adjoint, so by what we proved above, we have

$$\|\varphi_2(\tilde{a})\| \leq \|\tilde{a}\|\|\varphi_2(1)\|,$$

or, equivalently, $\|\varphi(a)\| \leq \|a\|\|\varphi(1)\|$. This implies that $\|\varphi\| \leq \|\varphi(1)\|$. We actually have equality. The rest follows easily. □

Remark 1.6.

(i) The following sharper result is true (see Corollary 2.9, Paulsen):
If A, B are unital C^*-algebras and $\varphi\colon A \to B$ is positive and linear, then

$$\|\varphi\| = \|\varphi(1)\|.$$

(ii) The statement that any c.p. map $\varphi\colon A \to B$ satisfies

$$\|\varphi\| = \|\varphi\|_{\text{cb}} = \|\varphi(1)\|$$

can also be obtained as a consequence of Stinespring's theorem below.

Lemma 1.7. *Let A be a unital C^*-algebra and $a \in A$. Then*

$$\|a\| \leq 1 \;\Leftrightarrow\; \begin{bmatrix} 1 & a \\ a^* & 1 \end{bmatrix} \in M_2(A)_+.$$

Proof. "⇒": If $\|a\| \leq 1$, then

$$\left\| \begin{bmatrix} 0 & a \\ a^* & 0 \end{bmatrix} \right\| = \max\{\|a\|, \|a^*\|\} = \|a\| \leq 1.$$

$\begin{bmatrix} 0 & a \\ a^* & 0 \end{bmatrix} \in M_2(A)_+$ and $1_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ is the unit in $M_2(A)$. Hence $-1_2 \leq \begin{bmatrix} 0 & a \\ a^* & 0 \end{bmatrix} \leq 1_2$, which implies that $0 \leq \begin{bmatrix} 1 & a \\ a^* & 1 \end{bmatrix}$.

"⇐": Suppose that $\begin{bmatrix} 1 & a \\ a^* & 1 \end{bmatrix} \geq 0$ in $M_2(A) \subset M_2(B(H))$ $(A \subset B(H))$. Then for all $\xi, \eta \in H$,

$$\begin{aligned} 0 \leq \left\langle \begin{bmatrix} 1 & a \\ a^* & 1 \end{bmatrix} \begin{bmatrix} \xi \\ \eta \end{bmatrix}, \begin{bmatrix} \xi \\ \eta \end{bmatrix} \right\rangle &= \langle \xi, \xi \rangle + \langle a\eta, \xi \rangle + \langle a^*\xi, \eta \rangle + \langle \eta, \eta \rangle \\ &= \|\xi\|^2 + 2\text{Re}\langle a\eta, \xi \rangle + \|\eta\|^2 \end{aligned}$$

Assume by contradiction that $\|a\| > 1$. Then there exist unit vectors $\xi, \eta \in H$ with $\langle a\eta, \xi\rangle < -1$. Then, with the above calculations,

$$0 \leq 1 + 2\mathrm{Re}\langle a\eta, \xi\rangle + 1 = 2 + 2\mathrm{Re}\langle a\eta, \xi\rangle < 2 - 2 = 0,$$

a contradiction! Thus $\|a\| \leq 1$. □

Lemma 1.8. *Let $v \in B(H)$. Then v is self-adjoint and $-1_H \leq v \leq 1_H$ if and only if*

$$\|v - it1_H\| \leq \sqrt{1+t^2}, \quad t \in \mathbb{R}.$$

Proof. "$\Rightarrow$": $\sigma(v) \subset [-1,1]$. By functional calculus, $v \in C^*(v) \subset B(H)$ corresponds to $f(\lambda) = \lambda$ and $v - it1_H$ corresponds to $g(\lambda) = \lambda - it$ in $C(\sigma(v))$, where $\lambda \in \sigma(v)$. Hence

$$\|v - it1_H\| = \|\lambda - it\|_\infty = \sup_{t \in \sigma(v)} |\lambda - it| = \sup_{\lambda \in \sigma(v)} \sqrt{\lambda^2 + t^2} \leq \sqrt{1+t^2}$$

since $\sigma(v) \subset [-1,1]$.

"$\Leftarrow$": Suppose $v = a + ib$, with a, b self-adjoint. Then

$$v - it1_H = a + i(b - t1_H) \;\Rightarrow\; \|(a + i(b-t1_H))^*(a + i(b - t1_H))\| = \|\underbrace{a + i(b - t1_H)}_{v - it1_H}\|^2 \leq 1 + t^2, \quad t \in \mathbb{R}$$

$$\Rightarrow\; 0 \leq (a + i(b - t1_H))^*(a + i(b - t1_H)) \leq (1 + t^2)1_H, \quad t \in \mathbb{R}$$

$$\Rightarrow\; 0 \leq \underbrace{a^*a + iab - iba + b^2}_{x} - 2bt \leq 1_H, \quad t \in \mathbb{R},$$

so $0 \leq x - 2bt \leq 1_H$ for all $t \in \mathbb{R}$. This implies that $b = 0$, and therefore $v = a$ is self-adjoint. Then $\|(v - it1_H)^*(v - it1_H)\| = \|v - it1_H\|^2 \leq 1 + t^2$ for all $t \in \mathbb{R}$. Since v is self-adjoint, $(v - it1_H)^*(v - it1_H) = v^2 + t^2 1_H$, so we get

$$0 \leq v^2 + t^2 1_H \leq 1_H + t^2 1_H \;\Rightarrow\; 0 \leq v^2 \leq 1_H \Rightarrow\; \|v\|^2 = \|v^*v\| = \|v^2\| \leq 1 \Rightarrow\; \|v\| \leq 1 \Rightarrow\; -1_H \leq v \leq 1_H,$$

completing the proof. □

Proposition 1.9. *Let A, B be unital C^*-algebras. If $\varphi \colon A \to B$ is unital (i.e., $\varphi(1_A) = 1_B$) and contractive (respectively, c.c.), then φ is positive (respectively, c.p.).*

Proof (Arveson, 1969). If $0 \leq a \leq 1_A$, then $0 \leq 1_A - a \leq 1_A$. Then by Lemma 1.8, $\|(1_A - a) - it1_A\| \leq \sqrt{1+t^2}$ for all $t \in \mathbb{R}$. Since φ is contractive, $\|\varphi((1_A - a) - it1_A)\| \leq \sqrt{1+t^2}$ for all $t \in \mathbb{R}$. As $\varphi(1_A) = 1_B$, this becomes

$$\|(1_B - \varphi(a)) - it1_B\| \leq \sqrt{1+t^2}$$

for all $t \in \mathbb{R}$. Using Lemma 1.8 again, we get $-1_B \leq 1_B - \varphi(a) \leq 1_B$, implying $\varphi(a) \geq 0$. For an arbitrary $a \geq 0$, we have $0 \leq \frac{a}{\|a\|} \leq 1_A$. The argument above shows that $\varphi(\frac{a}{\|a\|}) \geq 0$, i.e., $\varphi(a) \geq 0$. □

Recall the GNS representation for positive linear functionals:

Let A be a unital C^*-algebra. If $\varphi \colon A \to \mathbb{C}$ is positive and linear, then there exists a Hilbert space K and a unital *-representation $\pi \colon A \to B(K)$ and $\xi \in K$ with $\|\xi\|^2 = \|\varphi\|$ such that

$$\varphi(a) = \langle \pi(a)\xi, \xi\rangle = \xi^*\pi(a)\xi, \quad a \in A,$$

where $\xi\colon \mathbb{C} \to K$ is given by $\alpha \mapsto \alpha\xi$ and ξ^* is the adjoint map $K \to \mathbb{C}$. Moreover, $K = \overline{\pi(A)\xi}^{\|\cdot\|}$, i.e., ξ is a cyclic vector for π (or, π is a cyclic representation with cyclic vector ξ).

Theorem 1.10 (Stinespring, 1955)**.** *Let A be a unital C^*-algebra, and let H be a Hilbert space. If $\varphi\colon A \to B(H)$ is completely positive linear map, then there exists a Hilbert space K, a unital *-representation $\pi\colon A \to B(K)$ and $V\colon H \to K$ such that*

$$\varphi(a) = V^*\pi(a)V, \quad a \in A.$$

In particular, $\|\varphi\| = \|V^*V\| = \|\varphi(1)\|$, which, applied to φ_n implies than $\|\varphi_n\| = \|\varphi_n(1_{M_n(A)})\| = \|\varphi(1)\|$ for all $n \geq 1$, so $\|\varphi\|_{\mathrm{cb}} = \|\varphi\|$.

Proof. Define a sesquilinear form $\langle\cdot,\cdot\rangle_\varphi$ on (the algebraic tensor product $A \odot H$) by

$$\left\langle \sum_{i=1}^n a_i \otimes \xi_i, \sum_{j=1}^m b_j \otimes \eta_j \right\rangle_\varphi := \left\langle \underbrace{[\varphi(b_j^* a_i)]}_{\in M_{m,n}(B(H))} \begin{bmatrix} \xi_1 \\ \vdots \\ \xi_n \end{bmatrix}, \begin{bmatrix} \eta_1 \\ \vdots \\ \eta_m \end{bmatrix} \right\rangle_{H^m} = \sum_{i,j} \langle \varphi(b_j^* a_i)\xi_i, \eta_j\rangle_H.$$

Note that $\langle\cdot,\cdot\rangle_\varphi$ is positive semidefinite, i.e.,

$$\left\langle \sum_{i=1}^n a_i \otimes \xi_i, \sum_{i=1}^n a_i \otimes \xi_i \right\rangle_\varphi := \left\langle \underbrace{[\varphi(a_j^* a_i)]}_{=\varphi_n([a_j^* a_i])} \begin{bmatrix} \xi_1 \\ \vdots \\ \xi_n \end{bmatrix}, \begin{bmatrix} \xi_1 \\ \vdots \\ \xi_n \end{bmatrix} \right\rangle_{H^n} \geq 0$$

since φ_n is positive and$[a_j^* a_i] \in M_n(A)_+$, as we have

$$[a_j^* a_i] = \underbrace{\begin{bmatrix} a_1^* \\ \vdots \\ a_n^* \end{bmatrix}}_{\in M_{n,1}(A)} \underbrace{[a_1, \ldots, a_n]}_{\in M_{1,n}(A)} = c^*c \geq 0,$$

where $c = [a_1, \ldots, a_n]$. Let

$$N = \left\{ \sum_{i=1}^n a_i \otimes \xi_i \in A \odot H \;\middle|\; \left\langle \sum_{i=1}^n a_i \otimes \xi_i, \sum_{i=1}^n a_i \otimes \xi_i \right\rangle_\varphi = 0,\ n \in \mathbb{N} \right\}.$$

Note that N is a left A-module, i.e., if $a \in A$ and $\sum_{i=1}^n a_i \otimes \xi_i \in N$, then

$$a\left(\sum_{i=1}^n a_i \otimes \xi_i\right) := \sum_{i=1}^n aa_i \otimes \xi_i \in N.$$

This follows from the following:

Claim. *If $x \in A \odot H$, $a \in A$, then*

$$\langle ax, ax\rangle_\varphi \leq \|a\|^2 \langle x, x\rangle_\varphi.$$

Proof of claim. Let $x = \sum_{i=1}^n a_i \otimes \xi_i \in A \odot H$, for some $n \in \mathbb{N}$. Set $c = [a_1, \dots, a_n] \in M_{1,n}(A)$, $\xi = [\xi_1, \dots, \xi_n] \in H^n$. Then

$$\begin{aligned}\langle \varphi_n(c^*c)\xi, \xi\rangle &= \left\langle \begin{bmatrix} \varphi(a_1^*a_1) & \cdots & \varphi(a_1^*a_n) \\ \vdots & & \vdots \\ \varphi(a_n^*a_1) & \cdots & \varphi(a_n^*a_n)\end{bmatrix} \begin{bmatrix}\xi_1 \\ \vdots \\ \xi_n\end{bmatrix}, \begin{bmatrix}\xi_1 \\ \vdots \\ \xi_n\end{bmatrix} \right\rangle_{H^n} \\ &= \sum_{i,j} \langle \varphi(a_j^*a_i)\xi_i, \xi_j\rangle_H = \langle x, x\rangle_\varphi.\end{aligned}$$

Similarly, we see that

$$\langle \varphi_n(c^*a^*ac)\xi, \xi\rangle = \sum_{i,j} \langle \varphi(a_j^*a^*aa_i)\xi_i, \xi_j\rangle_H = \langle ax, ax\rangle_\varphi.$$

Now, $c^*a^*ac \leq \|a\|^2 c^*c$, so $\varphi_n(c^*a^*ac) \leq \varphi_n(c^*c)\|a\|^2$ (since φ_n is positive), hence $\langle \varphi_n(c^*a^*ac)\xi, \xi\rangle \leq \|a\|^2 \langle \varphi_n(c^*c)\xi, \xi\rangle$. Done!

Now consider ${}^{A\odot H}/_N$. Then $\langle \cdot, \cdot\rangle_\varphi$ is an inner product on it. Let K be the Hilbert space completion of $({}^{A\odot H}/_N, \langle \cdot, \cdot\rangle_\varphi)$. Given $a \in A$, define $\pi_0(a) \colon {}^{A\odot H}/_N \to {}^{A\odot H}/_N$ by

$$\left[\sum_{i=1}^n a_i \otimes \xi_i\right] \mapsto \left[\sum_{i=1}^n aa_i \otimes \xi_i\right].$$

$\pi_0(a)$ is a well-defined linear map (since N is a left A-module). Now, by the Claim,

$$\left\| \pi_0(a)\left(\left[\sum_{i=1}^n a_i \otimes \xi_i\right]\right)\right\|^2 = \left\| \left[\sum_{i=1}^n aa_i \otimes \xi_i\right]\right\|^2 \leq \|a\|^2 \left\| \left[\sum_{i=1}^n a_i \otimes \xi_i\right]\right\|^2.$$

So $\|\pi_0(a)\| \leq \|a\|$, i.e., $\pi_0(a)$ is a bounded operator on ${}^{A\odot H}/_N$. Hence we can extend it uniquely to $\pi(a) \in B(K)$, satisfying $\|\pi(a)\| \leq \|a\|$. We thus obtain a map $\pi \colon A \to B(K)$, satisfying $\|\pi(a)\| \leq \|a\|, \forall a \in A$.

Note now that π is a unital *-homomorphism. Indeed,

- $\pi(1)([\sum a_i \otimes \xi_i]) = [\sum 1 \cdot a_i \otimes \xi_i] = [\sum a_i \otimes \xi_i]$ implies $\pi(1) = \mathrm{Id}_K$, i.e., π is unital.
- $\pi(ab)([\sum a_i \otimes \xi_i]) = [\sum aba_i \otimes \xi_i] = \pi(a)([\sum ba_i \otimes \xi_i]) = \pi(a)\pi(b)([\sum a_i \otimes \xi_i])$, implying $\pi(ab) = \pi(a)\pi(b)$.
- $\pi(a^*) = \pi(a)$ by similar computations.

Now let $V \colon H \to K$ be defined by

$$V(\xi) = [1 \otimes \xi], \quad \xi \in H.$$

Let $a \in A$. Then for all $\xi, \eta \in H$,

$$\begin{aligned}\langle V^*\pi(a)V\xi, \eta\rangle_H &= \langle \pi(a)V\xi, V\eta\rangle_K \\ &= \langle \pi(a)[1 \otimes \xi], [1 \otimes \eta]\rangle_K \\ &= \langle [a \otimes \xi], [1 \otimes \eta]\rangle_K \\ &= \langle \varphi(1^*a)\xi, \eta\rangle_H = \langle \varphi(a)\xi, \eta\rangle_H,\end{aligned}$$

implying $V^*\pi(a)V = \varphi(a), \forall a \in A$. It follows that $\|V^*V\| = \|\varphi(1)\|$. On the other hand, for all $a \in A$,

$$\|\varphi(a)\| = \|V^*\pi(a)V\| \leq \|V\|^2 \|\pi(a)\| \leq \|a\| \|V\|^2,$$

so $\|\varphi\| \leq \|V\|^2 = \|\varphi(1)\| \leq \|\varphi\|$, hence $\|\varphi\| = \|\varphi(1)\|$. So $\|\varphi\| = \|\varphi(1)\| = \|V^*V\|$. Applying this to the c.p. map φ_n, we get $\|\varphi_n\| = \|\varphi_n(1)\| = \|\varphi(1)\| = \|\varphi\|$, hence φ is c.b. with $\|\varphi\|_{\mathrm{cb}} = \|\varphi\|$. □

Remark 1.11 (Remark 1.5.5 [BO])**.** (π, K, V) *is called a* Stinespring dilation *of* φ*. If* φ *is unital, then* $V^*V = \varphi(1) = 1$*, so* V *is an isometry. The projection* $VV^* \in B(K)$ *is called the* Stinespring projection*.*

A Stinespring dilation is <u>*not unique*</u>*. We may assume that* (π, K, V) *is* minimal*, in the sense that*

$$\overline{\pi(A)VH}^{\|\cdot\|} = K.$$

(This condition holds for the construction above.) Note that under the minimality assumption, a Stinespring dilation is unique up to unitary equivalence (Paulsen, Proposition 4.2).

Lecture 2, GOADyn
September 9, 2021

Multiplicative domains

Proposition 2.1 (Proposition 1.5.7 [BO])**.** *Let A, B be C^*-algebras and $\varphi\colon A \to B$ be c.c.p. (contractive c.p.) Then the following holds:*

(1) *(Schwarz inequality): $\varphi(a)^*\varphi(a) \leq \varphi(a^*a)$ for all $a \in A$.*
(2) *(Bimodule property): Given $a \in A$, if $\varphi(a)^*\varphi(a) = \varphi(a^*a)$, then $\varphi(ba) = \varphi(b)\varphi(a)$ for all $b \in A$, respectively, if $\varphi(a)\varphi(a)^* = \varphi(aa^*)$, then $\varphi(ab) = \varphi(a)\varphi(b)$ for all $b \in A$.*
(3) *$A_\varphi = \{a \in A : \varphi(a)^*\varphi(a) = \varphi(a^*a)$ and $\varphi(a)\varphi(a)^* = \varphi(aa)^*\}$ is a C^*-subalgebra of A.*

Proof. (1) Let $B \subset B(H)$ be a faithful *-representation and (π, K, V) a minimal Stinespring dilation of $\varphi\colon A \to B \subset B(H)$. Then, for all $a \in A$,

$$\varphi(a^*a) - \varphi(a)^*\varphi(a) = V^*\pi(a)^*(1_K - VV^*)\pi(a)V \geq 0$$

since $\|V\| \leq 1$.

(2) Let $a \in A$ with $\varphi(a^*a) = \varphi(a)^*\varphi(a)$. This is equivalent to $(1_K - VV^*)^{1/2}\pi(a)V = 0$. Then $\forall b \in A$,

$$\varphi(ba) - \varphi(b)\varphi(a) = V^*\pi(b)(1_K - VV^*)\pi(a)V = 0.$$

The other statement follows similarly.

(3) Follows from (2). $\square$

Definition 2.2 (Definition 1.5.8 [BO])**.** Let $\varphi\colon A \to B$ be a c.p. map. The C^*-algebra A_φ is called the *multiplicative domain* of φ.

Note that A_φ is the largest C^*-subalgebra C of A such that $\varphi|_C$ is a *-homomorphism.

Conditional expectations (important examples of c.c.p. maps)

Definition 2.3 (Definition 1.5.9 [BO])**.** *Let $B \subset A$ be (unital) C^*-algebras (if they are unital, then $1_B = 1_A$ does not necessarily hold).*

- *A* projection from *A onto B is a linear map $E\colon A \to B$ such that $E(b) = b$, for all $b \in B$.*
- *A* conditional expectation from *A* onto *B is a c.c.p. projection $E\colon A \to B$ onto such that $E(bxb') = bE(x)b'$, for all $x \in A$, $b, b' \in B$, i.e., E is a B-bimodule map.*

Theorem 2.4 (Tomiyama, Theorem 1.5.10 [BO])**.** *Let $B \subset A$ be (unital) C^*-algebras and $E\colon A \to B$ be a projection onto. The following are equivalent:*

(1) *E is a conditional expectation.*
(2) *E is c.c.p.*
(3) *E is contractive.*

Proof. Clearly (1) $\Rightarrow$ (2) $\Rightarrow$ (3).

We show (3) $\Rightarrow$ (1). By passing to second duals, we may assume that A and B are von Neumann algebras with units 1_A and 1_B, respectively (again, $1_A = 1_B$ may not be true). (One needs to check that $E\colon A \to B$ being a contractive projection implies that $E^{**}\colon A^{**} \to B^{**}$ is a contractive projection.)

First, we prove E is a B-bimodule map. Since von Neumann algebras are the norm-closed linear span of their projections, it suffices to check the module property on projections. Let $p \in B$ be a projection, and set $p^\perp = 1_A - p$. For every $x \in A$, $t \in \mathbb{R}$,

$$\begin{aligned} (1+t)^2\|pE(p^\perp x)\|^2 &= \|pE(p^\perp x + tpE(p^\perp x))\|^2 \\ &\leq \|p^\perp x + tpE(p^\perp x)\|^2 \\ &\overset{(\star)}{\leq} \|p^\perp x\|^2 + t^2\|pE(p^\perp x)\|^2, \end{aligned} \tag{2.3}$$

since $B \ni pE(p^\perp x) = E(pE(p^\perp x))$ and E is contractive. Inequality $(\star)$ follows from the following computations: Set $y = p^\perp x + tpE(p^\perp x)$, so

$$\begin{aligned} \|y\|^2 &= \|y^*y\| \\ &= \|x^*p^\perp x + t^2E(p^\perp x)^*pE(p^\perp x)\| \\ &\leq \|x^*p^\perp x\| + t^2\|E(p^\perp x)^*pE(p^\perp x)\| \\ &= \|p^\perp x\|^2 + t^2\|pE(p^\perp x)\|^2 \end{aligned}$$

(using $p^\perp p = 0 = pp^\perp$ at second equality), so $(\star)$ is justified. By (2.3) we therefore have

$$\|pE(p^\perp x)\|^2 + 2t\|pE(p^\perp x)\|^2 \leq \|p^\perp x\|^2$$

for all $t \in \mathbb{R}$, so $pE(p^\perp x) = 0$, for all projections $p \in B$ and all $x \in A$. In particular, for $p = 1_B$ we get

$$0 = 1_B \underbrace{E(1_B^\perp x)}_{\in B} = E(1_B^\perp x), \quad x \in A.$$

Respectively, for any projection $p \in B$, $1_B - p$ is also a projection in B, hence

$$(1_B - p)E((1_B - p)^\perp x) = 0$$

for all $x \in A$. But $(1_B - p)^\perp = 1_A - 1_B + p = 1_B^\perp + p$, implying $E((1_B - p)^\perp x) = E((1_B^\perp + p)x) = E(px)$, since $E(1_B^\perp x) = 0$ from above. Hence, for all $x \in A$, $(1_B - p)E(px) = 0$ which implies

$$E(px) = 1_BE(px) = pE(px) = pE(x - p^\perp x) = pE(x),$$

since $pE(p^\perp x) = 0$ from above. Therefore we have proved that $E(px) = pE(x)$, for all projections $p \in B$ and all $x \in A$. Similarly, $E(xp) = E(x)p$, for all projections $p \in B$ and all $x \in A$. We conclude that E is a B-bimodule map.

Note that E is a unital map, since $bE(1_A) = E(b) = b$, for all $b \in B$, so $E(1_A) = 1_B$. Since E is then a unital contraction ($\|E\| = 1$), E is positive (by Proposition 1.10, Lecture 1).

It remains to show that E is c.p. For this, we will use the following:

Lemma 2.5. *Let A be a unital C^*-algebra, let $n \in \mathbb{N}$ and $x \in M_n(A)$. Then $x \in M_n(A)_+$ if and only if $b^*xb \in A_+$ for all $b \in M_{n,1}(A)$.*

Proof. "$\Rightarrow$": Well-known.

"$\Leftarrow$": Suppose by contradiction that $x = [x_{ij}]$ is not positive in $M_n(A)$. Set $B = C^*(x_{ij}, 1 : 1 \leq i, j \leq n) \subset A$. Then B is separable and unital, $x \in M_n(B)$ and x is not positive in $M_n(B)$. Choose a faithful state $\rho \in S(B)$ and let $(\pi_\rho, H_\rho, \xi_\rho)$ be the corresponding GNS representation. Then $\pi_\rho \colon B \to B(H_\rho)$ is faithful (inj), and so is also $(\pi_\rho)_n \colon M_n(B) \to B(H_\rho^n)$. Hence $(\pi_\rho)_n(x)$ is not positive in $B(H_\rho^n)$. Note that

$$K = \left\{ \begin{bmatrix} \pi_\rho(b_1)\xi_\rho \\ \vdots \\ \pi_\rho(b_n)\xi_\rho \end{bmatrix} : b_j \in B \right\} \subset H_\rho^n$$

is <u>dense</u>. Hence $\langle (\pi_\rho)_n(x)\xi, \xi\rangle \not\geq 0$ for some

$$\xi = \begin{bmatrix} \pi_\rho(b_1)\xi_\rho \\ \vdots \\ \pi_\rho(b_n)\xi_\rho \end{bmatrix} \in K.$$

By letting

$$b = \begin{bmatrix} b_1 \\ \vdots \\ b_n \end{bmatrix} \in M_{n,1}(B) \subset M_{n,1}(A)$$

and observing that

$$\langle (\pi_\rho)_n(x)\xi, \xi\rangle = \sum_{i,j} \langle \pi_\rho(b_j^* x_{ji} b_i)\xi_\rho, \xi_\rho\rangle = \langle \pi_\rho(b^* x b)\xi_\rho, \xi_\rho\rangle,$$

we now get a contradiction. □

We return to the proof of the statement that $E \colon A \to B$ is completely positive: Take $x \in M_n(A)_+$. We must show that $E_n(x) \in M_n(B)_+$. By the above lemma, it is enough to show that

$$b^* E_n(x) b \in B_+ \quad \text{for all } b = \begin{bmatrix} b_1 \\ \vdots \\ b_n \end{bmatrix} \in M_{n,1}(B).$$

We have

$$b^* E_n(x) b = \sum_{i,j} b_j^* E(x_{ji}) b_i = \sum_{i,j} E(b_j^* x_{ji} b_i) = E\left(\sum_{i,j} b_j^* x_{ji} b_i\right) = E(b^* x b) \geq 0.$$

The proof is complete. □

Lemma 2.6 (Lemma 1.5.11 [BO])**.** *Let M be a von Neumann algebra with a faithful, normal, tracial state τ. Let $1_M \in N \subset M$ be a von Neumann subalgebra. Then there exists a unique normal conditional expectation $E \colon M \to N$ that is* τ-preserving, *i.e., $\tau \circ E = \tau$.*

Proof. Let $a, y \in M$. Let $a = u|a|$ be the polar decomposition of a in M. Note that $u, |a| \in M$. (The fact that $|a| \in M$ is clear, while $u \in M$, since $M \ni u|a|^{1/n} \to u$ SOT.)

Claim 1. $|\tau(ya)| \leq \|y\|\tau(|a|)$. Indeed,

$$\begin{aligned}
|\tau(ya)| = |\tau(yu|a|)| &= |\tau(yu|a|^{1/2}|a|^{1/2})| \\
&\overset{(1)}{\leq} \tau(yu\underbrace{|a|^{1/2}|a|^{1/2}}_{=|a|}u^*y^*)^{1/2}\tau(|a|)^{1/2} \\
&= \tau(yu|a|(yu)^*)^{1/2}\tau(|a|)^{1/2} \\
&\overset{(2)}{\leq} \|yu\|\tau(|a|) \\
&\overset{(3)}{\leq} \|y\|\tau(|a|).
\end{aligned}$$

(1) Here we use the Cauchy-Schwarz inequality for a positive linear functional $\varphi\colon A \to \mathbb{C}$:

$$|\varphi(x^*z)|^2 \leq \varphi(z^*z)\varphi(x^*x), \quad x, z \in A,$$

obtained by defining $\langle x, z\rangle_\varphi := \varphi(z^*x)$ and using the Cauchy-Schwarz inequality for positive definite sesquilinear forms. Here $x = (yu|a|^{1/2})^*$, $z = |a|^{1/2}$.

(2) With $w = yu$, we have $\tau(w|a|w^*) = \tau(w^*w|a|) = \tau(|a|^{1/2}w^*w|a|^{1/2}) \leq \|w^*w\|\tau(|a|) = \|w\|^2\tau(|a|)$.

(3) u is a partial isometry, so u^*u is a projection, implying $\|u\| = 1$.

For each $a \in N$, define $\tau_a\colon N \to \mathbb{C}$ by

$$\tau_a(y) = \tau(ya), \quad y \in N.$$

Then by Claim 1, $|\tau_a(y)| = |\tau(ya)| \leq \|y\|\tau(|a|)$, implying $\tau_a \in N^*$ with $\|\tau_a\| \leq \tau(|a|)$. In fact, $\|\tau_a\| = \tau(|a|)$ (since $|\tau_a(u^*)| = |\tau(u^*a)| = \tau(|a|)$, since $u^*a = |a|$ and $\|u^*\| = \|u\| = 1$).

Note that τ_a is normal. Suppose first that $a \geq 0$. Then τ_a is a positive linear functional, so to prove normality, it suffices to show that if $0 \leq y_\alpha \nearrow y$ SOT, then

$$\tau(y_\alpha a) = \tau_a(y_\alpha) \to \tau_a(y) = \tau(ya).$$

This follows from normality of τ. For the general case, an arbitrary $a \in N$ is a linear combination of 4 positive elements, and a linear combination of normal functionals is normal.

Claim 2. *$\{\tau_a : a \in N\}$ is a norm-dense subspace of N_*.*

If it were not norm-dense, then by Hahn-Banach, there would exist $0 \neq n \in (N_*)^* = N$ such that $\tau_a(n) = 0$ for all $a \in N$. In particular, $\tau_{n^*}(n) = 0$ implying $\tau(n^*n) = 0$. But τ is faithful, so $n = 0$, contradiction!

Construct $E\colon M \to N$ as follows:

For all $x \in M$, let $E(x)(\tau_a) := \tau(xa)$, for all $a \in N$. Recall that

$$|E(x)(\tau_a)| = |\tau(xa)| \leq \|x\|\tau(|a|) = \|x\|\|\tau_a\|,$$

where the latter equality was shown above. Hence $\|E(x)\| \leq \|x\|$. Use Claim 2 to conclude that $E(x)$ extends uniquely to a linear functional (still denoted by $E(x)$) on N_*, which is bounded with $\|E(x)\| \leq \|x\|$. So $E\colon M \to N$ is a well-defined contraction. Note also that for all $x \in M$ and $a \in N$,

$$\tau(E(x)a) = E(x)(\tau_a) = \tau(xa). \tag{2.4}$$

In particular, for $a = 1_M$, we get $\tau \circ E = \tau$ (E is τ-preserving). Next we show that E is a projection, i.e., for all $x \in M$, $E(E(x)) = E(x)$. Indeed, for all $a \in N$,

$$E(E(x))(\tau_a) = \tau(E(x)a) = \tau(xa) = E(x)(\tau_a)$$

by definition of $E(x)$ and (2.4). By uniqueness, $E(E(x)) = E(x)$.

Since E is a contractive projection, it follows from Tomiyama's theorem that E is a conditional expectation.

Furthermore, we show that E is normal, i.e., for all $x \in M_+$ and $0 \leq x_\alpha \nearrow x$ SOT, $\sup E(x_\alpha) = E(x)$. For this it suffices to show that for all $a \in N$,

$$\tau(E(x)a) = \tau((\sup_\alpha E(x_\alpha))a),$$

which follows from normality of τ.

Now assume that E' is another τ-preserving conditional expectation. Then for all $x \in M$, $a \in N$,

$$\tau(E'(x)a) = \tau(E'(xa)) = \tau(xa) = \tau(E(xa)) = \tau(E(x)a).$$

This implies $E = E'$. □

Examples 2.7. (1) Let $D_n(\mathbb{C}) \subset M_n(\mathbb{C})$ be the subalgebra of diagonal matrices. Then the conditional expectation $E\colon M_n(\mathbb{C}) \to D_n(\mathbb{C})$ is given by

$$E([a_{ij}]) = \begin{bmatrix} a_{11} & & 0 \\ & \ddots & \\ 0 & & a_{nn} \end{bmatrix}.$$

(2) Let $M = M_m(\mathbb{C})$, $N = M_n(\mathbb{C})$, $m, n \in \mathbb{N}$. The conditional expectation $E\colon M \otimes N \to M \otimes 1_n$ is given by

$$E(x \otimes y) = \int_{\mathcal{U}(\mathbb{C}^n)} x \otimes u^* y u \, du, \quad x \in M,\ y \in N,$$

where $\mathcal{U}(\mathbb{C}^n)$ is the group of unitary operators on $\mathbb{C}^n$ (compact in norm) and du is the Haar measure on $\mathcal{U}(\mathbb{C}^n)$.

Lecture 3, GOADyn
September 14, 2021

Proposition 3.1 (Proposition 1.5.12, [BO]). *Let A be a C^*-algebra and (e_{ij}) be matrix units in $M_n(\mathbb{C})$. A map $\varphi\colon M_n(\mathbb{C}) \to A$ is c.p. if and only if $[\varphi(e_{ij})] \in M_n(A)_+$. In other words,*

$$\mathrm{CP}(M_n(\mathbb{C}), A) \ni \varphi \longmapsto [\varphi(e_{ij})] \in M_n(A)_+$$

is a bijective correspondence. (Here $\mathrm{CP}(M_n(\mathbb{C}), A)$ denotes the set of c.p. maps from $M_n(\mathbb{C})$ into A.)

Proof. "$\Rightarrow$": Suppose that $\varphi\colon M_n(\mathbb{C}) \to A$ is c.p., in particular, φ is n-positive. Note that $e := [e_{ij}] \in (M_n(M_n(\mathbb{C})))_+$, since we can show that

$$e^2 = ne. \tag{$\star$}$$

Since e is self adjoint, this implies that e is positive. To prove ($\star$), note that for all $1 \le i, j \le n$,

$$(e^2)_{ij} = \sum_{k=1}^{n} \underbrace{e_{ik}e_{kj}}_{e_{ij}} = ne_{ij},$$

as wanted. Since φ_n is positive, $[\varphi(e_{ij})] = \varphi_n([e_{ij}]) \in M_n(A)_+$.

"$\Leftarrow$": Assume that $a = [\varphi(e_{ij})] \in M_n(A)_+$. Let $a^{1/2} := [b_{ij}]$. Then $\varphi(e_{ij}) = \sum_{k=1}^n b_{ki}^* b_{kj}$. Let $A \subset B(H)$ be a faithful *-representation and define $V\colon H \to \ell_n^2 \otimes \ell_n^2 \otimes H$ by

$$V\xi = \sum_{j,k=1}^{n} \zeta_j \otimes \zeta_k \otimes b_{kj}\xi, \quad \xi \in H,$$

where $(\zeta_j)_{j=1}^n$ is the canonical unit vector basis in ℓ_n^2. Then for $T = [t_{ij}] \in M_n(\mathbb{C})$, we have for all $\xi, \eta \in H$,

$$\begin{aligned}
\langle V^*(T \otimes 1_n \otimes 1_{B(H)})V\eta, \xi\rangle &= \langle (T \otimes 1 \otimes 1)V\eta, V\xi\rangle \\
&= \left\langle (T \otimes 1 \otimes 1) \sum_{j,k} \zeta_j \otimes \zeta_k \otimes b_{kj}\eta, \sum_{i,l} \zeta_i \otimes \zeta_l \otimes b_{li}\xi \right\rangle \\
&= \sum_{i,j,k,l} \langle T\zeta_j \otimes \zeta_k \otimes b_{kj}\eta, \zeta_i \otimes \zeta_\ell \otimes b_{\ell i}\xi\rangle \\
&= \sum_{i,j,k,l} \underbrace{\langle T\zeta_j, \zeta_i\rangle}_{t_{ij}} \underbrace{\langle \zeta_k, \zeta_\ell\rangle}_{\substack{1 \text{ if } k=\ell, \\ 0 \text{ else}}} \langle b_{kj}\eta, b_{\ell i}\xi\rangle \\
&= \sum_{i,j=1}^{n} t_{ij} \left\langle \sum_{k=1}^{n} b_{ki}^* b_{kj}\eta, \xi \right\rangle \\
&= \left\langle \varphi\left(\sum_{i,j} t_{ij}e_{ij}\right)\eta, \xi \right\rangle \\
&= \langle \varphi(T)\eta, \xi\rangle.
\end{aligned}$$

Hence $\varphi(T) = V^*(T \otimes 1 \otimes 1)V$ for all $T \in M_n(\mathbb{C})$. Clearly, φ is positive and for all $n \ge 1$, if

$$V_n = \begin{pmatrix} V & & 0 \\ & \ddots & \\ 0 & & V \end{pmatrix},$$

then

$$\varphi_n(T) = \begin{pmatrix} V^*\varphi(T)V & & 0 \\ & \ddots & \\ 0 & & V^*\varphi(T)V \end{pmatrix} = V_n^*(\varphi \otimes 1_n)(T)V_n$$

which is positive. Hence φ is c.p. □

Example 3.2 (Example 1.5.19, [BO])**.** *Let $a_1, \dots, a_n \in A$, and define $\varphi\colon M_n(\mathbb{C}) \to A$ by*

$$\varphi(e_{ij}) = a_i a_j^*.$$

By Proposition 3.1, φ is c.p. since

$$[\varphi(e_{ij})] = \begin{bmatrix} a_1a_1^* & \cdots & a_1a_n^* \\ \vdots & & \vdots \\ a_na_1^* & \cdots & a_na_n^* \end{bmatrix} = \begin{bmatrix} a_1 & \\ \vdots & 0 \\ a_n & \end{bmatrix} \begin{bmatrix} a_1 & \\ \vdots & 0 \\ a_n & \end{bmatrix}^* \geq 0.$$

Remark 3.3. *Let A be a C^*-algebra, $n \in \mathbb{N}$. A linear map $\varphi\colon M_n(\mathbb{C}) \to A$ is c.p. if and only if φ is n-positive.*

Proof. Suppose that φ is n-positive. By Proposition 3.1, it suffices to show that $[\varphi(e_{ij})] \in M_n(A)_+$. But $[\varphi(e_{ij})] = \varphi_n([e_{ij}])$. We have seen in the proof of Proposition 3.1 that $[e_{ij}] \in M_n(M_n(\mathbb{C}))_+$. Use the fact that φ_n is positive to get the conclusion. □

There is a *similar characterization of c.p. maps from A into $M_n(\mathbb{C})$*:

Given a linear map $\varphi\colon A \to M_n(\mathbb{C})$, define $\hat{\varphi}\colon M_n(A) \to \mathbb{C}$ by

$$\hat{\varphi}([a_{ij}]) := \sum_{i,j=1}^{n} \varphi(a_{ij})_{ij},$$

where $\varphi(a_{ij})_{ij}$ is the $(i,j)^{\text{th}}$ entry of the matrix $\varphi(a_{ij})$. Put differently, if $(\zeta_i)_{i=1}^n$ is the canonical ONB for ℓ_n^2, $\zeta = [\zeta_1, \dots, \zeta_n]^T \in (\ell_n^2)^n$, then

$$\hat{\varphi}([a_{ij}]) = \langle \varphi_n([a_{ij}])\zeta, \zeta \rangle, \quad [a_{ij}] \in M_n(A).$$

Proposition 3.4 (Proposition 1.5.14, [BO])**.** *Let A be a unital C^*-algebra. A linear map $\varphi\colon A \to M_n(\mathbb{C})$ is c.p. if and only if $\hat{\varphi} \in M_n(A)_+^*$, meaning that $\hat{\varphi}$ is a positive (thus bounded) linear functional on $M_n(A)$. Moreover,*

$$\mathrm{CP}(A, M_n(\mathbb{C})) \ni \varphi \longmapsto \hat{\varphi} \in M_n(A)_+^*$$

is a bijective correspondence.

Proof. "⇒": This is easy: If φ is c.p., then φ_n is positive, so with $\zeta \in (\ell_n^2)^n$ defined above,

$$\hat{\varphi}([a_{ij}]) = \langle \varphi_n([a_{ij}])\zeta, \zeta \rangle \geq 0,$$

whenever $[a_{ij}] \in M_n(A)_+$.

"⇐": Suppose that $\hat{\varphi}\colon M_n(A) \to \mathbb{C}$ is a positive linear functional. Let (π, H, ξ) be the GNS triple for $\hat{\varphi}$, i.e., $\pi\colon M_n(A) \to B(H)$ is a unital *-representation, $\xi \in H$ with $\|\xi\|^2 = \|\varphi\|$ and $\hat{\varphi}(x) = \langle \pi(x)\xi, \xi \rangle$ for all $x \in M_n(A)$. Let (e_{ij}) be the matrix units in $M_n(\mathbb{C})$, which can be viewed as elements in $M_n(A)$, i.e., we identify

$$e_{ij} \cong {}_i \begin{bmatrix} 0 & \overset{j}{\cdots} & 0 \\ \vdots & 1_A & \vdots \\ 0 & \cdots & 0 \end{bmatrix} \in M_n(A).$$

Define $V\colon \ell_n^2 \to H$ by

$$V(\zeta_j) = \pi(e_{1j})\xi, \quad 1 \le j \le n,$$

extended by linearity. We claim that the following holds,

$$\varphi(a) = V^*\pi\left(\begin{bmatrix} a & & 0 \\ & \ddots & \\ 0 & & a \end{bmatrix}\right)V, \quad a \in A,$$

which implies that φ is c.p. For $1 \le i, j \le n$, $\varphi(a)_{ij} = \langle \varphi(a)\zeta_j, \zeta_i\rangle$. Hence

$$\begin{aligned}
\left\langle V^*\pi\left(\begin{bmatrix} a & & 0 \\ & \ddots & \\ 0 & & a \end{bmatrix}\right)V\zeta_j, \zeta_i \right\rangle &= \left\langle \pi\left(\begin{bmatrix} a & & 0 \\ & \ddots & \\ 0 & & a \end{bmatrix}\right)V\zeta_j, V\zeta_i \right\rangle \\
&= \left\langle \pi\left(\begin{bmatrix} a & & 0 \\ & \ddots & \\ 0 & & a \end{bmatrix}\right)\pi(e_{1j})\xi, \underbrace{\pi(e_{1i})}_{(\pi(e_{i1}))^*}\xi \right\rangle \\
&= \left\langle \pi\left({}_i \begin{bmatrix} 0 & \overset{j}{\cdots} & 0 \\ \vdots & a & \vdots \\ 0 & \cdots & 0 \end{bmatrix}\right)\xi, \xi \right\rangle \\
&= \hat{\varphi}\left({}_i \begin{bmatrix} 0 & \overset{j}{\cdots} & 0 \\ \vdots & a & \vdots \\ 0 & \cdots & 0 \end{bmatrix}\right) = \varphi(a)_{ij},
\end{aligned}$$

as wanted. □

Lemma 3.5 (Lemma 1.5.15, [BO])**.** *Let A be a unital C^*-algebra, $E \subset A$ an operator subsystem and $\psi\colon E \to \mathbb{C}$ be a positive linear functional. Then $\|\psi\| = \psi(1_A)$. Hence any norm-preserving extension of ψ to A is also positive.*

Proof. Let $\varepsilon > 0$. Fix $x \in E$, $\|x\| \le 1$ such that $|\psi(x)| \ge \|\psi\| - \varepsilon$. Upon replacing x by αx for some $\alpha \in \mathbb{C}$, $|\alpha| = 1$, we may assume that $\psi(x) = |\psi(x)| \in \mathbb{R}$. Set

$$y = \frac{1}{2}(x + x^*).$$

Then $\psi(y) = \frac{1}{2}\psi(x) + \frac{1}{2}\overline{\psi(x)} = \psi(x)$ (since $\psi(x) \in \mathbb{R}$). As $\|y\| \le 1$ and $y = y^*$, we have $y \le 1_A$. Thus

$$\|\psi\| - \varepsilon \le \psi(x) = \psi(y) \le \psi(1_A) \le \|\psi\|.$$

As $\varepsilon > 0$ was arbitrary, we conclude that $\|\psi\| = \psi(1_A)$.

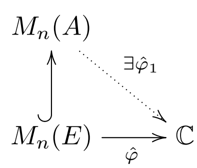
Now, let $\tilde{\psi}\colon A \to \mathbb{C}$, $\tilde{\psi}|E = \psi$ and $\|\tilde{\psi}\| = \|\psi\| = \psi(1_A) = \tilde{\psi}(1_A)$. Therefore $\frac{1}{\tilde{\psi}(1_A)}\tilde{\psi}$ is a unital contraction, and hence positive (by Proposition 1.9, Lecture 1). $\square$

Remark 3.6. *Using Proposition 3.4, one can now prove an analogue of the result in Remark 3.3, namely:*

If A is a unital C^*-algebra and $E \subset A$ is an operator subsystem, then a linear map $\varphi\colon E \to M_n(\mathbb{C})$ is c.p. if and only if φ is n-positive.

Corollary 3.7 (Corollary 1.5.16, [BO])**.** *Let $E \subset A$ be an operator subsystem and $\varphi\colon E \to M_n(\mathbb{C})$ a c.p. map. Then there exists a c.p. map $\psi\colon A \to M_n(\mathbb{C})$ extending φ.*

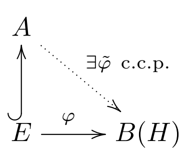
Proof. If $\varphi\colon E \to M_n(\mathbb{C})$ is c.p., then $\hat{\varphi}\colon M_n(E) \to \mathbb{C}$ is positive (as $M_n(E) \subset M_n(A)$ is an operator subsystem). (The argument is the same as in the proof of "$\Rightarrow$" in Proposition 3.4.) By Hahn-Banach, there exists $\hat{\varphi}_1\colon M_n(A) \to \mathbb{C}$ such that $\hat{\varphi}_1|_{M_n(E)} = \hat{\varphi}$ and $\|\hat{\varphi}_1\| = \|\hat{\varphi}\|$, yielding the following commutative diagram:



By Lemma 3.5, $\hat{\varphi}_1$ is positive. By Proposition 3.4, applying the 1-1 correspondence in reverse, there exists $\psi : A \to M_n(\mathbb{C})$ c.p. such that $\hat{\psi} = \hat{\varphi}_1$, which will imply that $\psi|_E = \varphi$. $\square$

Arveson's extension theorem

Theorem 3.8 (Theorem 1.6.1, [BO])**.** *Let A be a unital C^*-algebra, $E \subset A$ an operator subsystem. Then every c.c.p. map $\varphi\colon E \to B(H)$ extends to a c.c.p. map $\tilde{\varphi}\colon A \to B(H)$, i.e., the following diagram commutes:*

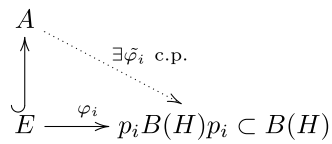


Proof. Let $(p_i)_{i\in I} \subset B(H)$ be an increasing net of finite rank projections such that $p_i \to 1_{B(H)}$ SOT. (If H is separable, and $(e_n)_{n\ge1}$ is an ONB for H, one can take p_n to be the projection onto $\mathrm{Span}\{e_1, \dots, e_n\}$. In the general case, one may take the net of all finite rank projections.)

For all $i \in I$, we define $\varphi_i\colon E \to p_i B(H) p_i$, by

$$\varphi_i(b) = p_i\varphi(b)p_i, \quad b \in E.$$

Then φ_i is a c.c.p. map and $p_i B(H) p_i \simeq B(p_i H)$ is a matrix algebra. By Corollary 3.7, φ_i extends to a c.p. map $\tilde{\varphi}_i$ on A:



such that $\tilde{\varphi}_{i|E} = \varphi_i$ and $\|\tilde{\varphi}_i\| = \|\tilde{\varphi}_i(1)\| = \|\varphi_i(1)\| \leq 1$ since $\|\varphi_i\| \leq 1$. Therefore $\tilde{\varphi}_i$ is a contraction. Hence $\tilde{\varphi}_i \in B(A, B(H))_1$ (the closed unit ball of $B(A, B(H))$). Note that $B(A, B(H))$ is a dual space.

In general, if X is a Banach space and M is a von Neumann algebra, consider $B(X, M)$. Let $E_0 \subset B(X, M)^*$ be the space

$$E_0 = \mathrm{Span}\{x \otimes \xi \in B(X, M)^* \, : \, x \in X, \ \xi \in M_*\},$$

where $(x \otimes \xi)(T) = \xi(Tx)$ for all $T \in B(X, M)$. Then $E = \overline{E_0}^{\|\cdot\|}$ is a Banach space and $E^* = B(X, M)$, in the sense that for every $\Lambda \in E^*$, there is a unique $T \in B(X, M)$ such that $\Lambda(\varphi) = \varphi(T)$, for all $\varphi \in E$. Moreover, $\|\Lambda\| = \|T\|$. We denote E by $B(X, M)_*$. By Alaoglu's theorem, the closed unit ball $B(X, M)_1$ of $B(X, M)$ is compact in the w*-topology coming from the duality $B(X, M) = E^*$. Hence, if (T_λ) is a net in $B(X, M)_1$, then it has a subnet (T_{λ_μ}) which converges w* to some $T \in B(X, M)_1$, i.e., $\varphi(T_{\lambda_\mu}) \to \varphi(T)$, for all $\varphi \in E$. In particular,

$$\xi(T_{\lambda_\mu}(x)) \to \xi(T(x)), \quad x \in X, \xi \in M_*,$$

i.e., the net (T_{λ_μ}) converges point-ultraweakly to T.

Back to our setting, there exists $\tilde{\varphi} \in B(A, B(H))_1$ such that $\tilde{\varphi}_i \to \tilde{\varphi}$ in the point-ultraweak topology.

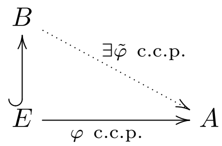
We have to show that ① $\tilde{\varphi}$ is c.c.p. and that ② $\tilde{\varphi}|_E = \varphi$.

② For any $b \in E$, $\tilde{\varphi}_i(b) = p_i \varphi(b) p_i$ since $\tilde{\varphi}_i|_E = \varphi_i$. Moreover, $\|\tilde{\varphi}_i(b)\| \leq \|b\|$ for all $i \in I$. Recall that $p_i \to 1_{B(H)}$ SOT. This will imply that $p_i \varphi(b) p_i \to \varphi(b)$ WOT, and we have that $\|p_i \varphi(b) p_i\| \leq \|\varphi(b)\|$, for all $i \in I$. Since on bounded sets in $B(H)$, WOT = ultraweak topology, we deduce that $\tilde{\varphi}(b) = \varphi(b)$. Hence $\tilde{\varphi}|_E = \varphi$.

① $\tilde{\varphi}$ is contractive (clear). $\tilde{\varphi}$ is also positive: Let $a \in A_+$. Since $\tilde{\varphi}_i(a) \to \tilde{\varphi}(a)$ ultraweakly and hence WOT, and $\tilde{\varphi}_i(a) \geq 0$ (since $\tilde{\varphi}_i$ is positive), we deduce $\tilde{\varphi}(a) \geq 0$. A similar argument applies to the amplifications $(\tilde{\varphi}_i)_n$ to conclude that $\tilde{\varphi}$ is c.p. □

Injectivity and Arveson's theorem (Remark 1.6.2, [BO]):

Definition 3.9 (Injective C^*-algebras)**.** *Let A be a C^*-algebra. We say that A is* injective *if whenever $E \subset B$ is an operator subsystem of a C^*-algebra B and $\varphi \colon E \to A$ is a c.c.p. map then there exists $\tilde{\varphi} \colon B \to A$ c.c.p. with $\tilde{\varphi}|_E = \varphi$:*



A von Neumann algebra is called injective if it is injective as a C^*-algebra.

By Arveson's extension theorem, $B(H)$ is an injective von Neumann algebra.

Injectivity of a von Neumann algebra can be characterized as follows:

Proposition 3.10. *Let $M \subset B(H)$ be a von Neumann algebra. Then M is injective if and only if there exists a contractive projection $P\colon B(H) \to M$ onto, i.e., a conditional expectation.*

Theorem 3.11 (Pisier/Christensen–Sinclair, 1994)**.** *Let $M \subset B(H)$ be a von Neumann algebra. Then M is injective if and only if there exists a c.b. projection $P\colon B(H) \to M$ onto.*

Lecture 4, GOADyn
September 21, 2021

Section 2.1 [BO]: Nuclear maps

Definition 4.1 (Definition 2.1.1, [BO]). *Let A, B be C^*-algebras. A bounded linear map $\theta\colon A \to B$ is called* nuclear *if there exist nets of contractive completely positive (c.c.p.) maps*

$$\varphi_n\colon A \to M_{k(n)}(\mathbb{C}), \quad \psi_n\colon M_{k(n)}(\mathbb{C}) \to B \quad (n \in I),$$

for some $k(n) \in \mathbb{N}$, such that $\psi_n \circ \varphi_n \to \theta$ in the point-norm topology, i.e., $\|\psi_n \circ \varphi_n(a) - \theta(a)\| \to 0$, for all $a \in A$.

Remark 4.2. *Since $\psi_n \circ \varphi_n\colon A \to B$ is c.c.p., for all $n \in \mathbb{N}$, then θ is also c.c.p., whenever θ is nuclear.*

Definition 4.3. *Let A be a C^*-algebra and N a von Neumann algebra. A bounded linear map $\theta\colon A \to N$ is called* weakly nuclear *if there exists c.c.p. maps*

$$\varphi_n\colon A \to M_{k(n)}(\mathbb{C}), \quad \psi_n\colon M_{k(n)}(\mathbb{C}) \to N \quad (n \in I)$$

for some $k(n) \in \mathbb{N}$, such that $\psi_n \circ \varphi_n \to \theta$ in the point-ultraweak topology, i.e.,

$$\psi_n \circ \varphi_n(a) \xrightarrow{uw} \theta(a), \quad a \in A,$$

or equivalently, $\eta(\psi_n \circ \varphi_n(a)) \to \eta(\theta(a))$, for all $a \in A$ and $\eta \in N_*$ (the predual of N).

Remark 4.4 (Remark 2.1.3, [BO]). *Assume $N \subset B(H)$ is a von Neumann algebra. For every $x, y \in H$, let $\varphi_{x,y} : N \to \mathbb{C}$ be the vector functional $\varphi_{x,y}(T) = \langle Tx, y\rangle$, $T \in N$. Then $\varphi_{x,y} \in N_*$, and, in fact, span$\{\varphi_{x,y} : x, y \in H\}$ is norm-dense in N_*.*

Let $a \in A$. Since $\psi_n \circ \varphi_n(a)$ is a bounded net (or sequence), then

$$\psi_n \circ \varphi_n(a) \xrightarrow{uw} \theta(a) \quad \Longleftrightarrow \quad \langle \psi_n \circ \varphi_n(a)v, w\rangle \to \langle \theta(a)v, w\rangle\,, \quad v, w \in H\,.$$

By the polarization identity, it is further sufficient to check $\langle \psi_n \circ \varphi_n(a)v, v\rangle \to \langle \theta(a)v, v\rangle$, for all $v \in H$.

Exercise: If θ is weakly nuclear, then θ is c.c.p.

Proposition 4.5 (Proposition 2.1.4, [BO]). *If $M \subset B(H)$ is a von Neumann algebra, then the inclusion map $i\colon M \hookrightarrow B(H)$ is always weakly nuclear.*

Proof. Choose an increasing net $(p_i)_{i \in I}$ of finite dimensional projections in $B(H)$, such that $p_i \nearrow 1$ SOT. Set $k_i = \dim(p_i(H))$. Then $p_i B(H) p_i \cong B(p_i(H)) \cong M_{k_i}(\mathbb{C})$. Further, set

$$\begin{aligned} \varphi_i(a) &= p_i a p_i, \quad && a \in M \\ \psi_i(b) &= b, && b \in B(p_i(H)). \end{aligned}$$

Then $\varphi_i\colon M \to B(p_i(H))$ and $\psi_i\colon B(p_i(H)) \to B(H)$ are c.c.p. and $\psi_i \circ \varphi_i(a) = p_i a p_i$, for all $a \in M$. For all $v \in H$,

$$\langle p_i a p_i v, w\rangle = \langle a \underbrace{p_i v}_{\to v}, \underbrace{p_i w}_{\to w}\rangle \longrightarrow \langle av, w\rangle.$$

Hence $p_i a p_i \to a$ in the WOT-topology. But $\|p_i a p_i\| \le \|a\|$ for all i, hence $p_i a p_i \to a$ ultraweakly. ∎

Note: By contrast, the identity map $i\colon M \to M$ may not necessarily be weakly nuclear! In fact, $i\colon M \to M$ is weakly nuclear if and only if M is an injective von Neumann algebra. Hence, if Γ is a non-amenable group, then the identity map $i\colon L(\Gamma) \to L(\Gamma)$ is not weakly nuclear. Here $L(\Gamma)$ denotes the group von Neumann algebra of Γ.

Section 2.2 [BO]: Non-unital technicalities

The purpose of this section is to provide some technical tools that will help passing from the case of not necessarily unital C*-algebras (or maps) to the unital one. Every non-unital C^*-algebra A has a *unitization* $\widetilde{A}$ which is a unital C^*-algebra with unit $1_{\tilde{A}}$, which contains A, and satisfies

$$\widetilde{A} = A + \mathbb{C}1_{\tilde{A}}.$$

The unitization $\widetilde{A}$ is unique with these properties. The original C^*-algebra A is a closed two-sided ideal in $\widetilde{A}$, and $\widetilde{A}/A \cong \mathbb{C}$. Moreover, if B is any unital C^*-algebra which contains A, then $\widetilde{A}$ is isomorphic to $A + \mathbb{C}1_B$. (Note that the latter always is a C^*-algebra.) A quick way to construct $\tilde{A}$ is by using the embedding $A \hookrightarrow A^{**}$ (the second dual). Recall that $A^{**} \cong \pi_u(A)''$ where π_u is the universal representation. Since A^{**} has a unit, we obtain $\widetilde{A} \cong A + \mathbb{C}1_{A^{**}}$.

We will not cover in lectures any of the results in this section, but only mention (without proof) the following:

Proposition 4.6 (Proposition 2.2.8, [BO])**.** *Let M, N be von Neumann algebras, and let $\theta\colon M \to N$ be a unital, weakly nuclear map. Then there exist nets of <u>normal</u> u.c.p. maps*

$$\varphi_n\colon M \to M_{k(n)}(\mathbb{C}), \quad \psi_n\colon M_{k(n)}(\mathbb{C}) \to N$$

for some $k(n) \in \mathbb{N}$, such that $\psi_n \circ \varphi_n \to \theta$ in the point-ultraweak topology.

Section 2.3 [BO]: Nuclear and exact C*-algebras

Definition 4.7 (Definition 2.3.1, [BO])**.** *A C^*-algebra A is <u>nuclear</u> if* $\mathrm{id}_A\colon A \to A$ *is nuclear.*

Definition 4.8 (Definition 2.3.2, [BO])**.** *A C^*-algebra A is <u>exact</u> if there exists a faithful representation $\pi\colon A \to B(H)$ such that π is nuclear.*

Remark 4.9. *A positive map $\varphi\colon A \to B$ (where A, B are C^*-algebras) is called <u>faithful</u> if $a \in A_+$ and $\varphi(a) = 0$ imply that $a = 0$. A representation π is faithful if and only if it is one-to-one. (That one-to-one implies faithful is obvious, and if π is faithful, then $\pi(a) = 0$ implies $\pi(a^*a) = \pi(a)^*\pi(a) = 0$, and thus $a^*a = 0$, implying $a = 0$).*

Remark 4.10. *Let $\pi : A \to B(H)$ be a faithful representation of a C^*-algebra A. Then, A is nuclear if and only if the map $\pi : A \to \pi(A)$ is nuclear, while A is exact if and only if π is nuclear when π is regarded as taking values in $B(H)$. In particular, nuclearity implies exactness. (The converse is false.)*

Definition 4.11 (Definition 2.3.3, [BO])**.** *A von Neumann algebra M is called <u>semidiscrete</u> if the identity map* $\mathrm{id}_M\colon M \to M$ *is weakly nuclear.*

Note: It is a deep and difficult result of A. Connes that a (separable) von Neumann algebra *factor* (i.e., has trivial center) is semidiscrete if and only if it is injective.

With the tools developed so far, one can prove (see textbook) the following:

Proposition 4.12 (Proposition 2.3.8, [BO]). *Let A be a C^*-algebra. If A^{**} is semidiscrete, then A is nuclear.*

Section 2.4 [BO]: First examples

First, look up Exercises 2.1.1 and 2.1.2 in [BO]. The latter implies that finite dimensional C*-algebras are nuclear. Furthermore, since inductive limits of nuclear C*-algebras are nuclear (see Exercise 2.3.7 [BO]), we obtain:

Proposition 4.13 (Proposition 2.4.1, [BO]). *Approximately finite-dimensional (AF) algebras are nuclear.*

A further important class of examples is given by:

Proposition 4.14 (Proposition 2.4.2, [BO]). *Every abelian C^*-algebra is nuclear.*

Proof. (In the unital case.) Each unital abelian C^*-algebra is isomorphic to $C(X)$ for some compact Hausdorff space. Hence it suffices to show that if F is a finite subset of $C(X)$ and if $\varepsilon > 0$, then, for some $n \geq 1$, there are ucp maps

$$C(X) \xrightarrow{\varphi} \mathbb{C}^n \xrightarrow{\psi} C(X)$$

such that $\|(\psi \circ \varphi)(f) - f\| \leq \varepsilon$ for all $f \in F$.

By compactness of X one can find a finite open cover $\{U_j\}_{j=1}^n$ of X such that

$$\forall j \; \forall x, y \in U_j \; \forall f \in F : |f(x) - f(y)| \leq \varepsilon.^{\ddagger}$$

Choose $x_j \in U_j$ for each j and define φ by

$$\varphi(f) = \big(f(x_1), f(x_2), \dots, f(x_n)\big), \qquad f \in C(X).$$

Note that φ is a unital *-homomorphism, and hence in particular a ucp map.

Let $\{h_i\}_{i=1}^n$ be a partition of the unit subordinate to the cover $\{U_i\}_{i=1}^n$ (so that each h_i is supported inside U_i, $0 \leq h_i \leq 1$, and $\sum_{i=1}^n h_i = 1$). Define ψ by

$$\psi(\lambda_1, \lambda_2, \dots, \lambda_n) = \sum_{i=1}^n \lambda_i h_i.$$

Then ψ is a ucp map. Observe that

$$(\psi \circ \varphi)(f)(x) = \sum_{i=1}^n f(x_i) h_i(x), \qquad f \in C(X), \quad x \in X.$$

It follows that

$$|f(x) - (\psi \circ \varphi)(f)(x)| \leq \sum_{i=1}^n |f(x) - f(x_i)| h_i(x), \qquad f \in C(X), \quad x \in X.$$

---

‡To see this, set

$$U_x = \{y \in X \mid |f(y) - f(x)| < \varepsilon \text{ for all } f \in F\}$$

for each $x \in X$, and then pick a finite subcover of $\{U_x\}_{x \in X}$.

If $f \in F$ and if $h_i(x) > 0$, then $x \in U_i$ whence $|f(x) - f(x_i)| \leq \varepsilon$. This shows that

$$|f(x) - f(x_i)|h_i(x) \leq \varepsilon h_i(x)$$

for *all* $x \in X$ and for all $f \in F$. Hence $\|(\psi \circ \varphi)(f) - f\| \leq \varepsilon$ holds for all $f \in F$. □

Note: In view of above result (and its proof), nuclearity is sometimes viewed as a noncommutative analogue of having a partition of unity.

Lecture 5, GOADyn
September 23, 2021

Section 2.5 [BO]: C^*-algebras associated to discrete groups

Let H be a Hilbert space. We denote by $\mathcal{U}(H)$ the set of all unitary operators in $B(H)$. Note that $\mathcal{U}(H)$ is a group: if $u_1, u_2 \in \mathcal{U}(H)$, then $u_1u_2 \in \mathcal{U}(H)$, the identity operator I is in $\mathcal{U}(H)$ and we have $u^{-1} = u^*$ for all $u \in \mathcal{U}(H)$.

Let Γ be a discrete group. A *unitary representation* of Γ on a Hilbert space H is a group homomorphism $u\colon \Gamma \to \mathcal{U}(H)$ for which we define $u_s = u(s) \in \mathcal{U}(H)$, for all $s \in \Gamma$. Note that $u_e = u(e) = I$. Moreover, we have $u_{s^{-1}} = (u_s)^{-1} = (u_s)^*$ for all $s \in \Gamma$. Consider now

$$\ell^2(\Gamma) = \left\{ f\colon \Gamma \to \mathbb{C} \,:\, \sum_{s\in\Gamma} |f(s)|^2 < \infty \right\},$$

equipped with the norm

$$\|f\|_2 = \left(\sum_{s\in\Gamma} |f(s)|^2 \right)^{1/2}, \quad f \in \ell^2(\Gamma).$$

Then $\ell^2(\Gamma)$ is a Hilbert space with orthonormal basis $\{\delta_s \,:\, s \in \Gamma\}$, where

$$\delta_s(t) = \begin{cases} 1 & \text{if } s = t \\ 0 & \text{else} \end{cases}$$

Two very important unitary representations of Γ on $\ell^2(\Gamma)$ are the following:

(1) The *left regular representation* $\lambda\colon \Gamma \to B(\ell^2(\Gamma))$, given by $\lambda_s(\delta_t) = \delta_{st}$, $s, t \in \Gamma$.

(2) The *right regular representation* $\rho\colon \Gamma \to B(\ell^2(\Gamma))$, given by $\rho_s(\delta_t) = \delta_{ts^{-1}}$, $s, t \in \Gamma$.

Note that λ and ρ are unitarily equivalent: the intertwining unitary $U\colon \ell^2(\Gamma) \to \ell^2(\Gamma)$ is given by

$$U\delta_t = \delta_{t^{-1}}.$$

Indeed, for all $s, t \in \Gamma$, $U^*\rho_s U\delta_t = U^*\rho_s\delta_{t^{-1}} = U^*\delta_{t^{-1}s^{-1}} = U^*\delta_{(st)^{-1}} = \delta_{st} = \lambda_s\delta_t$, which shows that $\lambda = U^*\rho U$.

Consider the group ring $\mathbb{C}\Gamma$ of Γ, i.e.,

$$\mathbb{C}\Gamma = \left\{ \sum_{s\in\Gamma} a_s s \,:\, a_s \in \mathbb{C}, \text{ only finitely many } a_s \text{ are non-zero} \right\}$$

We want to view $\mathbb{C}\Gamma$ as a vector space over Γ. By defining

$$\left(\sum_{s\in\Gamma} a_s s\right)\left(\sum_{t\in\Gamma} b_t t\right) = \sum_{s,t\in\Gamma} a_s b_t st \qquad \text{(multiplication)}$$

$$\left(\sum_{s\in\Gamma} a_s s\right)^* = \sum_{s\in\Gamma} \overline{a_s} s^{-1} \qquad \text{(}^*\text{-involution)}$$

we make $\mathbb{C}\Gamma$ into a *-algebra. Given a unitary representation $s \in \Gamma \mapsto u_s \in \mathcal{U}(H)$ of Γ on some Hilbert space H, it gives rise to a unital *-homomorphism $\pi\colon \mathbb{C}\Gamma \to B(H)$ such that

$$\pi(s) = u_s, \quad s \in \Gamma.$$

This is a 1-1 correspondence. We let

$$C^*_\lambda(\Gamma) := \overline{\lambda(\mathbb{C}\Gamma)}^{\|\cdot\|} \subset B(\ell^2(\Gamma)).$$

The above inclusion is an embedding, since $\lambda\colon \mathbb{C}\Gamma \to B(\ell^2(\Gamma))$ is injective: indeed, if $x = \sum_{t\in\Gamma} a_t t \in \mathbb{C}\Gamma$ satisfies $\lambda(x) = 0$, then for all $s \in \Gamma$,

$$0 = \langle \lambda(x)\delta_e, \delta_s\rangle = \sum_{t\in\Gamma} a_t \langle \lambda(t)\delta_e, \delta_s\rangle = \sum_{t\in\Gamma} a_t \langle \delta_t, \delta_s\rangle = a_s.$$

Hence $x = 0$.

We call $C^*_\lambda(\Gamma)$ the *reduced group C^*-algebra* of Γ (it is sometimes also denoted by $C^*_r(\Gamma)$). $C^*_\lambda(\Gamma)$ is thus the completion of $\mathbb{C}\Gamma$ with respect to

$$\|x\|_r = \|\lambda(x)\|_{B(\ell^2(\Gamma))}.$$

Similarly, we let $C^*_\rho(\Gamma)$ be the completion of $\mathbb{C}\Gamma$ with respect to ρ.

Definition 5.1. The *full (universal) group C^*-algebra of* Γ, denoted by $C^*(\Gamma)$, is the completion of $\mathbb{C}\Gamma$ with respect to

$$\|x\|_u := \sup\{\|\pi(x)\| \,:\, \pi \text{ is a (cyclic) representation of } \Gamma\}.$$

Note that $\|x\|_u \le \|x\|_1$ (since $\|\pi(s)\| = 1$ for all $s \in \Gamma$).

Remark 5.2. If Γ is an abelian discrete group, then $C^*_\lambda(\Gamma) = C^*(\Gamma)$. (This holds more generally for amenable groups, see Theorem 2.6.8, [BO].)

Example 5.3 (Example 2.5.1, [BO])**.** If $\Gamma = \mathbb{Z}$, then $C^*_\lambda(\Gamma) \cong C(\mathbb{T})$. To prove this, let $u = \lambda(1) \in \mathcal{U}(\ell^2(\Gamma))$. If $\sum_{k\in\mathbb{Z}} a_k k \in \mathbb{C}\mathbb{Z}$, then

$$\lambda\left(\sum_{k\in\mathbb{Z}} a_k k\right) = \sum_{k\in\mathbb{Z}} a_k u^k \in C^*(u).$$

Hence $u \in \lambda(\mathbb{C}\mathbb{Z}) \subset C^*(u)$, so $C^*_\lambda(\mathbb{Z}) = C^*(u) \cong C(\sigma(u))$. We claim that $\sigma(u) = \mathbb{T}$. Note that u is the bilateral shift on $\ell^2(\mathbb{Z})$, i.e., $u\delta_n = \delta_{n+1}$ for all $n \in \mathbb{Z}$. Since u is a unitary, we have $\sigma(u) \subset \mathbb{T}$. To show equality, let $z \in \mathbb{T}$. For $k \in \mathbb{N}$, set

$$\zeta_{k,z} := k^{-1/2} \sum_{j=1}^{k} (\overline{z})^j \delta_j.$$

Then $\|\zeta_{k,z}\| = 1$ and one can check that

$$u\zeta_{k,z} = k^{-1/2} \sum_{j=1}^{k} (\overline{z})^j \delta_{j+1} = z k^{-1/2} \sum_{j=1}^{k} (\overline{z})^{j+1} \delta_{j+1} = z k^{-1/2} \sum_{j=2}^{k+1} (\overline{z})^j \delta_j.$$

Hence $\|(u - z\cdot 1)\zeta_{k,z}\| = \sqrt{\frac{2}{k}} \to 0$ as $k \to \infty$, so $z \in \sigma(u)$. (We say that $\zeta_{k,z}$ is a sequence of approximate eigenvectors for z.)

A more general approach. Let Γ be an abelian discrete group. Its (Pontryagin) dual is defined to be

$$\hat{\Gamma} = \{\varphi\colon \Gamma \to \mathbb{T} \,:\, \varphi \text{ is a group homomorphism}\}.$$

Note that $\hat{\mathbb{Z}} = \mathbb{T}$. Indeed, for $z \in \mathbb{T}$, let $\varphi_z \in \hat{\mathbb{Z}}$ be given by $\varphi_z(n) = z^n$, $n \in \mathbb{Z}$. Then $z \mapsto \varphi_z$ defines a homomorphism $\mathbb{T} \to \hat{\mathbb{Z}}$. The fact that this map is onto follows from this: Given $\varphi \in \hat{\mathbb{Z}}$, set $z = \varphi(1) \in \mathbb{T}$. Then $\varphi(n) = \varphi(1)^n = z^n = \varphi_z(n)$ for all $n \in \mathbb{Z}$.

Theorem 5.4. *If Γ is an abelian discrete group, then $C^*_\lambda(\Gamma) \cong C(\hat{\Gamma})$.*

Proof. If Γ is abelian, then $C^*_\lambda(\Gamma)$ is abelian, so $C^*_\lambda(\Gamma) \cong C(\Omega)$, where Ω is the space of characters on $C^*_\lambda(\Gamma)$.

Claim. Ω is homeomorphic to $\hat{\Gamma}$.

Any $\varphi \in \Omega$ induces a $\hat{\varphi} \in \hat{\Gamma}$ defined by $\hat{\varphi}(t) = \varphi(t)$, $t \in \Gamma$. (If $\varphi \in \Omega$, then $t \mapsto \varphi(t)$ belongs to $\hat{\Gamma}$.) Set $\Phi\colon \Omega \to \hat{\Gamma}$, where $\Phi(\varphi) = \hat{\varphi}$, $\varphi \in \Omega$. Then we must show that

(i) Φ is continuous,
(ii) Φ is 1-1, and
(iii) Φ is onto,

so that Φ is a homeomorphism.

(i) Let $(\varphi_\alpha)_\alpha, \varphi \in \Omega$. Assume that $\varphi_\alpha \to \varphi$. Then $\varphi_\alpha(x) \to \varphi(x)$, for all $x \in C^*_\lambda(\Gamma)$, which implies that $\varphi_\alpha(t) \to \varphi(t)$, for all $t \in \Gamma$. Hence $\Phi(\varphi_\alpha)(t) \to \Phi(\varphi)(t)$, for all $t \in \Gamma$, i.e., $\Phi(\varphi_\alpha) \to \Phi(\varphi)$.

(ii) Let $\varphi, \psi \in \Omega$. Then $\Phi(\varphi) = \Phi(\psi)$ implies that $\varphi(t) = \psi(t)$, for all $t \in \Gamma$, hence $\varphi(x) = \psi(x)$, for all $x \in \mathbb{C}\Gamma$. So $\varphi(x) = \psi(x)$, for all $x \in C^*_\lambda(\Gamma)$, and thus $\varphi = \psi$.

(iii) Let $\varphi_0 \in \hat{\Gamma}$. Then $\varphi_0\colon \Gamma \to B(\mathbb{C})$ is a one-dimensional representation. It extends to a *-homomorphism $\varphi\colon \mathbb{C}\Gamma \to B(\mathbb{C})$, and further to a *-representation $\varphi\colon C^*(\Gamma) \to B(\mathbb{C}) = \mathbb{C}$. Since $C^*_\lambda(\Gamma) = C^*(\Gamma)$ (which holds because Γ is abelian), we deduce that $\varphi \in \Omega$ and $\Phi(\varphi) = \varphi_0$. □

Remark 5.5. $C^*(\Gamma)$ has the following <u>universal property</u>: Given any unitary representation $u\colon \Gamma \to \mathcal{U}(H)$ of Γ, there exists a unique *-homomorphism $\pi_u\colon C^*(\Gamma) \to B(H)$ such that $\pi_u(s) = u_s$ for all $s \in \Gamma$.

Proposition 5.6 (Proposition 2.5.3, [BO])**.** *The vector state $\tau_e\colon C^*_\lambda(\Gamma) \to \mathbb{C}$ given by $\tau_e(x) = \langle x\delta_e, \delta_e\rangle$, $x \in C^*_\lambda(\Gamma)$ defines a faithful tracial state on $C^*_\lambda(\Gamma)$.*

Proof. τ_e *is positive,* since $\tau_e(x^*x) = \langle x^*x\delta_e, \delta_e\rangle = \|x\delta_e\|^2 \geq 0$. Hence τ_e is a positive linear functional on $C^*_\lambda(\Gamma)$ with $\|\tau_e\| = \tau_e(e) = 1$. Furthermore, τ_e *is tracial,* since

$$\tau_e(\lambda_s\lambda_t) = \langle \lambda_{st}\delta_e, \delta_e\rangle = \langle \delta_{st}, \delta_e\rangle = \begin{cases} 1 & \text{if } st = e \\ 0 & \text{else} \end{cases}$$

$$\tau_e(\lambda_t\lambda_s) = \langle \lambda_{ts}\delta_e, \delta_e\rangle = \langle \delta_{ts}, \delta_e\rangle = \begin{cases} 1 & \text{if } ts = e \\ 0 & \text{else.} \end{cases}$$

Since $st = e$ if and only if $s = t^{-1}$ if and only if $ts = e$, it folllows that $\tau_e(\lambda_s\lambda_t) = \tau_e(\lambda_t\lambda_s)$ for all $s, t \in \Gamma$. Use that $\lambda(\mathbb{C}\Gamma)$ is dense in $C^*_\lambda(\Gamma)$ to deduce that

$$\tau_e(xy) = \tau_e(yx), \quad x, y \in C^*_\lambda(\Gamma).$$

Also, τ_e *is faithful:* Let $0 \leq x \in C^*_\lambda(\Gamma)$ with $\tau_e(x) = \langle x\delta_e, \delta_e\rangle = 0$. Then $x^{1/2}\delta_e = 0$. Note that δ_e is a separating vector, i.e., if $x\delta_e = y\delta_e$ for $x, y \in C^*_\lambda(\Gamma)$, then $x = y$. Indeed, we claim that if $x\delta_e = y\delta_e$, then for all $s \in \Gamma$,

$$x\delta_s = x\delta_{e(s^{-1})^{-1}} = x\rho_{s^{-1}}\delta_e = \rho_{s^{-1}}x\delta_e = \rho_{s^{-1}}y\delta_e = y\delta_s,$$

since $\rho_{s^{-1}}$ commutes with $C^*_\lambda(\Gamma)$, and then use that $\{\delta_s : s \in \Gamma\}$ is an orthonormal basis for $\ell^2(\Gamma)$ to conclude $x = y$. So $x^{1/2}\delta_e = 0$ does imply $x = 0$. □

Definition 5.7. The *group von Neumann algebra associated to* Γ is

$$L(\Gamma) = \mathrm{vN}(\Gamma) := C^*_\lambda(\Gamma)'' \subset B(\ell^2(\Gamma)).$$

Theorem 5.8 (Fell's Absorption Principle, Theorem 2.5.5, [BO])**.** *Let π be a unitary representation of Γ on H. Then $\lambda \otimes \pi$ is unitarily equivalent to $\lambda \otimes 1_H$, i.e., there exists a unitary operator $U\colon \ell^2(\Gamma) \otimes H \to \ell^2(\Gamma) \otimes H$ such that $\lambda \otimes 1_H = U^*(\lambda \otimes \pi)U$. (Roughly speaking, the left regular representation absorbs all other representations tensorially.)*

Theorem 5.9 (Proposition 2.5.9 and Corollary 2.5.12, [BO])**.** *Let $\Lambda \subset \Gamma$ be a subgroup. Then $C^*_\lambda(\Lambda) \subset C^*_\lambda(\Gamma)$ (inclusion of C^*-algebras). Moreover, there exists a c.c.p. projection $E^\Gamma_\Lambda\colon C^*_\lambda(\Gamma) \to C^*_\lambda(\Lambda)$ onto, i.e., a conditional expectation.*

Proof. We follow Pisier (Proposition 8.5, Introduction to Operator Spaces). Define a map $J : \mathbb{C}\Lambda \to \mathbb{C}\Gamma$ by $J(\lambda_\Lambda(t)) = \lambda_\Gamma(t)$, $t \in \Lambda$. We claim that J extends to an isometric (C*-algebraic) embedding of $C^*_\lambda(\Lambda)$ into $C^*_\lambda(\Gamma)$. We know that $\mathbb{C}\Lambda \ni \sum \alpha_t \lambda_\Lambda(t)$ (finite sum) $\overset{J}{\longmapsto} \sum \alpha_t \lambda_\Gamma(t) \in \mathbb{C}\Gamma$. We need to check that

$$\|\sum \alpha_t \lambda_\Lambda(t)\| = \|\sum \alpha_t \lambda_\Gamma(t)\|\,. \tag{5.1}$$

Then, by the density of $\mathbb{C}\Lambda$ in $C^*_\lambda(\Lambda)$, respectively, of $\mathbb{C}\Gamma$ in $C^*_\lambda(\Gamma)$, it will follow that J extends to an isometric map from $C^*_\lambda(\Lambda)$ into $C^*_\lambda(\Gamma)$.

To prove (5.1), let $(\delta^\Gamma_t)_{t\in\Gamma}$ be an orthonormal basis for $\ell^2(\Gamma)$ and $(\delta^\Lambda_t)_{t\in\Gamma}$ be an orthonormal basis for $\ell^2(\Lambda)$. Define $Q = \Gamma/\Lambda$ (the right cosets). For all $q \in Q$, pick a transversal $s(q) \in q$. Then $\Gamma = \dot{\bigcup}_{q\in Q} \Lambda s(q)$ (disjoint union). Then we have the <u>set</u> identification $\Gamma = \Lambda \times Q$ by means of the map $ts(q) \mapsto (t, q)$.

Define a unitary map $U\colon \ell^2(\Lambda) \otimes \ell^2(Q) \to \ell^2(\Gamma)$ by

$$U(\delta^\Lambda_t \otimes \delta^Q_q) = \delta^\Gamma_{ts(q)}.$$

We claim that

$$U^*\lambda_\Gamma(r)U = \lambda_\Lambda(r) \otimes I_{\ell^2(Q)}, \quad r \in \Lambda. \tag{5.2}$$

Indeed, $U^*\lambda_\Gamma(r)U(\delta^\Lambda_t \otimes \delta^Q_q) = U^*\lambda_\Gamma(r)\delta^\Gamma_{ts(q)} = U^*\delta^\Gamma_{rts(q)} = \delta^\Lambda_{rt} \otimes \delta^Q_q = (\lambda_\Lambda(r) \otimes I_{\ell^2(Q)})(\delta^\Lambda_t \otimes \delta^Q_q)$, $r \in \Lambda$. By (5.2) it follows that $U^*(\sum \alpha_t \lambda_\Gamma(t))U = (\sum \alpha_t \lambda_\Lambda(t)) \otimes I_{\ell^2(Q)}$. This implies that

$$\|\sum \alpha_t \lambda_\Gamma(t)\| = \|U^*(\sum \alpha_t \lambda_\Gamma(t))U\| = \|(\sum \alpha_t \lambda_\Gamma(t)) \otimes I_{\ell^2(Q)}\| = \|\sum \alpha_t \lambda_\Lambda(t)\|,$$

as wanted.

It remains to prove the existence of a conditional expectation. Let $V\colon \ell^2(\Lambda) \to \ell^2(\Gamma)$ be defined by $V\delta^\Lambda_t = \delta^\Gamma_t$ for all $t \in \Lambda \subset \Gamma$. Then V is an isometry and

$$V^*\delta^\Gamma_t = \begin{cases} \delta^\Lambda_t & t \in \Lambda \\ 0 & t \notin \Lambda, \end{cases}$$

since

$$\langle V^*\delta^\Gamma_t, \delta^\Lambda_s\rangle = \langle \delta^\Gamma_t, V\delta^\Lambda_s\rangle = \langle \delta^\Gamma_t, \delta^\Gamma_s\rangle = \begin{cases} 0 & t \neq s \\ 1 & t = s. \end{cases}$$

Now let $E^\Gamma_\Lambda\colon C^*_\lambda(\Gamma) \to B(\ell^2(\Lambda))$ be defined by

$$E^\Gamma_\Lambda(x) = V^*xV, \quad x \in C^*_\lambda(\Gamma).$$

Then E_Λ^Γ is c.c.p. We claim that

$$C_\lambda^*(\Lambda) \ni E_\Lambda^\Gamma(\lambda_\Gamma(t)) = \begin{cases} \lambda_\Lambda(t) & t \in \Lambda \\ 0 & t \notin \Lambda. \end{cases}$$

So $E_\Lambda^\Gamma(\mathbb{C}\Gamma) \subset C_\lambda^*(\Lambda)$. Then

$$E_\Lambda^\Gamma(\underbrace{\overline{\mathbb{C}\Gamma}}_{C_\lambda^*(\Gamma)}) \subset \overline{C_\lambda^*(\Lambda)} = C_\lambda^*(\Lambda).$$

Hence E_Λ^Γ is a c.c.p. projection onto $C_\lambda^*(\Lambda)$ (E_Λ^Γ acts as the identity on $\mathbb{C}\Gamma$, which is dense in $C_\lambda^*(\Gamma)$), i.e., a conditional expectation. □

Definition 5.10 (Definition 2.5.6, [BO])**.** A function $\varphi\colon \Gamma \to \mathbb{C}$ is called *positive definite* if the matrix $[\varphi(s^{-1}t)]_{s,t\in F} \in M_F(\mathbb{C})_+$ for every finite set $F \subset \Gamma$.

Fix a positive definite function $\varphi\colon \Gamma \to \mathbb{C}$ and recall that $C_c(\Gamma)$ denotes the set of finitely supported functions on Γ. Define $\langle\cdot,\cdot\rangle_\varphi\colon C_c(\Gamma) \times C_c(\Gamma) \to \mathbb{C}$ by

$$\langle f, g\rangle_\varphi = \sum_{s,t\in F} \varphi(s^{-1}t)f(t)\overline{g(s)}, \quad f, g \in C_c(\Gamma).$$

One can check that $\langle\cdot,\cdot\rangle_\varphi$ is positive semidefinite (use that φ is positive definite). Let $\ell^2_\varphi(\Gamma)$ be the Hilbert space completion of $C_c(\Gamma)/\{f \in C_c(\Gamma) : \langle f, f\rangle_\varphi = 0\}$. We write $\hat{f} = [f] \in \ell^2_\varphi(\Gamma)$, for all $f \in C_c(\Gamma)$.

Definition 5.11 (Definition 2.5.7, [BO])**.** Let $\varphi\colon \Gamma \to \mathbb{C}$ be positive definite. Define $\lambda^\varphi\colon \Gamma \to B(\ell^2_\varphi(\Gamma))$ by

$$\lambda_s^\varphi(\hat{f}) = \widehat{s.f}, \quad s \in \Gamma,$$

where $(s.f)(t) = f(s^{-1}t)$ for all $t \in \Gamma$. Then λ^φ is a unitary representation satisfying $\lambda_s^\varphi \circ \lambda_t^\varphi = \lambda_{st}^\varphi$, for all $s, t \in \Gamma$, and λ_s^φ is an isometry for all s, as

$$\|\lambda_s^\varphi(\hat{f})\|^2 = \sum_{x,y\in\Gamma} \varphi(x^{-1}y)f(s^{-1}x)\overline{f(s^{-1}y)} = \sum_{x',y'\in\Gamma} \varphi((x')^{-1}y')f(x')\overline{f(y')} = \|\hat{f}\|^2$$

where $x' = s^{-1}x$, $y' = s^{-1}y$ and hence $x^{-1}y = (x')^{-1}y'$. Moreover,

$$\langle \lambda_s^\varphi \hat{\delta}_e, \hat{\delta}_e\rangle_\varphi = \langle \hat{\delta}_s, \hat{\delta}_e\rangle_\varphi = \varphi(s), \quad s \in \Gamma,$$

so we can recover φ from $\langle \cdot\hat{\delta}_e, \hat{\delta}_e\rangle_\varphi$.

Remark 5.12. Suppose that $\varphi\colon C^*(\Gamma) \to \mathbb{C}$ is a positive linear functional. Then $s \mapsto \varphi(s)$ is positive definite on Γ. Indeed, for all $s_1, \ldots, s_n \in \Gamma$, we have

$$[\varphi(s_i^{-1}s_j)] = (\mathrm{id}_n \otimes \varphi)\left(\begin{bmatrix} s_1 & \cdots & s_n \\ & 0 & \end{bmatrix}^* \begin{bmatrix} s_1 & \cdots & s_n \\ & 0 & \end{bmatrix}\right) \geq 0,$$

since φ is completely positive. The GNS space of $C^*(\Gamma)$ with respect to φ is $\ell^2_\varphi(\Gamma)$.

Definition 5.13 (Definition 2.5.10, [BO])**.** Let $\varphi\colon \Gamma \to \mathbb{C}$ be any function. Define $\omega_\varphi\colon \mathbb{C}\Gamma \to \mathbb{C}$ by

$$\omega_\varphi\left(\sum_{t\in\Gamma} \alpha_t t\right) = \sum_{t\in\Gamma} \varphi(t)\alpha_t$$

and a multiplier $m_\varphi\colon \mathbb{C}\Gamma \to \mathbb{C}\Gamma$ by

$$m_\varphi\left(\sum_{t\in\Gamma}\alpha_t t\right) = \sum_{t\in\Gamma}\varphi(t)\alpha_t t.$$

Theorem 5.14 (Theorem 2.5.11, [BO])**.** *Let $\varphi\colon \Gamma \to \mathbb{C}$ be a function with $\varphi(e) = 1$. Then the following are equivalent:*

(1) *φ is positive definite.*

(2) *There exists a unitary representation λ_φ of Γ on a Hilbert space H_φ and a unit vector ξ_φ such that*

$$\varphi(s) = \langle \lambda_\varphi(s)\xi_\varphi, \xi_\varphi\rangle, \quad s \in \Gamma.$$

(3) *The functional ω_φ extends to a state on $C^*(\Gamma)$.*

(4) *The multiplier m_φ extends to a u.c.p. map on either $C^*(\Gamma)$ or $C^*_\lambda(\Gamma)$, or extends to a normal u.c.p. map on $L(\Gamma)$.*

Proof. We show (1) ⇒ (2) ⇒ (3) ⇒ (4) ⇒ (1).

(1) ⇒ (2): Let $\varphi\colon \Gamma \to \mathbb{C}$ be positive definite. Then, as in Definition 5.11, there exists a Hilbert space $\ell^2_\varphi(\Gamma)$, a unitary representation $\lambda^\varphi\colon \Gamma \to \mathcal{U}(\ell^2_\varphi(\Gamma))$ and a unit vector, namely $\hat{\delta}_e \in \ell^2_\varphi(\Gamma)$ such that

$$\varphi(s) = \langle \lambda^\varphi(s)\hat{\delta}_e, \hat{\delta}_e\rangle, \quad s \in \Gamma.$$

This proves (2).

(2) ⇒ (3): By universality of $C^*(\Gamma)$, the unitary representation λ^φ from above extends to a unital *-homomorphism $\lambda^\varphi\colon C^*(\Gamma) \to B(\ell^2_\varphi(\Gamma))$. It is easy to see that

$$\omega(x) = \langle \lambda^\varphi(x)\hat{\delta}_e, \hat{\delta}_e\rangle, \quad x \in C^*(\Gamma),$$

defines a state on $C^*(\Gamma)$. Moreover, if $x = \sum_{t\in\Gamma}\alpha_t t \in \mathbb{C}\Gamma$, then

$$\omega(x) = \sum_{t\in\Gamma}\alpha_t\langle \lambda^\varphi(t)\hat{\delta}_e, \hat{\delta}_e\rangle = \sum_{t\in\Gamma}\varphi(t)\alpha_t = \omega_\varphi.$$

Hence ω is the desired extension of ω_φ to a state on $C^*(\Gamma)$.

(3) ⇒ (4): We first consider the $L(\Gamma)$ case. Let $C^*(\Gamma) \subset B(H)$ be a faithful representation such that ω_φ extends to a normal state ω on $B(H)$. (One can take $\pi\colon C^*(\Gamma) \to B(H)$ to be the universal representation, which is faithful, and where each state on $C^*(\Gamma)$ is represented by a vector state. This will do the job, because each vector state is normal.) So we assume that $\omega(T) = \langle T\xi, \xi\rangle$, $T \in B(H)$, for some unit vector $\xi \in H$. Let $V_\xi : H \to \mathbb{C}$ be the projection onto $\mathbb{C}\xi \simeq \mathbb{C}$. ($V_\xi$ is the adjoint of the map $\mathbb{C} \ni \beta \mapsto \beta\xi \in H$.) Note that $\|V_\xi\| = 1$, and that

$$\omega(T) = V_\xi T V_\xi^*\,, \quad T \in B(H).$$

By Fell's absorption principle, the two representations $\Gamma \to \mathcal{U}(\ell^2(\Gamma)\otimes H)$ given by $t \mapsto \lambda_t \otimes t$ and $t \mapsto \lambda_t \otimes 1_H$ are unitarily equivalent. Hence there exists $U \in \mathcal{U}(\ell^2(\Gamma)\otimes H)$ such that

$$U(\lambda_t \otimes 1_H)U^* = \lambda_t \otimes t, \quad t \in \Gamma.$$

Let $\sigma\colon L(\Gamma) \to B(\ell^2(\Gamma)) \otimes B(H)$ be the normal *-homomorphism given by

$$\sigma(x) = U(x \otimes 1_H)U^*, \quad x \in L(\Gamma).$$

Next, note that there is a normal u.c.p. map $\psi\colon B(\ell^2(\Gamma))\otimes B(H) \to B(\ell^2(\Gamma))$ satisfying $\psi(S\otimes T) = \omega(T)S$, for all $S \in B(\ell^2(\Gamma))$, $T \in B(H)$. Namely, the map defined by

$$\psi(x) = (I_{\ell^2(\Gamma)} \otimes V_\xi)x(I_{\ell^2(\Gamma)} \otimes V_\xi^*), \quad x \in B(\ell^2(\Gamma)) \otimes B(H)\,.$$

This is called a *slice map*, and is usually denoted by $\mathrm{id}_{B(\ell^2(\Gamma))} \otimes \omega$. We deduce that

$$m = (\mathrm{id}_{B(\ell^2(\Gamma))} \otimes \omega) \circ \sigma \colon L(\Gamma) \to B(\ell^2(\Gamma))$$

is a normal u.c.p. map, as well. We claim that $m\colon L(\Gamma) \to L(\Gamma)$ is the normal u.c.p. extension of $m_\varphi\colon \mathbb{C}\Gamma \to \mathbb{C}\Gamma$. To verify this, it suffices to show that $m(\lambda_t) = \varphi(t)\lambda_t$ for all $t \in \Gamma$. Indeed,

$$\begin{aligned} m(\lambda_t) = (\mathrm{id}_{B(\ell^2(\Gamma))} \otimes \omega)U(\lambda_t \otimes 1_H)U^* &= (\mathrm{id}_{B(\ell^2(\Gamma))} \otimes \omega)(\lambda_t \otimes t) \\ &= \omega(t)\lambda_t = \omega_\varphi(t)\lambda_t = \varphi(t)\lambda_t. \end{aligned}$$

By normality of m, since $\mathbb{C}\Gamma$ is ultraweakly dense in $L(\Gamma)$, it follows that $m(L(\Gamma)) \subset L(\Gamma)$, so $m\colon L(\Gamma) \to L(\Gamma)$, as wanted. The restriction of m to $C^*_\lambda(\Gamma) \subset L(\Gamma)$ gives a u.c.p. map $m\colon C^*_\lambda(\Gamma) \to L(\Gamma)$ that extends $m_\varphi\colon \mathbb{C}\Gamma \to \mathbb{C}\Gamma$. As m is norm-continuous, and $\mathbb{C}\Gamma$ is norm-dense in $C^*_\lambda(\Gamma)$, we conclude that $m(C^*_\lambda(\Gamma)) \subset C^*_\lambda(\Gamma)$. So $m\colon C^*_\lambda(\Gamma) \to C^*_\lambda(\Gamma)$ is the desired u.c.p. extension of m_φ.

We finally show that m_φ extends to a u.c.p. map $m\colon C^*(\Gamma) \to C^*(\Gamma)$. The unitary representation

$$\Gamma \ni s \mapsto s \otimes s \in \mathcal{U}(C^*(\Gamma) \otimes C^*(\Gamma))$$

extends, by universality of $C^*(\Gamma)$, to a unital *-homomorphism

$$\Delta\colon C^*(\Gamma) \to C^*(\Gamma) \otimes C^*(\Gamma).$$

As before, let $\mathrm{id}_{C^*(\Gamma)} \otimes \omega_\varphi\colon C^*(\Gamma) \otimes C^*(\Gamma) \to C^*(\Gamma)$ be the slice map determined by the condition $(\mathrm{id}_{C^*(\Gamma)} \otimes \omega_\varphi)(x \otimes y) = \omega_\varphi(y)x$, for all $x, y \in C^*(\Gamma)$. This is a u.c.p. map. Set

$$m = (\mathrm{id}_{C^*(\Gamma)} \otimes \omega_\varphi) \circ \Delta\colon C^*(\Gamma) \to C^*(\Gamma)$$

and note that m is u.c.p. Then for all $t \in \Gamma$,

$$m(t) = (\mathrm{id}_{C^*(\Gamma)} \otimes \omega_\varphi)(t \otimes t) = \omega_\varphi(t) = \varphi(t)t = m_\varphi(t).$$

Hence $m(x) = m_\varphi(x)$ for all $x \in \mathbb{C}\Gamma$, so m extends m_φ.

(4)$\Rightarrow$(1): If (4) holds, then $m_\varphi\colon \mathbb{C}\Gamma \to \mathbb{C}\Gamma$ is u.c.p., where we view $\mathbb{C}\Gamma$ as a subalgebra of $C^*(\Gamma)$, $C^*_\lambda(\Gamma)$ or $L(\Gamma)$, respectively. We show that φ is positive definite. Take $F = \{s_1, \ldots, s_n\} \subset \Gamma$. Set

$$S = \begin{bmatrix} s_1 & \cdots & s_n \\ & 0 & \end{bmatrix} \in M_n(\mathbb{C}\Gamma), \quad U = \begin{bmatrix} s_1 & & \\ & \ddots & \\ & & s_n \end{bmatrix} \in M_n(\mathbb{C}\Gamma).$$

Then $S^*S = [s_i^{-1}s_j]_{i,j} \in M_n(\mathbb{C}\Gamma)_+$. Since m_φ is u.c.p., we get $[m_\varphi(s_i^{-1}s_j)]_{i,j} \in M_n(\mathbb{C}\Gamma)_+$. We claim that

$$[\varphi(s_i^{-1}s_j)]_{i,j} = U[m_\varphi(s_i^{-1}s_j)]_{i,j}U^*.$$

This will imply that $[\varphi(s_i^{-1}s_j)]_{i,j} \in M_n(\mathbb{C})_+$, as wanted. To prove the claim, note that for $k, \ell \in \{1, \ldots, n\}$,

$$\begin{aligned} (U[m_\varphi(s_i^{-1}s_j)]_{i,j}U^*)_{k,\ell} &= s_k m_\varphi(s_k^{-1}s_\ell)s_\ell^{-1} \\ &= s_k\varphi(s_k^{-1}s_\ell)s_k^{-1}s_\ell s_\ell^{-1} = \varphi(s_k^{-1}s_\ell), \end{aligned}$$

which is the (k, ℓ) entry in $[\varphi(s_i^{-1}s_j)]_{i,j}$. $\square$

We will need the next result in the proof of Proposition 5.16 below:

Lemma 5.15. *Let $\Lambda \subset \Gamma$ be a subgroup and let $\varphi_0 \colon \Lambda \to \mathbb{C}$ be a positive definite function. Let $\varphi \colon \Gamma \to \mathbb{C}$ be given by*

$$\varphi(s) = \begin{cases} \varphi_0(s) & \text{if } s \in \Lambda \\ 0 & \text{if } s \notin \Lambda. \end{cases}$$

Then φ is positive definite.

Proof. Let $F \subset \Gamma$ be a finite set. As Γ is the disjoint union of left cosets, there exist $g_1, \dots, g_n \in \Gamma$ such that

$$F \subseteq g_1\Lambda \,\dot\cup\, \cdots \,\dot\cup\, g_k\Lambda.$$

Set $F_i = F \cap g_i\Lambda$, $i = 1, \dots, k$. If $s \in F_i$ and $t \in F_j$, $i \neq j$, then $s^{-1}t \notin \Lambda$, so $\varphi(s^{-1}t) = 0$. This shows that

$$[\varphi(s^{-1}t)]_{s,t\in F} = \bigoplus_{i=1}^{k} [\varphi(s^{-1}t)]_{s,t\in F_i},$$

where the right hand side is the block-diagonal matrix with k blocks $[\varphi(s^{-1}t)]_{s,t\in F_i}$. This matrix is positive if and only if $[\varphi(s^{-1}t)]_{s,t\in F_i}$ is positive for all $i = 1, \dots, k$. Set $G_i = g_i^{-1}F_i \subset \Lambda$. If $s, t \in F_i$, then $s = g_i s_0$, $t = g_i t_0$, where $s_0, t_0 \in G_i$ and $s^{-1}t = s_0^{-1}t_0$. This shows that

$$[\varphi(s^{-1}t)]_{s,t\in F_i} = [\varphi_0(s_0^{-1}t_0)]_{s_0,t_0\in G_i}$$

for all $i = 1, \dots, k$. The right hand side is positive because φ_0 is positive definite on Λ. □

We are now ready to prove the following:

Proposition 5.16 (Proposition 2.5.8, [BO])**.** *Let $\Lambda \subset \Gamma$ be a subgroup. Then there exists a canonical inclusion*

$$C^*(\Lambda) \subset C^*(\Gamma).$$

Proof. By universality of $C^*(\Lambda)$, whenever B is a unital C^*-algebra and $\pi_0 \colon \Lambda \to \mathcal{U}(B)$ is a unitary representation, there exists a (unique) *-homomorphism $\pi \colon C^*(\Lambda) \to B$ such that $\pi(t) = \pi_0(t)$ for all $t \in \Lambda$. Hence there exists a unique *-homomorphism $\pi \colon C^*(\Lambda) \to C^*(\Gamma)$ such that $\pi(t) = t$ for all $t \in \Lambda \subset \Gamma$. We must show that π is injective.

Let $x \in C^*(\Lambda)$, $x \geq 0$, $x \neq 0$. It suffices to show that $\pi(x) \neq 0$. There is a state ω on $C^*(\Lambda)$ such that $\omega(x) \neq 0$. Let $\varphi_0 \colon \Lambda \to \mathbb{C}$ be given by $\varphi_0(t) = \omega(t)$, $t \in \Lambda$. Then φ_0 is positive definite on Λ (by the Remark after Definition 5.11). By the Lemma above, φ_0 extends to a positive definite function $\varphi \colon \Gamma \to \mathbb{C}$, and by (1) ⇒ (3) in Theorem 5.14, ω_φ extends to a state on $C^*(\Gamma)$. For each $t \in \Lambda$,

$$(\omega_\varphi \circ \pi)(t) = \omega_\varphi(t) = \varphi(t) = \varphi_0(t) = \omega(t).$$

By continuity and linearity, this implies that $\omega_\varphi \circ \pi = \omega$. Hence $(\omega_\varphi \circ \pi)(x) = \omega(x) \neq 0$, so $\pi(x) \neq 0$, as desired. □

Lecture 6 (continued), GOADyn
September 27, 2021

Comments on sections 3.4–3.9

Recall from Section 3.3 (see hand-written notes–Lecture 6–on Tensor products):

Let A and B be C*-algebras.

Definition 6.1 (Definition 3.3.1, [BO]). A C*-norm $\|\cdot\|_\alpha$ on $A \odot B$ is a norm such that

$$\|xy\|_\alpha \leq \|x\|_\alpha \|y\|_\alpha, \quad \|x^*\|_\alpha = \|x\|_\alpha, \quad \|x^*x\|_\alpha = \|x\|_\alpha^2, \quad x, y \in A \odot B.$$

We denote by $A \odot_\alpha B$ the completion of $A \odot B$ with respect to $\|\cdot\|_\alpha$.

Lemma 6.2 (Lemma 3.4.10, [BO]). *Any C***-norm* $\|\cdot\|_\alpha$ *on* $A \odot B$ *is a cross-norm, i.e.,*

$$\|x \otimes y\|_\alpha = \|a\|\|b\|, \quad a \otimes b \in A \odot B.$$

Proof. To be discussed in lecture. □

C*-norms on algebraic tensor products do exist. The two most natural of them are the following:

- Maximal norm (Definition 3.3.3. [BO]): Given $x \in A \odot B$, set
$$\|x\|_{\max} := \sup\{\|\pi(x)\| \mid \pi\colon A \odot B \to B(H) \text{ is a (cyclic) } {}^*\text{-homomorphism}\}.$$
- Minimal (or spatial) norm (Definition 3.3.4. [BO]): If $\pi\colon A \to B(H)$, $\sigma\colon B \to B(K)$ are faithful representations and $x = \sum_{i=1}^n a_i \otimes b_i \in A \odot B$, then
$$\Big\|\sum a_i \otimes b_i\Big\|_{\min} := \Big\|\sum \pi(a_i) \otimes \sigma(b_i)\Big\|_{B(H \otimes K)}.$$

We have seen that $\|\cdot\|_{\min}$ is independent of the choice of faithful representations, and that $\|\cdot\|_{\min}$ and $\|\cdot\|_{\max}$ are, indeed, (C*-)norms on $A \odot B$. We denote by $A \otimes_{\max} B$ and $A \otimes_{\min} B$ (or, simply $A \otimes B$ in [BO]) the completion of $A \odot B$ with respect to the norms $\|\cdot\|_{\max}$ and $\|\cdot\|_{\min}$, respectively. Another important feature of the maximal tensor product is the following universal property:

Proposition 6.3 (Proposition 3.3.7, [BO]). *If* $\pi : A \odot B \to C$ *is a* $*$*-homomorphism, then there is a unique* $*$*-homomorphism* $\tilde{\pi} : A \otimes_{\max} B \to C$ *which extends* π*.*

As a consequence, we have the following result:

Corollary 6.4 (Corollary 3.3.8, [BO]). *The maximal norm* $\|\cdot\|_{\max}$ *is the largest C***-norm on* $A \odot B$*.*

A much harder result to prove is the following:

Theorem 6.5 (Takesaki, Theorem 3.4.8, [BO]). *The minimal norm* $\|\cdot\|_{\min}$ *is the smallest C***-norm on* $A \odot B$*.*

Proof. To be discussed in lecture. □

As a consequence of Takesaki's theorem and the universality property of $\|\cdot\|_{\max}$, we obtain the following:

Corollary 6.6 (Corollary 3.4.9, [BO]). *For any C^*-norm $\|\cdot\|_\alpha$ on $A \odot B$, there are natural surjective homomorphisms*

$$A \otimes_{\max} B \to A \otimes_\alpha B \to A \otimes_{\min} B,$$

where $A \otimes_\alpha B$ is the completion of $A \odot B$ in the norm $\|\cdot\|_\alpha$.

The next result, whose proof we omit, concerns continuity of tensor product maps:

Theorem 6.7 (Continuity of tensor product maps, Theorem 3.5.3, [BO]). *Let A, B, C, D be C^*-algebras and $\varphi\colon A \to C$, $\psi\colon B \to D$ be c.p. maps. Then $\varphi \odot \psi\colon A \odot B \to C \odot D$ extends to c.p. maps*

$$\varphi \otimes_{\max} \psi\colon A \otimes_{\max} B \to C \otimes_{\max} D,$$
$$\varphi \otimes_{\min} \psi\colon A \otimes_{\min} B \to C \otimes_{\min} D.$$

Moreover, $\|\varphi \otimes_{\max} \psi\| = \|\varphi \otimes_{\min} \psi\| = \|\varphi\|\|\psi\|$.

We are now ready to discuss nuclearity in terms of tensor products:

Proposition 6.8 (Proposition 3.6.12, [BO]). *If A is nuclear, then for all C^*-algebras C,*

$$A \otimes_{\max} C = A \otimes_{\min} C.$$

Proof. The proof below follows the proof of Lemma 3.6.2 in Brown-Ozawa. First note that

$$M_n(\mathbb{C}) \otimes_{\max} C = M_n(\mathbb{C}) \otimes_{\min} C \tag{$\star$}$$

because $M_n(\mathbb{C}) \odot C \cong M_n(C)$ (cf. Exercise 3.1.3, [BO]) and $M_n(C)$ has a unique C^*-norm. Since A is nuclear, there exist nets $(\varphi_i)_{i\in I}$, $(\psi_i)_{i\in I}$ of c.c.p. maps

$$\varphi_i\colon A \to M_{k(i)}(\mathbb{C}), \quad \psi_i\colon M_{k(i)}(\mathbb{C}) \to A$$

such that $\|\psi_i \circ \varphi_i(a) - a\| \to 0$ for all $a \in A$. Using ($\star$) and Theorem 6.7 we get that

$$\sigma_i = (\psi_i \otimes_{\max} \mathrm{id}_C) \circ (\varphi_i \otimes_{\min} \mathrm{id}_C)$$

is a well-defined c.c.p. map from $A \otimes_{\min} C$ to $A \otimes_{\max} C$. In particular, $\|\sigma_i\| \le 1$. Since

$$\sigma_i(a \otimes c) = (\psi_i \circ \varphi_i)(a) \otimes c, \quad a \in A,\ c \in C,$$

we have for all $n \in \mathbb{N}$, $a_1, \dots, a_n \in A$ and $c_1, \dots, c_n \in C$ that

$$\sigma_i\left(\sum_{k=1}^n a_k \otimes c_k\right) = \sum_{k=1}^n (\varphi_i \circ \psi_i)(a_k) \otimes c_k$$

and hence

$$\left\|\sum_{k=1}^n (\varphi_i \circ \psi_i)(a_k) \otimes c_k\right\|_{\max} \le \left\|\sum_{k=1}^n a_k \otimes c_k\right\|_{\min}.$$

But since $\|\psi_i \circ \varphi_i(a_k) - a_k\| \to 0$ and since $\|\cdot\|_{\max}$ is a cross-norm (by Lemma 8.2 above), we get

$$\left\|\sum_{k=1}^n a_k \otimes c_k\right\|_{\max} = \lim_i \left\|\sum_{k=1}^n (\varphi_i \circ \psi_i)(a_k) \otimes c_k\right\|_{\max} \le \left\|\sum_{k=1}^n a_k \otimes c_k\right\|_{\min}.$$

Hence $\|\cdot\|_{\max} \le \|\cdot\|_{\min}$ on $A \odot C$, and therefore the two norms coincide. In other words, we have proved that $A \otimes_{\max} C = A \otimes_{\min} C$. □

Theorem 6.9 (Choi/Effros, Kirchberg 1973, Theorem 3.8.7, [BO])**.** *For a C^*-algebra A, the following are equivalent:*

(1) *A is nuclear (i.e., id_A is a nuclear map).*
(2) *For every C^*-algebra C,*

$$A \otimes_{\max} C = A \otimes_{\min} C.$$

Remark 6.10. (1) $\Rightarrow$ (2) is already proved above. The proof of (2) $\Rightarrow$ (1) is very involved (see Section 3.8).

Remark 6.11. Condition (2) above was the original definition of a nuclear C^*-algebra A, due to C. Lance (1973).

§3.7. Exact sequences

A sequence

$$X_0 \xrightarrow{\delta_1} X_1 \xrightarrow{\delta_2} X_2 \xrightarrow{\delta_3} \cdots \xrightarrow{\delta_n} X_n$$

of vector spaces $(X_i)_{i=0}^n$ and linear maps $\delta_i \colon X_{i-1} \to X_i$ is called *exact* if

$$\mathrm{Im}(\delta_i) = \mathrm{Ker}(\delta_{i+1}), \quad i = 1, \ldots, n-1.$$

If

$$0 \xrightarrow{\delta_1} X_1 \xrightarrow{\delta_2} X_2 \xrightarrow{\delta_3} X_3 \xrightarrow{\delta_4} 0 \text{ is a short exact sequence,} \tag{$\star$}$$

then $\delta_1 = \delta_4 = 0$, δ_2 is one-to-one, δ_3 is surjective, and since $\mathrm{Im}(\delta_2) = \mathrm{Ker}(\delta_3)$, we have

$$X_3 \cong X_2/\mathrm{Im}(\delta_2).$$

If we think of δ_2 as an inclusion map and δ_3 as a quotient map, then ($\star$) is just another way of writing $X_1 \subset X_2$ and $X_3 = X_2/X_1$.

Definition 6.12. We call

$$0 \longrightarrow J \longrightarrow A \longrightarrow C \longrightarrow 0$$

a short exact sequence of C^*-algebras if J is a closed two-sided ideal in A and $C = A/J$.

Remark 6.13. Let

$$0 \longrightarrow J \longrightarrow A \longrightarrow A/J \longrightarrow 0$$

be a short exact sequence of C^*-algebras. Then it is easy to check that for all C^*-algebras B,

$$0 \longrightarrow J \odot B \longrightarrow A \odot B \longrightarrow (A/J) \odot B \longrightarrow 0$$

is an exact sequence of algebras, i.e. $J \odot B$ is a two-sided ideal in $A \odot B$, and

$$(A \odot B)/(J \odot B) = (A/J) \odot B.$$

Proposition 6.14 (Proposition 3.7.1, [BO]). *Let*

$$0 \longrightarrow J \longrightarrow A \longrightarrow {}^{A}\!/\!_{J} \longrightarrow 0$$

be a short exact sequence of C^*-algebras. Then for all every C^*-algebra B,

$$0 \longrightarrow J \otimes_{\max} B \longrightarrow A \otimes_{\max} B \longrightarrow ({}^{A}\!/\!_{J}) \otimes_{\max} B \longrightarrow 0$$

is also a short exact sequence of C^*-algebras.

Proposition 6.15 (Proposition 3.7.2, [BO]). *Given $J \triangleleft A$ and B as above, then there exists a C^*-norm $\|\cdot\|_\alpha$ on $({}^{A}\!/\!_{J}) \odot B$ such that*

$$0 \longrightarrow J \otimes_{\min} B \longrightarrow A \otimes_{\min} B \longrightarrow ({}^{A}\!/\!_{J}) \otimes_{\alpha} B \longrightarrow 0$$

is an exact sequence.

Theorem 6.16 (Kirchberg, Theorem 3.9.1, [BO]). *Let B be a C^*-algebra. Then the following are equivalent:*

(1) *B is exact.*

(2) *For every pair (A, J) of a C^*-algebra A and a closed two-sided ideal $J \triangleleft A$, the sequence*

$$0 \longrightarrow J \otimes_{\min} B \longrightarrow A \otimes_{\min} B \longrightarrow ({}^{A}\!/\!_{J}) \otimes_{\min} B \longrightarrow 0$$

is exact.

Remark 6.17. (1) $\Rightarrow$ (2) is proved in Proposition 3.7.8 [BO]. The proof of (2) $\Rightarrow$ (1) is very involved (see Section 3.9).

Remark 6.18. Condition (2) above was Kirchberg's original definition of an exact C^*-algebra B.

Remark 6.19. A C^*-algebra B is exact if and only if (2) holds for the pair

$$(A, J) = (\mathbb{B}(H), \mathbb{K}(H))$$

where H is a separable, infinite-dimensional Hilbert space (cf. Exercise 3.9.7, [BO]).

Lectures 7 and 8, GOADyn
September 30 and October 5, 2021

Section 2.6: Amenable groups

In the following, let Γ be a discrete group.

Definition 7.1 (Definition 2.6.1, [BO])**.** The group Γ is called *amenable* if there exists a state (=mean) μ on $\ell^\infty(\Gamma)$ such that for all $s \in \Gamma$ and all $f \in \ell^\infty(\Gamma)$,

$$\mu(s.f) = \mu(f),$$

where $(s.f)(t) = f(s^{-1}t)$, $t \in \Gamma$, i.e., μ is invariant under the left action of Γ.

We will begin by proving that this is equivalent to the original definition of amenability given by John von Neumann. (In what follows, $\mathcal{P}(\Omega)$ denotes the power set of Ω.)

Theorem 7.2. *A group Γ is amenable if and only if there exists a finitely additive left-invariant measure $\mu\colon \mathcal{P}(\Gamma) \to [0,1]$ such that $\mu(\Gamma) = 1$.*

Definition 7.3. Let Ω be a set. A map $\mu\colon \mathcal{P}(\Omega) \to [0,1]$ is a *finitely additive* probability measure on Ω if $\mu(\Omega) = 1$ and $\mu(A \cup B) = \mu(A) + \mu(B)$ whenever A and B are disjoint subsets of Ω.

Let $\mathrm{PM}(\Omega)$ denote the set of all finitely additive probability measures on Ω.

Example 7.4. Let $F \subseteq \Omega$ be a finite subset, and define $\mu_F\colon \mathcal{P}(\Omega) \to [0,1]$ by

$$\mu_F(A) = \frac{|A \cap F|}{|F|}, \qquad A \subset \Omega.$$

Then $\mu_F \in PM(\Omega)$.

Let $\mathrm{M}(\Omega)$ be the set of all states (means) on $\ell^\infty(\Omega)$, so that $\mathrm{M}(\Omega) \subset \big(\ell^\infty(\Omega)\big)^*_1$. We shal see that there exists a one-to-one correspondence between $\mathrm{M}(\Omega)$ and $\mathrm{PM}(\Omega)$:

For each $m \in \mathrm{M}(\Omega)$ let $\hat{m}\colon \mathcal{P}(\Omega) \to [0,1]$ be defined by $\widehat{m}(A) = m(1_A)$, for all $A \subset \Omega$. Note that $\widehat{m} \in \mathrm{PM}(\Omega)$. Let $\Phi\colon \mathrm{M}(\Omega) \to \mathrm{PM}(\Omega)$ be given by $\Phi(m) = \widehat{m}$.

Claim. Φ is bijective.

For the proof, we need several facts.

(1) Let $\mathrm{E}(\Omega)$ denote the collection of all *simple* maps on Ω, i.e., maps $x\colon \Omega \to \mathbb{R}$ such that $x(\Omega)$ is finite. Then $\mathrm{E}(\Omega)$ is a subspace of $\ell^\infty(\Omega)$, which moreover is *dense* in $\ell^\infty(\Omega)$ with respect to $\|\cdot\|_\infty$. This means that each $x \in \ell^\infty(\Omega)$ is the uniform limit of simple functions.

(2) Let $\mu \in \mathrm{PM}(\Omega)$. Define $\overline{\mu}\colon \mathrm{E}(\Omega) \to \mathbb{R}$ by

$$\overline{\mu}(x) = \sum_{i=1}^n \alpha_i \mu(A_i), \qquad x = \sum_{i=1}^n \alpha_i 1_{A_i},$$

where $(A_i)_{i=1}^n$ is a finite partition of Ω. Note that $\overline{\mu}(x) \geq 0$ whenever $x \geq 0$. Then $\overline{\mu}\colon \mathrm{E}(\Omega) \to \mathbb{R}$ is a linear contraction. The latter follows from

$$|\overline{\mu}(x)| \leq \sup_{\omega\in\Omega} |x(\omega)| = \|x\|_\infty, \quad x \in \mathrm{E}(\Omega).$$

By (1), $\overline{\mu}$ extends uniquely to some $\widetilde{\mu} \in \mathrm{M}(\Omega)$.

We show $\Phi(\widetilde{\mu}) = \mu$, for all $\mu \in \mathrm{PM}(\Omega)$, which will prove that Φ is surjective. Indeed, for $A \subset \Omega$,

$$\Phi(\widetilde{\mu})(A) = \widetilde{\mu}(1_A) = \overline{\mu}(1_A) = \mu(A).$$

To show that Φ is injective, let $m_1, m_2 \in \mathrm{M}(\Omega)$ be such that $\Phi(m_1) = \Phi(m_2)$. Then $\widehat{m}_1 = \widehat{m}_2$, i.e., $m_1(1_A) = m_2(1_A)$ for all $A \subset \Omega$. By linearity, $m_1 = m_2$ on $\mathrm{E}(\Omega)$, which by (1) and continuity implies $m_1 = m_2$ on $\ell^\infty(\Omega)$. $\square$

Now assume that $\Omega = \Gamma$ is a group. Given $\mu \in \mathrm{PM}(\Gamma)$ and $g \in \Gamma$, define $g\mu\colon \mathcal{P}(\Gamma) \to [0,1]$ by

$$g\mu(A) = \mu(g^{-1}A), \quad A \subset \Gamma.$$

Note that $g\mu \in \mathrm{PM}(\Gamma)$. Indeed, $g\mu(\Gamma) = \mu(g^{-1}\Gamma) = \mu(\Gamma) = 1$, and if $A, B \subset \Gamma$ are disjoint, then

$$g\mu(A \cup B) = \mu\big(g^{-1}(A \cup B)\big) = \mu(g^{-1}A \cup g^{-1}B) = \mu(g^{-1}A) + \mu(g^{-1}B) = g\mu(A) + g\mu(B).$$

We say that μ *is left-invariant* if $g\mu = \mu$, for all $g \in \Gamma$.

Some further constructions:

- For $x \in \ell^\infty(\Gamma)$ and $g \in \Gamma$ let $gx\colon \Gamma \to \mathbb{R}$ be given by $(gx)(t) = x(g^{-1}t)$, $t \in \Gamma$. Then $gx \in \ell^\infty(\Gamma)$ with $\|gx\|_\infty = \|x\|_\infty$.
- For $u \in \ell^\infty(\Gamma)^*$ and $g \in \Gamma$, let $gu\colon \ell^\infty(\Gamma) \to \mathbb{R}$ be given by $(gu)(x) = u(g^{-1}x)$, $x \in \ell^\infty(\Gamma)$. Then $gu \in \ell^\infty(\Gamma)^*$ with $\|gu\| = \|u\|$.

Proof of Theorem 7.2: Let $m \in \mathrm{M}(\Gamma)$. Then m is left-invariant if and only if the associated finitely additive probability measure $\widehat{m}$ is left-invariant. This follows from the fact that $\widehat{gm} = g\,\widehat{m}$ for all $g \in \Gamma$, which can be verified as follows. For all $A \subset \Gamma$:

$$\widehat{gm}(A) = gm(1_A) = m(g^{-1}1_A) \overset{(*)}{=} m(1_{g^{-1}A}) = \widehat{m}(g^{-1}A) = g\,\widehat{m}(A),$$

where $(*)$ holds because $g^{-1}1_A = 1_{g^{-1}A}$. $\square$

Definition 7.5 (Definition 2.6.2, [BO])**.** The set of probability measures on Γ is denoted by $\mathrm{Prob}(\Gamma)$, i.e.,

$$\mathrm{Prob}(\Gamma) = \left\{ \mu \in \ell^1(\Gamma) \,:\, \mu \geq 0, \ \sum_{t \in \Gamma} \mu(t) = 1 \right\}.$$

Definition 7.6 (Definition 2.6.3, [BO])**.** Γ has an *approximate invariant mean* if for any finite set $E \subset \Gamma$ and every $\varepsilon > 0$, there exists $\mu \in \mathrm{Prob}(\Gamma)$ such that

$$\max_{s \in E} \|s.\mu - \mu\|_1 < \varepsilon.$$

Recall that for two sets $E, F \subset \Gamma$,

$$E \triangle F = (E \cup F) \setminus (E \cap F) = (E \setminus F) \cup (F \setminus E) = (E \setminus (E \cap F)) \cup (F \setminus (E \cap F)).$$

Definition 7.7 (Definition 2.6.4, [BO])**.** Γ satisfies *the Følner condition* if for every finite set $E \subset \Gamma$ and $\varepsilon > 0$ there exists a finite set $F \subset \Gamma$ such that

$$\max_{s \in E} \frac{|sF \triangle F|}{|F|} < \varepsilon,$$

where $sF = \{st \,:\, t \in F\}$. A sequence $(F_n)_{n \geq 1}$ of finite subsets of Γ is called a *Følner sequence* if

$$\frac{|sF_n \triangle F_n|}{|F_n|} \to 0 \quad \text{as } n \to \infty$$

for all $s \in \Gamma$.

Remark 7.8. Since $|sF \triangle F| = |sF| + |F| - 2|sF \cap F|$, the Følner condition is equivalent to

$$\max_{s \in E} \frac{|sF \cap F|}{|F|} > 1 - \frac{\varepsilon}{2}\,.$$

If Γ satisfies the Følner condition, then Γ has an approximate invariant mean given by normalized characteristic functions of finite subsets. Given $F \subset \Gamma$ a finite subset, then $(1/|F|)1_F \in \mathrm{Prob}(\Gamma)$ and

$$\left\| s.\frac{1}{|F|}1_F - \frac{1}{|F|}1_F \right\|_1 = \frac{|sF \triangle F|}{|F|}\,.$$

Example 7.9 (Example 2.6.7, [BO])**.** $\mathbb{F}_2$ (the free group on 2 generators a, b) is non-amenable:

$$\mathbb{F}_2:\ e, a, a^{-1}, b, b^{-1}, ab, ab^{-1}, a^2, a^{-1}b, a^{-1}b^{-1}, \ldots$$

If $x \in \mathbb{F}_2$, $x \neq e$, then $x = s_1 s_2 \cdots s_n$ (uniquely), where $s_i \in \{a, a^{-1}, b, b^{-1}\}$ and

$$(s_i, s_{i+1}) \neq (a, a^{-1}), (a^{-1}, a), (b, b^{-1}), (b^{-1}, b).$$

$s_1 s_2 \cdots s_n$ is called the reduced word of x and $|x| = n$ is called the length of x. To multiply $s_1 \cdots s_n t_1 \cdots t_k$, make a reduction by (successively) removing pairs of the form $(a, a^{-1}), (a^{-1}, a), (b, b^{-1}), (b^{-1}, b)$. Put

$$\begin{aligned} A^+ &= \{\text{all reduced words starting with } a\} \subset \mathbb{F}_2, \\ A^- &= \{\text{all reduced words starting with } a^{-1}\} \subset \mathbb{F}_2, \\ B^+ &= \{\text{all reduced words starting with } b\} \subset \mathbb{F}_2, \\ B^- &= \{\text{all reduced words starting with } b^{-1}\} \subset \mathbb{F}_2. \end{aligned}$$

Then

(a) $\mathbb{F}_2 = A^+ \cup aA^-$ (if $x \notin A^+$, then either $x = e \in aA^-$ or x has the reduced form $x = s_1 \cdots s_n$, $s_1 \neq a$, so that

$$x = s_1 \cdots s_n = a(a^{-1}s_1 \cdots s_n) \in aA^-,$$

since $a^{-1}s_1 \cdots s_n$ is reduced).

(b) $\mathbb{F}_2 = B^+ \cup bB^-$.

(c) $\mathbb{F}_2 = \{e\} \,\dot\cup\, A^+ \,\dot\cup\, A^- \,\dot\cup\, B^+ \,\dot\cup\, B^-$.

Assume that μ is a left invariant mean on $\mathbb{F}_2$. Consider $m = \hat{\mu} \in PM(\mathbb{F}_2)$. Then m is left-invariant, so

$$m(sE) = m(E), \quad s \in \mathbb{F}_2,\ E \in \mathcal{P}(\mathbb{F}_2).$$

By (a) and (b), $m(A^+) + m(A^-) \geq m(\mathbb{F}_2) = 1$ and $m(B^+) + m(B^-) \geq m(\mathbb{F}_2) = 1$, and by (c),

$$1 + 1 \leq m(A^+) + m(A^-) + m(B^+) + m(B^-) \leq 1,$$

which is obviously wrong! Hence $\mathbb{F}_2$ is not amenable.

Theorem 7.10 (Theorem 2.6.8, [BO])**.** *Let Γ be a discrete group. Then the following are equivalent:*

(1) Γ *is amenable.*

(2) Γ *has an approximate invariant mean.*

(3) Γ *satisfies the Følner condition.*

(4) *The trivial representation* τ_0 *is weakly contained in the regular representation, i.e., there exists a net of unit vectors* $\xi_i \in \ell^2(\Gamma)$ *such that for all* $s \in \Gamma$,

$$\lim_i \|\lambda_s \xi_i - \xi_i\|_2 = 0.$$

(5) *There exists a net* $(\varphi_i)_{i\in I}$ *of finitely supported positive definite functions on* Γ *such that* $\lim_i \varphi_i(s) = 1$, *for all* $s \in \Gamma$. *(Note: Without loss of generality, we may assume* $\varphi_i(e) = 1$, *for all* $i \in I$.)

(6) $C^*(\Gamma) = C^*_\lambda(\Gamma)$.

(7) $C^*_\lambda(\Gamma)$ *has a character (a one-dimensional representation).*

(8) *For any finite set* $E \subset \Gamma$,

$$\left\| \frac{1}{|E|} \sum_{s\in E} \lambda_s \right\| = 1.$$

(9) $C^*_r(\Gamma)$ *is nuclear.*

(10) $L(\Gamma)$ *is semidiscrete.*

Proof. **(1)** $\Rightarrow$ **(2):** We will first prove the following statement:

Claim: For every state μ on $\ell^\infty(\Gamma)$ there exists a net $(\nu_i)_{i\in I}$ in $\mathrm{Prob}(\Gamma)$ such that $\nu_i \xrightarrow{w^*} \mu$, meaning that for all $f \in \ell^\infty(\Gamma)$,

$$\lim_i \underbrace{\left(\sum_{s\in\Gamma} f(s)\nu_i(s) \right)}_{\nu_i(f)} = \mu(f).$$

This is equivalent to showing that $\mu \in \overline{\mathrm{Prob}(\Gamma)}^{w^*}$ (the w^*-closure in $\ell^\infty(\Gamma)^*$). If this was not true, then by the Hahn-Banach separation theorem we could find $f \in \ell^\infty(\Gamma)$ such that

$$\mathrm{Re}\mu(f) > \sup\{\mathrm{Re}\,\nu(f) \,:\, \nu \in \mathrm{Prob}(\Gamma)\}.$$

Replacing f by $\mathrm{Re}(f)$ we have a <u>real</u> function $f \in \ell^\infty(\Gamma)$ such that $\mu(f) > \sup\{\nu(f) \,:\, \nu \in \mathrm{Prob}(\Gamma)\}$. Since the Dirac measures δ_s given by

$$\delta_s(t) = \begin{cases} 1 & s = t \\ 0 & s \neq t \end{cases}$$

are in $\mathrm{Prob}(\Gamma)$, we have $\mu(f) > \sup\{f(t) \,:\, t \in \Gamma\}$. Set $f_0 = f - \sup\{f(t) \,:\, t \in \Gamma\}$. Then $f_0 \leq 0$, but $\mu(f_0) = \mu(f) - \sup\{f(t) \,:\, t \in \Gamma\} > 0$, a contradiction! This proves the Claim.

Let μ be a left-invariant state on $\ell^\infty(\Gamma)$. By the Claim, let $(\nu_i)_{i\in I}$ be a net in $\mathrm{Prob}(\Gamma)$ such that $\nu_i \to \mu$ weak* (in $\ell^\infty(\Gamma)^*$). Given $s \in \Gamma$ and $f \in \ell^\infty(\Gamma)$, we have

$$(s.\nu_i)(f) = \sum_{t\in\Gamma} (s.\nu_i)(t) f(t) = \sum_{t\in\Gamma} \nu_i(s^{-1}t) f(t) = \sum_{u\in\Gamma} \nu_i(u) f(su) = \nu_i(s^{-1}.f)\,,$$

which shows that $(s.\nu_i)(f) \to \mu(s^{-1}.f)$. Since μ is left invariant, it follows that for all $s \in \Gamma$,

$$s.\nu_i - \nu_i \to 0 \;\; \text{weak}^*.$$

But since $s.\nu_i - \nu_i \in \ell^1(\Gamma)$ and $\ell^1(\Gamma)^* = \ell^\infty(\Gamma)$, then $s.\nu_i - \nu_i$ actually converges to 0 *weakly* in $\ell^1(\Gamma)$. Now, let $E \subset \Gamma$ be finite, with $E = \{s_1, \ldots, s_n\}$. Then

$$(0, \ldots, 0) \in \overline{\{(s_1.\nu_i - \nu_i, \ldots, s_n.\nu_i - \nu_i) : i \in I\}}^{\text{weak}}.$$

where the weak closure is in

$$\underbrace{\ell^1(\Gamma) \oplus \cdots \oplus \ell^1(\Gamma)}_{n \text{ times}} \simeq \ell^1(\underbrace{\Gamma \dot\cup \cdots \dot\cup \Gamma}_{n \text{ times}})$$

Since convex sets in a Banach space have the same closure in norm and weak topology, we have

$$(0, \ldots, 0) \in \overline{\operatorname{conv}\{(s_1.\nu_i - \nu_i, \ldots, s_n.\nu_i - \nu_i) : i \in I\}}^{\text{norm}}.$$

Therefore, there exists a net $(\mu_j)_{j\in J}$ in $\operatorname{conv}\{\nu_i : i \in I\}$ such that

$$\|s_1.\mu_j - \mu_j\|_1 + \ldots + \|s_n.\mu_j - \mu_j\|_1 \to 0.$$

Hence for all $\varepsilon > 0$ there exists $j \in J$ such that such that

$$\max_{s\in E} \|s.\mu_j - \mu_j\|_1 \leq \sum_{s\in E} \|s.\mu_j - \mu_j\|_1 < \varepsilon.$$

This completes the proof of (1) ⇒ (2).

(2) ⇒ (3): Let $E \subset \Gamma$ be finite and let $\varepsilon > 0$ be given. Choose $\mu \in \operatorname{Prob}(\Gamma)$ such that $\max_{s\in E} \|s.\mu - \mu\|_1 < \varepsilon/|E|$ and hence

$$\sum_{s\in E} \|s.\mu - \mu\|_1 < \varepsilon.$$

Assume that $f \in \ell^1(\Gamma)_+$, $r \geq 0$. Set

$$F(f, r) = \{t \in \Gamma : f(t) > r\}.$$

Note that for $f, h \in \ell^1(\Gamma)_+$,

$$|1_{F(f,r)}(t) - 1_{F(h,r)}(t)| = \begin{cases} 0 & \text{if } f(t), h(t) \leq r \text{ or } f(t), h(t) > r \\ 1 & \text{if } h(t) \leq r < f(t) \text{ or } f(t) \leq r < h(t) \end{cases}$$

Thus if both $f \leq 1$ and $h \leq 1$, we can see (after some computations) that

$$|f(t) - h(t)| = \int_0^1 |1_{F(f,r)}(t) - 1_{F(h,r)}(t)| dr.$$

Let $\mu \in \operatorname{Prob}(\Gamma)$. Then $\mu \in \ell^1(\Gamma)_+$ and $\sum_{s\in\Gamma} \mu(s) = 1$. Hence $\mu(s) \leq 1$ for all $s \in \Gamma$. Therefore

$$\begin{aligned} \|s.\mu - \mu\|_1 &= \sum_{t\in\Gamma} |(s.\mu(t) - \mu(t)| \\ &= \sum_{t\in\Gamma} \int_0^1 |1_{F(s.\mu,r)}(t) - 1_{F(\mu,r)}(t)| dr \\ &= \int_0^1 \sum_{t\in\Gamma} |1_{F(s.\mu,r)}(t) - 1_{F(\mu,r)}(t)| dr \\ &= \int_0^1 |F(s.\mu, r) \bigtriangleup F(\mu, r)| dr. \end{aligned}$$

Since $F(s.\mu, r) = \{t \in \Gamma : (s.\mu)(t) > r\} = \{t \in \Gamma : \mu(s^{-1}t) > r\} = \{t \in \Gamma : s^{-1}t \in F(\mu, r)\} = \{t \in \Gamma : t \in sF(\mu, r)\} = sF(\mu, r)$, we have

$$\|s.\mu - \mu\|_1 = \int_0^1 |sF(\mu, r) \bigtriangleup F(\mu, r)| dr.$$

Using that

$$1_{F(\mu,r)}(t) = \begin{cases} 0 & \text{if } \mu(t) \leq r \\ 1 & \text{if } \mu(t) > r, \end{cases}$$

a similar (but simpler) computation gives

$$1 = \|\mu\|_1 = \sum_{t\in\Gamma} \mu(t) = \sum_{t\in\Gamma} \int_0^1 1_{F(\mu,r)}(t)dr = \int_0^1 |F(\mu,r)|dr.$$

Therefore,

$$\varepsilon \int_0^1 |F(\mu,r)|dr = \varepsilon > \sum_{s\in E} \|s.\mu - \mu\|_1 = \int_0^1 \sum_{s\in E} |sF(\mu,r) \triangle F(\mu,r)|dr.$$

Hence for some $r \in (0,1)$,

$$\varepsilon |F(\mu,r)| > \sum_{s\in E} |sF(\mu,r) \triangle F(\mu,r)|.$$

Hence with $F = F(\mu,r)$, for this particular r, we have

$$\varepsilon|F| > |sF \triangle F|, \quad s \in E.$$

In particular, $|F| > 0$. Moreover $|F| < \infty$, because when $r > 0$,

$$|F(\mu,r)| \leq \frac{1}{r} \sum_{t\in F(\mu,r)} \mu(t) \leq \frac{1}{r} \sum_{t\in\Gamma} \mu(t) = \frac{1}{r} < \infty.$$

Thus we have found a non-empty set F such that

$$\frac{|sF \triangle F|}{|F|} < \varepsilon$$

for all $s \in E$, i.e., Γ satisfies the Følner condition.

(3) ⇒ (4): By (3), there exists a net (F_i) of non-empty finite subsets of Γ such that

$$\frac{|sF_i \triangle F_i|}{|F_i|} \to 0$$

for all $s \in \Gamma$. Now put $\xi_i = |F_i|^{-1/2} 1_{F_i}$. Then $\|\xi_i\|_2 = 1$ and

$$\lambda_s \xi_i - \xi_i = s.\xi_i - \xi_i = \frac{1}{|F_i|^{1/2}} (1_{sF_i} - 1_{F_i}).$$

Thus

$$\|\lambda_s \xi_i - \xi_i\|_2^2 = \frac{1}{|F_i|} \sum_{t\in\Gamma} (1_{sF_i} - 1_{F_i})^2(t) = \frac{|sF_i \triangle F_i|}{|F_i|} \to 0,$$

which proves the assertion.

(4) ⇒ (5): Put $\varphi_i(s) = \langle \lambda_s \xi_i, \xi_i \rangle$, $s \in \Gamma$. Then by Theorem 5.14 (Theorem 2.5.11, [BO]), φ_i is positive definite and $\varphi_i(e) = \|\xi_i\|^2 = 1$. Moreover, $\varphi_i(s) = \langle \lambda_s \xi_i - \xi_i, \xi_i \rangle + \langle \xi_i, \xi_i \rangle = \langle \lambda_s \xi_i - \xi_i, \xi_i \rangle + 1$. Hence, for all $s \in \Gamma$,

$$|\varphi_i(s) - 1| = |\langle \lambda_s \xi_i - \xi_i, \xi_i \rangle| \leq \|\lambda_s \xi_i - \xi_i\| \|\xi_i\| = \|\lambda_s \xi_i - \xi_i\| \to 0$$

Does φ_i have finite support? NO, not in general. But since $\xi_i \in \ell^2(\Gamma)$, then for all $n \in \mathbb{N}$ there exists a finitely supported $\xi_{i,n} \in \ell^2(\Gamma)$ such that $\|\xi_i - \xi_{i,n}\|_2 < 1/n$ and $\|\xi_{i,n}\|_2 = 1$. Set

$$\varphi_{i,n}(s) = \langle \lambda_s \xi_{i,n}, \xi_{i,n} \rangle.$$

Clearly $\varphi_{i,n}(s) \to \varphi_i(s)$ as $n \to \infty$ (for all $s \in \Gamma$) and $\varphi_{i,n}$ has finite support because

$$\operatorname{supp}(\varphi_{i,n}) \subset \{s \in \Gamma \,:\, \exists x, y \in \operatorname{supp}(\xi_{i,n}) : sx = y\} = \{yx^{-1} \,:\, x, y \in \operatorname{supp}(\xi_{i,n})\}$$

and the latter set is finite. Let $P_1(\Gamma)$ be the set of positive definite functions φ on Γ with $\varphi(e) = 1$ and $C_c(\Gamma)$ be the set of finitely supported functions on Γ. Again by Theorem 6.14 (Theorem 2.5.11, [BO]), $\varphi_{i,n} \in P_1(\Gamma) \cap C_c(\Gamma)$, so $\varphi_i \in \overline{P_1(\Gamma) \cap C_c(\Gamma)}$ and finally $1 \in \overline{P_1(\Gamma) \cap C_c(\Gamma)}$ (here 1 is the constant function 1), where the closures are in the topology of pointwise convergence of functions. Hence there exists a net $(\psi_j)_{j \in J}$ in $P_1(\Gamma) \cap C_c(\Gamma)$ such that $\psi_j(s) \to 1$ for all $s \in \Gamma$, proving (5).

(5) $\Rightarrow$ (6): Since λ is a unitary representation of Γ, λ extends to a *-homomorphism $\tilde{\lambda}$:

$$\begin{array}{ccc} \Gamma & \hookrightarrow & C^*(\Gamma) \\ & \searrow^{\lambda} & \downarrow \tilde{\lambda} \\ & & C^*_\lambda(\Gamma) \end{array}$$

and the range of $\tilde{\lambda}$ is dense in $C^*_\lambda(\Gamma)$ because it contains $\mathbb{C}\Gamma$. Hence by standard C^*-algebra theory (see, e.g., Zhu's book, Theorem 11.1), $\tilde{\lambda}$ maps $C^*(\Gamma)$ <u>onto</u> $C^*_\lambda(\Gamma)$. To prove (5) $\Rightarrow$ (6) we will show that if there exists a net $(\varphi_i)_{i \in I}$ in $P_1(\Gamma) \cap C_c(\Gamma)$, converging pointwise to 1, then

$$\operatorname{Ker}(\tilde{\lambda}) = 0,$$

so that $\tilde{\lambda}$ becomes a *-isomorphism. Let $(\varphi_i)_{i \in I}$ be such a net and let

$$m_{\varphi_i} \colon C^*(\Gamma) \to C^*(\Gamma), \quad \overline{m}_{\varphi_i} \colon C^*_\lambda(\Gamma) \to C^*_\lambda(\Gamma)$$

be the corresponding u.c.p. multipliers on $C^*(\Gamma)$ and $C^*_\lambda(\Gamma)$ from Theorem 6.14 (4). Then the diagram

$$\begin{array}{ccc} C^*(\Gamma) & \xrightarrow{m_{\varphi_i}} & C^*(\Gamma) \\ \downarrow \tilde{\lambda} & & \downarrow \tilde{\lambda} \\ C^*_r(\Gamma) & \xrightarrow[\overline{m}_{\varphi_i}]{} & C^*_r(\Gamma) \end{array} \qquad (\star)$$

commutes. For this, it is enough to check that for $s \in \Gamma \subset C^*(\Gamma)$,

$$\overline{m}_{\varphi_i} \circ \tilde{\lambda}(s) = \overline{m}_{\varphi_i}(\tilde{\lambda}(s)) = \varphi_i(s)\lambda(s) = m_{\varphi_i}(\tilde{\lambda}(s)).$$

From the commutativity of $(\star)$ we have

$$m_{\varphi_i}(\operatorname{Ker}(\tilde{\lambda})) \subset \operatorname{Ker}(\tilde{\lambda}) \qquad (\star\star)$$

Set $E_i = \operatorname{supp}(\varphi_i)$. Then $|E_i| < \infty$ and since for $s \in \Gamma \subset C^*(\Gamma)$, $m_{\varphi_i}(s) = \varphi_i(s)s \in \operatorname{Span}\{s \,:\, s \in E_i\}$, we have

$$m_{\varphi_i}(C^*(\Gamma)) \subset \overline{\operatorname{Span}\{s \,:\, s \in E_i\}} = \operatorname{Span}\{s \,:\, s \in E_i\} \qquad (\star\star\star)$$

since finite-dimensional subspaces are automatically closed. Note also that since $\lim_i \|m_{\varphi_i}(s) - s\| = \lim_i |\varphi_i(s) - 1| = 0$, for all $s \in \Gamma$ and $\|m_{\varphi_i}\| \leq 1$ for all i, we have

$$\lim_i \|m_{\varphi_i}(a) - a\| = 0, \quad a \in C^*(\Gamma). \qquad (4\star)$$

Assume now that $a \in \operatorname{Ker}(\tilde{\lambda})$. By $(\star\star\star)$, $m_{\varphi_i}(a) = \sum_{s\in E_i} c_s^{(i)} s$ for suitable complex numbers $c_s^{(i)}$. Moreover, by $(\star\star)$, $\tilde{\lambda}(m_{\varphi_i}(a)) = 0$. Hence

$$0 = \tilde{\lambda}\left(\sum_{s\in E_i} c_s^{(i)} s\right) = \sum_{s\in E_i} c_s^{(i)} \lambda(s)$$

and thus

$$\sum_{s\in E_i} c_s^{(i)} \delta_s = \left(\sum_{s\in E_i} c_s^{(i)} \lambda(s)\right) \delta_e = 0$$

which clearly implies that $c_s^{(i)} = 0$ for all $s \in \Gamma$ (and all $i \in I$). Therefore $m_{\varphi_i}(a) = 0$ for all $i \in I$ and hence by $(4\star)$, $a = 0$, i.e., we have proved that $\operatorname{Ker}(\tilde{\lambda}) = 0$ and hence $\tilde{\lambda}\colon C^*(\Gamma) \to C^*_\lambda(\Gamma)$ is a *-isomorphism.

(6) $\Rightarrow$ **(7)**: The trivial representation τ_0 gives a character on $C^*(\Gamma)$, by universality of $C^*(\Gamma)$. If (6) holds, then $C^*_\lambda(\Gamma) = C^*(\Gamma)$ also has a character.

(7) $\Rightarrow$ **(1)**: Let $\tau\colon C^*_\lambda(\Gamma) \to \mathbb{C}$ be a *-homomorphism. Then τ is a state on $C^*_\lambda(\Gamma)$. Use Hahn-Banach to extend it to a state $\tilde{\tau}$ on $B(\ell^2(\Gamma))$. Note that $\tilde{\tau}$ may not be a *-homomorphism anymore, but $C^*_\lambda(\Gamma)$ is contained in the multiplicative domain of $\tilde{\tau}$ in the sense of Definition 1.14 (1.5.8 [BO]):

$$A_{\tilde{\tau}} = \{a \in B(\ell^2(\Gamma)) \,|\, \tilde{\tau}(a^*a) = \tilde{\tau}(a)^*\tilde{\tau}(a),\ \tilde{\tau}(aa^*) = \tilde{\tau}(a)\tilde{\tau}(a)^*\}.$$

Hence, by Proposition 1.13,

$$\tilde{\tau}(\lambda_s a \lambda_t) = \tilde{\tau}(\lambda_s)\tilde{\tau}(a)\tilde{\tau}(\lambda_t), \quad s,t \in \Gamma,\ a \in B(\ell^2(\Gamma))$$

Consider now $\ell^\infty(\Gamma) \subset B(\ell^2(\Gamma))$ acting as multiplication operators. Then for $s \in \Gamma$, $f \in \ell^\infty(\Gamma)$,

$$\tilde{\tau}(s.f) \overset{?}{=} \tilde{\tau}(\lambda_s f \lambda_s^{-1}) = \tilde{\tau}(\lambda_s)\tilde{\tau}(f)\tilde{\tau}(\lambda_s^{-1}) = \tilde{\tau}(f),$$

since $\tilde{\tau}(\lambda_s)^{-1} = \tilde{\tau}(\lambda_s^{-1})$. Thus we have shown that $\tilde{\tau}$ is an invariant mean on $\ell^\infty(\Gamma)$. Therefore we need to check the formula

$$s.f = \lambda_s f \lambda_s^{-1}, \quad s \in \Gamma, f \in \ell^\infty(\Gamma).$$

It is enough to check on $\{\delta_t \,:\, t \in \Gamma\}$. Since $(s.f)\delta_t = (s.f)(t)\delta_t = f(s^{-1}t)\delta_t$ and

$$\lambda_s f \lambda_s^{-1}\delta_t = \lambda_s f \lambda_{s^{-1}}\delta_t = \lambda_s f \delta_{s^{-1}t} = \lambda_s f(s^{-1}t)\delta_{s^{-1}t} = f(s^{-1}t)\delta_t,$$

the formula holds.

We have now proved that (1), (2), ..., (7) are equivalent. To add (8), (9) and (10) we prove (4) $\Leftrightarrow$ (8), (3) $\Rightarrow$ (9) $\Rightarrow$ (1) and (10) $\Leftrightarrow$ (1).

(4) $\Rightarrow$ **(8)**: This is "easy". Choose a net of unit vectors $\xi_i \in \ell^2(\Gamma)$ such that

$$\lim_i \|\lambda_s\xi_i - \xi_i\|_2 = 0\,, \quad s \in \Gamma.$$

Then for every finite set $E \subset \Gamma$,

$$\left\|\sum_{s\in E} \lambda_s\right\| \leq |E|$$

and $\langle \sum_{s\in E} \lambda_s\xi_i, \xi_i\rangle \to \sum_{s\in E} 1 = |E|$. Thus $\|\sum_{s\in E}\lambda_s\| = |E|$.

(8) ⇒ (4): Let $E \subset \Gamma$ be a finite set. Let $F = E \cup E^{-1} \cup \{e\}$, and let $S = \sum_{g \in F} \lambda_g$. Then S is self-adjoint and $\|S\| = |F|$. Let $\varepsilon > 0$ ($\varepsilon < 2$) be given. There exists a unit vector $\xi \in \ell^2(\Gamma)$ such that

$$|\langle S\xi, \xi\rangle| \geq |F| - \varepsilon\,.$$

As $\langle S\xi, \xi\rangle \in \mathbb{R}$, we either have $\langle S\xi, \xi\rangle \leq -|F| + \varepsilon$, or $\langle S\xi, \xi\rangle \geq |F| - \varepsilon$. But

$$\langle S\xi, \xi\rangle = \|\xi\|^2 + \sum_{g \in F\setminus\{e\}} \langle \lambda_g \xi, \xi\rangle \geq 1 - (|F| - 1) = 2 - |F| > -|F| + \varepsilon\,,$$

hence $\langle S\xi, \xi\rangle \geq |F| - \varepsilon$. Now, for each $g \in E$ we have

$$\begin{aligned} \langle S\xi, \xi\rangle &= \langle \lambda_g \xi, \xi\rangle + \sum_{h \in F\setminus\{g\}} \langle \lambda_h \xi, \xi\rangle \\ &= \mathrm{Re}\langle \lambda_g \xi, \xi\rangle + \sum_{h \in F\setminus\{g\}} \mathrm{Re}\langle \lambda_h \xi, \xi\rangle \\ &\leq \mathrm{Re}\langle \lambda_g \xi, \xi\rangle + (|F| - 1)\,. \end{aligned}$$

We deduce that

$$\mathrm{Re}\langle \lambda_g \xi, \xi\rangle \geq \langle S\xi, \xi\rangle - (|F| - 1) \geq 1 - \varepsilon\,.$$

Hence

$$\|\lambda_g \xi - \xi\|^2 = \|\lambda_g \xi\|^2 + \|\xi\|^2 - 2\mathrm{Re}\langle \lambda_g \xi, \xi\rangle = 2 - 2\mathrm{Re}\langle \lambda_g \xi, \xi\rangle \leq 2 - 2(1 - \varepsilon) = 2\varepsilon.$$

Equivalently, $\|\lambda_g \xi - \xi\| \leq \sqrt{2\varepsilon}$, for all $g \in E$. By standard arguments, we can now find a net $(\xi_i)_{i \in I}$ of unit vectors in $\ell^2(\Gamma)$ such that $\|\lambda_s \xi_i - \xi_i\| \to 0$ for all $s \in \Gamma$.

(3) ⇒ (9): Let $(F_i)_{i \in I}$ be a Følner net (Følner sequences only exist if Γ is countable). Let $(e_{p,q})_{p,q \in \Gamma}$ be the matrix units of $B(\ell^2(\Gamma))$, i.e.,

$$e_{p,q}\delta_t = \begin{cases} \delta_p & q = t \\ 0 & q \neq t. \end{cases}$$

Let p_i denote the projection of $\ell^2(\Gamma)$ onto $\mathrm{Span}\{\delta_g \,|\, g \in F_i\}$. Recall that $|F_i| < \infty$. Then

$$p_i B(\ell^2(\Gamma)) p_i \cong M_{|F_i|}(\mathbb{C})$$

(in the book $M_{|F_i|}(\mathbb{C})$ is denoted by $M_{F_i}(\mathbb{C})$) with matrix units $(e_{p,q})_{p,q \in F_i}$. Define $\varphi_i \colon C^*_\lambda(\Gamma) \to M_{F_i}(\mathbb{C})$ by $x \mapsto p_i x p_i$ and $\psi_i \colon M_{F_i}(\mathbb{C}) \to C^*_\lambda(\Gamma)$ by

$$e_{p,q} \mapsto \frac{1}{|F_i|} \lambda_p \lambda_q^{-1}, \quad p, q \in F_i.$$

By Example 3.2 (1.5.13 [BO]), ψ_i is completely positive. Clearly φ_i is unital, and ψ_i is also unital, since

$$\psi_i(1) = \sum_{p \in F_i} \psi_i(e_{p,p}) = \sum_{p \in F_i} \frac{1}{|F_i|} \lambda_p \lambda_p^{-1} = 1.$$

To see that $\|\psi_i \circ \varphi_i(a) - a\| \to 0$ for all $a \in C^*_\lambda(\Gamma)$, it is enough to check on elements of the form $a = \lambda_s$, $s \in \Gamma$. We have

$$\varphi_i(\lambda_s) = p_i \lambda_s p_i \overset{(\star)}{=} \sum_{\substack{p,q \in F_i \\ p = sq}} e_{p,q}$$

(where the formula ($\star$) can be checked by evaluating on δ_t, $t \in \Gamma$). Hence

$$\psi_i \circ \varphi_i(\lambda_s) = \frac{1}{|F_i|} \sum_{\substack{p,q \in F_i \\ p = sq}} \lambda_p \lambda_q^{-1} = \frac{1}{|F_i|} \sum_{\substack{p,q \in F_i \\ p = sq}} \lambda_s = \frac{|F_i \cap sF_i|}{|F_i|} \lambda_s.$$

But $|F_i \triangle sF_i| = |F_i| + |sF_i| - 2|F_i \cap sF_i|$, and hence

$$\frac{|F_i \cap sF_i|}{|F_i|} = 1 - \frac{1}{2} \frac{|F_i \triangle sF_i|}{|F_i|} \to 1,$$

which proves that $\|\psi_i \circ \varphi_i(\lambda_s) - \lambda_s\| \to 0$.

(1) ⇒ (10): ψ_i and φ_i above are also well-defined u.c.p. maps

$$\varphi_i \colon L(\Gamma) \to M_{|F_i|}(\mathbb{C}), \quad \psi_i \colon M_{|F_i|}(\mathbb{C}) \to L(\Gamma).$$

We have to check that $\psi_i \circ \varphi_i(x) \to x$ ultraweakly for all $x \in L(\Gamma)$. By Remark 4.4 (2.1.3 [BO]), it suffices to prove that for all $g, h \in \Gamma$,

$$\langle (\psi_i \circ \varphi_i)(x)\delta_g, \delta_h \rangle \to \langle x\delta_g, \delta_h \rangle.$$

Let $x \in L(\Gamma)$ and put $\alpha_s = \langle x\delta_e, \delta_s \rangle$, $s \in \Gamma$. Then $\alpha_s \in \mathbb{C}$ and for $g, h \in \Gamma$,

$$\langle x\delta_g, \delta_h \rangle = \langle x\rho(g^{-1})\delta_e, \delta_h \rangle = \langle \rho(g^{-1})x\delta_e, \delta_h \rangle = \langle x\delta_e, \rho(g)\delta_h \rangle = \langle x\delta_e, \delta_{hg^{-1}} \rangle = \alpha_{hg^{-1}}$$

where we have used that λ and ρ are commuting representations of Γ on $\ell^2(\Gamma)$, so $L(\Gamma) = \lambda(\Gamma)''$ commutes with $\rho(g)$ for all $g \in \Gamma$. Therefore

$$\varphi_i(x) = \sum_{p,q \in F_i} \langle x\delta_q, \delta_p \rangle e_{p,q} = \sum_{p,q \in F_i} \alpha_{pq^{-1}} e_{p,q},$$

and hence

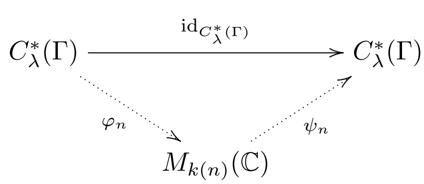
$$(\psi_i \circ \varphi_i)(x) = \frac{1}{|F_i|} \sum_{p,q \in F_i} \alpha_{pq^{-1}} \lambda_{pq^{-1}} = \frac{1}{|F_i|} \sum_s |F_i \cap sF_i| \alpha_s \lambda_s,$$

since each $s \in \Gamma$ can be written as pq^{-1} in exactly $|F_i \cap sF_i|$ ways with $p, q \in F_i$. Also note that the latter sum is finite (it has at most $|F_i|^2$ non-zero elements). We now have

$$\begin{aligned} \langle (\psi_i \circ \varphi_i)(x)\delta_g, \delta_h \rangle &= \sum_s \frac{|F_i \cap sF_i|}{|F_i|} \alpha_s \langle \lambda_s \delta_g, \delta_h \rangle \\ &= \frac{|F_i \cap (hg^{-1})F_i|}{|F_i|} \alpha_{hg^{-1}} \\ &= \frac{|F_i \cap (hg^{-1})F_i|}{|F_i|} \langle x\delta_g, \delta_h \rangle \to \langle x\delta_g, \delta_h \rangle \end{aligned}$$

since $\langle \lambda_s \delta_g, \delta_h \rangle = 1$ only if $s = hg^{-1}$ and is 0 for all other s.

(9) ⇒ (1): Assume that $C^*_\lambda(\Gamma)$ is nuclear. Let



be a u.c.p. approximate factorization. (The existence of such factorization is ensured by Proposition 2.2.6 [BO], since $A = C^*_\lambda(\Gamma)$ is unital.) Hence

$$\|\psi_n \circ \varphi_n(a) - a\| \to 0, \quad a \in C^*_\lambda(\Gamma). \tag{$\star$}$$

By Arveson's extension theorem, we can extend φ_n to a u.c.p. map $\widetilde{\varphi_n}$ on all of $B(\ell^2(\Gamma))$. Put

$$\Phi_n = \psi_n \circ \widetilde{\varphi_n} \colon B(\ell^2(\Gamma)) \to C^*_\lambda(\Gamma).$$

As explained in the proof of Arveson's theorem (see also Theorem 1.3.7 [BO]), the net Φ_n has a point-ultraweak limit

$$\Phi \colon B(\ell^2(\Gamma)) \to \overline{C^*_\lambda(\Gamma)}^{\text{u.w.}} = L(\Gamma)$$

which by ($\star$) satisfies

$$\Phi(a) = a\,, \quad a \in C^*_\lambda(\Gamma). \tag{7.1}$$

Let $\tau(T) = \langle T\delta_e, \delta_e\rangle$, $T \in L(\Gamma)$, be the canonical trace on $L(\Gamma)$, and set $\eta \colon = \tau \circ \Phi \colon B(\ell^2(\Gamma)) \to \mathbb{C}$. Then η is a state on $B(\ell^2(\Gamma))$. Moreover, for all $T \in B(\ell^2(\Gamma))$ and $s \in \Gamma$,

$$\eta(\lambda_s T \lambda_s^*) = \tau(\Phi(\lambda_s T \lambda_s^*)) = \tau(\lambda_s \Phi(T) \lambda_s^*),$$

which follows since $C^*_\lambda(\Gamma)$ is contained in the multiplicative domain of Φ, together with (7.1). By the trace property of τ we now have

$$\eta(\lambda_s T \lambda_s^*) = \tau(\lambda_s^*(\lambda_s \Phi(T))) = \tau(\Phi(T)) = \eta(T)$$

for all $T \in B(\ell^2(\Gamma))$ and all $s \in \Gamma$. Now let $f \in \ell^\infty(\Gamma)$ considered as a multiplication operator on $\ell^2(\Gamma)$. Then we have previously checked that

$$\lambda_s f \lambda_s^* = s.f, \quad f \in \ell^\infty(\Gamma),\ s \in \Gamma.$$

Hence $\eta(s.f) = \eta(\lambda_s f \lambda_s^*) = \eta(f)$, i.e., η restricted to $\ell^\infty(\Gamma)$ is a left invariant mean, which proves (1).

(10) ⇒ (1): The proof of (9) ⇒ (1) can be repeated almost word by word. Actually we get in this case that $\Phi(a) = a$ for all $a \in L(\Gamma)$ so that Φ is a conditional expectation of $B(\ell^2(\Gamma))$ onto $L(\Gamma)$. □

Lecture 9, GOADyn
October 7, 2021

Section 4.1: Crossed products

Definition 8.1 (Definition 4.1.1, [BO])**.** Let A be a C^*-algebra, Γ be a discrete group and $\alpha\colon \Gamma \to \mathrm{Aut}(A)$ be an action of Γ on A, i.e., α is a group homomorphism from Γ into the group of *-automorphisms of A. A C^*-algebra equipped with a Γ-action is called a Γ-*C^*-algebra*, and the triple (A, Γ, α) is called a *C^*-dynamical system.*

Let A be a Γ-C^*-algebra with the action of Γ on A denoted by α. Our goal is to construct a C^*-algebra $A \rtimes_\alpha \Gamma$ that encodes the Γ-action of Γ on A.

The model we have in mind is that we should have $A \rtimes_\alpha \Gamma := C^*(A, \{u_s\}_{s\in\Gamma})$ such that

$$u_s a u_s^* = \alpha_s(a),\ u_s u_t = u_{st}, \quad a \in A,\ s, t \in \Gamma. \tag{1}$$

In the case when A is unital, we want to think of u_s, $s \in \Gamma$, as unitaries implementing the action. Let

$$C_c(\Gamma, A) := \left\{ \sum_{s\in\Gamma} a_s s :\ a_s \in A,\ \text{sum is finite} \right\},$$

i.e., $C_c(\Gamma, A)$ is the space of finitely supported functions on Γ with values in A, so that if $S \in C_c(\Gamma, A)$, then by writing $a_t = S(t) \in A$ for all $t \in \Gamma$ we then write $S = \sum_{t\in\Gamma} a_t t$ (where the sum is finite). In the above model (if $1_A \in A$), then for all $s \in \Gamma$, we define

$$u_s := 1_A s \in C_c(\Gamma, A)$$

so that $u_s(t) = 1_A$ if $t = s$ and $u_s(t) = 0$ else.

We now want to make $C_c(\Gamma, A)$ into a *-algebra (and then complete it with respect to some appropriate norm to get $A \rtimes_\alpha \Gamma$). In the above model, if we want to implement relations (1), then we should have

$$(1_A s)(1_A t) = 1_A st, \quad s, t \in \Gamma,$$

and $sas^{-1} = \alpha_s(a)$ or $(1_A s)a = \alpha_s(a)s$ for all $s \in \Gamma$ and $a \in A$. Hence for $\sum_{s\in\Gamma} a_s s, \sum_{t\in\Gamma} b_t t \in C_c(\Gamma, A)$ we define

$$\left(\sum_{s\in\Gamma} a_s s\right)\left(\sum_{t\in\Gamma} b_t t\right) := \sum_{s,t\in\Gamma} (a_s s)(b_t t) = \sum_{s,t} a_s \alpha_s(b_t) st$$

(the latter equality following from noting that $sb_t = \alpha_s(b_t)s$) and

$$\left(\sum_{s\in\Gamma} a_s s\right)^* := \sum_s s^{-1} a_s^* = \sum_s \alpha_{s^{-1}}(a_s^*) \cdot s^{-1}.$$

(noting that $s^{-1}a_s^* = \alpha_{s^{-1}}(a_s^*)s^{-1}$ by the above).

Remark 8.2. Let $1 := 1_A e \in C_c(\Gamma, A)$, so that $1(t) = 1_A$ if $t = e$ and $1(t) = 0$ otherwise (where e is the unit of Γ). Then 1 is the unit in $C_c(\Gamma, A)$ with respect to the above multiplication, so $C_c(\Gamma, A)$ is now a unital *-algebra. The map $a \in A \mapsto ae \in C_c(\Gamma, A)$ is an injective *-homomorphism. Further, if $u_s := 1_A s \in C_c(\Gamma, A)$, $s \in \Gamma$, then we can check that

- $u_s^* u_s = 1 = u_s u_s^*$, $s \in \Gamma$,
- $u_s u_t = u_{st}$, $s, t \in \Gamma$,
- $u_s a u_s^* = \alpha_s(a)$, $s \in \Gamma$, $a \in A$.

Therefore $C_c(\Gamma, A)$ is the *-algebra generated by A and $\{u_s,\ s \in \Gamma\}$ and we have

$$C_c(\Gamma, A) \ni \sum_{s\in\Gamma} a_s s = \sum_{s\in\Gamma} a_s u_s.$$

Definition 8.3. We say that (u, π, H) is a covariant representation of (A, Γ, α) if $u\colon \Gamma \to \mathcal{U}(H)$ is a unitary representation, $\pi\colon A \to B(H)$ is a *-representation and they satisfy

$$u_s \pi(a) u_s^* = \pi(\alpha_s(a)), \quad s \in \Gamma,\ a \in A.$$

Remark 8.4. Let (u, π, H) be a covariant representation of (A, Γ, α). Define

$$(u \times \pi)\left(\sum_{s\in\Gamma} a_s s\right) := \sum_{s\in\Gamma} \pi(a_s) u(s), \quad \sum_{s\in\Gamma} a_s s \in C_c(\Gamma, A).$$

Then $u \times \pi$ is a *-representation of $C_c(\Gamma, A)$. Note that for all $x = \sum_{s\in\Gamma} a_s s \in C_c(\Gamma, A)$ we have

$$\|(u \times \pi)(x)\| = \left\|\sum_{s\in\Gamma} \pi(a_s) u_s\right\| \le \sum_{s\in\Gamma} \|\pi(a_s) u_s\| \le \sum_{s\in\Gamma} \|a_s\| := \|x\|_1. \tag{2}$$

Conversely, we can show that any non-degenerate *-representation φ of $C_c(\Gamma, A)$ arises in this way. (Recall that a *-representation $\varphi\colon B \to B(H)$ is called *non-degenerate* if

$$\overline{\operatorname{span}(\varphi(B)H)} = H. \tag{3}$$

If B is unital, with unit 1_B, then (3) holds if and only if $\varphi(1_B) = I_H$.) We justify this statement under the assumption that A is unital, with unit 1_A.

Thus, let $\varphi\colon C_c(\Gamma, A) \to B(H)$ be a non-degenerate *-representation, so that $\varphi(1_A) = I_H$. If we define $\pi(a) := \varphi(a)$ for $a \in A \subseteq C_c(\Gamma, A)$ then π is a unital *-representation of A, and if we let $u_s := \varphi(1_A s) \in \mathcal{U}(H)$ for $s \in \Gamma$, then $u\colon s \in \Gamma \mapsto u_s \in \mathcal{U}(H)$ is a unitary representation of Γ. Finally,

$$u_s \pi(a) u_s^* = \varphi(1_A s)\varphi(a)\varphi((1_A s)^*) = \pi(\alpha_s(a)),$$

so that (u, π, H) is a covariant representation of (A, Γ, α) and $\varphi = u \times \pi$.

Definition 8.5 (Definition 4.1.2, [BO])**.** The *full crossed product* (sometimes called the *universal crossed product*) of a Γ-C^*-algebra A with Γ-action α, denoted by $A \rtimes_\alpha \Gamma$, is the completion of $C_c(\Gamma, A)$ with respect to the norm

$$\|x\|_u = \sup \|\pi(x)\|, \quad x \in C_c(\Gamma, A). \tag{4}$$

where the supremum is taken over all (cyclic) *-representations $\pi\colon C_c(\Gamma, A) \to B(H)$.

Remark 8.6. We will show that *-representations of $C_c(\Gamma, A)$ *do exist*. For this (cf. Remark 9.4) it will suffice to construct a concrete example of a covariant representation of (A, Γ, α). Note that in computing $\|\cdot\|_u$ by formula (4), we can restrict ourselves to considering non-degenerate *-representations. Then, by Remark 9.4, we have

$$\|x\|_u \le \|x\|_1 < \infty, \quad x \in C_c(\Gamma, A).$$

Further, the fact that $\|\cdot\|_u$ defined by (4) is a seminorm on $C_c(\Gamma, A)$ follows immediately (as the supremum over a family of seminorms is a seminorm itself). We *will show* that $\|\cdot\|_u$ is actually a norm on $C_c(\Gamma, A)$.

Before proving the two assertions above, note the following:

Proposition 8.7 (Universal property, Proposition 4.1.3, [BO])**.** *For every covariant representation* (u, π, H) *of a* Γ*-*C^**-algebra* A*, there is a* **-homomorphism* $\sigma\colon A \rtimes_\alpha \Gamma \to B(H)$ *such that*

$$\sigma\left(\sum_{s\in\Gamma} a_s s\right) = \sum_{s\in\Gamma} \pi(a_s)u(s), \quad \sum_{s\in\Gamma} a_s s \in C_c(\Gamma, A).$$

We now construct a concrete example of a covariant representation of (A, Γ, α). Suppose that $A \subseteq B(H)$ is a faithful representation of A. Define a new representation $\pi\colon A \to B(H \otimes \ell^2(\Gamma))$ by

$$\pi(a)(v \otimes \delta_g) = (\alpha_{g^{-1}}(a)v) \otimes \delta_g, \quad a \in A,\ g \in \Gamma,\ v \in H, \tag{5}$$

where $\{\delta_g\}_{g\in\Gamma}$ is the canonical orthonormal basis in $\ell^2(\Gamma)$. (Under the identification $\bigoplus_{g\in\Gamma} H \cong H \otimes \ell^2(\Gamma)$, we have taken the direct sum representation $\pi(a) = \bigoplus_{g\in\Gamma} \alpha_{g^{-1}}(a) \in B(\bigoplus_{g\in\Gamma} H)$.)

Now let $\lambda\colon \Gamma \to B(\ell^2(\Gamma))$ be the left regular representation of Γ, i.e., $\lambda_s \delta_t = \delta_{st}$ for all $t \in \Gamma$. We claim that $(1 \otimes \lambda, \pi, H \otimes \ell^2(\Gamma))$ is a covariant representation of A on $H \otimes \ell^2(\Gamma)$. Indeed, this follows from the computations

$$\begin{aligned}(1 \otimes \lambda_s)\pi(a)(1 \otimes \lambda_s^*)(v \otimes \delta_g) &= (1 \otimes \lambda_s)\pi(a)(v \otimes \delta_{s^{-1}g}) \\ &= (1 \otimes \lambda_s)(\alpha_{g^{-1}s}(a)v \otimes \delta_{s^{-1}g}) \\ &= \alpha_{g^{-1}s}(a)v \otimes \delta_g \\ &= \pi(\alpha_s(a))(v \otimes \delta_g),\end{aligned}$$

which show that

$$(1 \otimes \lambda_s)\pi(a)(1 \otimes \lambda_s^*) = \pi(\alpha_s(a)), \quad s \in \Gamma,\ a \in A,$$

hence proving the claim. By Remark 9.4, let $(1 \otimes \lambda) \times \pi$ be the associated *-representation of $C_c(\Gamma, A)$. This is called a *left regular representation.*

Example 8.8. Let $\Gamma = \mathbb{Z}$ and let A be a C^*-algebra with a $\mathbb{Z}$-action $\alpha\colon \mathbb{Z} \to \mathrm{Aut}(A)$. Suppose that $A \subseteq B(H)$ is faithfully represented. Noting that $H \otimes \ell^2(\mathbb{Z}) = \bigoplus_{n\in\mathbb{Z}} H$, we construct $\pi\colon A \to B(H \otimes \ell^2(\mathbb{Z})$ by (5), i.e.,

$$\pi(a) = \begin{pmatrix} \ddots & & & & \\ & \alpha_1(a) & & & \\ & & a & & \\ & & & \alpha_{-1}(a) & \\ & & & & \ddots \end{pmatrix}$$

is an infinite diagonal matrix with $\pi(a)_{00} = a$. Then for all $n \in \mathbb{Z}$, we have $u_n = U^n \in \mathcal{U}(H \otimes \ell^2(\mathbb{Z}))$, where the unitary U is the shift

$$U = \begin{pmatrix} \ddots & \ddots & & & \\ & 0 & 1 & & \\ & & 0 & 1 & \\ & & & 0 & 1 \\ & & & & \ddots & \ddots \end{pmatrix} = 1 \otimes \lambda_1,$$

so $U(1 \otimes \delta_n) = 1 \otimes \delta_{n-1}$ for $n \in \mathbb{Z}$. One can check that $u_n\pi(a)u_n^* = \pi(\alpha_n(a))$ for all $n \in \mathbb{Z}$ and $a \in A$.

Let's go back to the general case and look at the regular representation

$$(1 \otimes \lambda) \times \pi \colon C_c(\Gamma, A) \to B(H \otimes \ell^2(\Gamma))$$

that we constructed.

Lemma 8.9. $(1 \otimes \lambda) \times \pi$ *is injective.*

Proof. Let $\sum_{t\in\Gamma} a_t t \in C_c(\Gamma, A)$. For any $v \in H$ and $g \in \Gamma$, we have

$$\begin{aligned}((1 \otimes \lambda) \times \pi)\left(\sum_{t\in\Gamma} a_t t\right)(v \otimes \delta_g) &= \sum_{t\in\Gamma} \pi(a_t)(1 \otimes \lambda_t)(v \otimes \delta_g) \\ &= \sum_{t\in\Gamma} \pi(a_t)(v \otimes \delta_{tg}) \\ &= \sum_{t\in\Gamma} \alpha_{g^{-1}t^{-1}}(a_t)v \otimes \delta_{tg}.\end{aligned}$$

Now, for every $g \in \Gamma$, let $P_g \in B(\ell^2(\Gamma))$ be the projection onto $\mathbb{C}\delta_g$. Then for all $h \in \Gamma$ we have

$$(1 \otimes P_g)((1 \otimes \lambda) \times \pi)\left(\sum_{t\in\Gamma} a_t t\right)(1 \otimes P_h) = \alpha_{g^{-1}}(a_{gh^{-1}}) \otimes P_g \lambda_{gh^{-1}} P_h. \tag{6}$$

Note now that if we set $e_{g,h} := P_g \lambda_{gh^{-1}} P_h$, $g, h \in \Gamma$, then $\{e_{g,h}\}_{g,h\in\Gamma}$ is a family of matrix units in $B(\ell^2(\Gamma))$, since $e_{g,h}e_{s,t} = \delta_{h,s}e_{g,t}$, $e_{g,h}^* = e_{h,g}$ and

$$\sum_{g\in\Gamma} e_{g,g} = I_{\ell^2(\Gamma)}.$$

In particular, if $((1 \otimes \lambda) \times \pi)(\sum_{t\in\Gamma} a_t t) = 0$, then (with $g = e$ and $h = s^{-1}$, we get

$$0 = (1 \otimes P_e)((1 \otimes \lambda) \times \pi)\left(\sum_{t\in\Gamma} a_t t\right)(1 \otimes P_{s^{-1}}) = a_s \otimes P_e \lambda_s P_{s^{-1}}.$$

Hence $a_s = 0$ for all $s \in \Gamma$, so $\sum_{s\in\Gamma} a_t t = 0$ and the claim is proved. □

Corollary 8.10. *The universal norm $\|\cdot\|_u$ defined by (4) is a norm on $C_c(\Gamma, A)$.*

Definition 8.11 (Definition 4.1.4, [BO])**.** The *reduced crossed product* of (A, Γ, α), denoted by $A \rtimes_{\alpha,r} \Gamma$, is the norm closure of the image of a regular representation $C_c(\Gamma, A) \to B(H)$.

We will abuse notation and denote an element $x \in C_c(\Gamma, A) \subseteq A \rtimes_{\alpha,r} \Gamma$ by $x = \sum_{s\in\Gamma} a_s \lambda_s$.

Proposition 8.12 (Proposition 4.1.5, [BO])**.** *The reduced crossed product $A \rtimes_{\alpha,r} \Gamma$ does not depend on the choice of faithful representation $A \subseteq B(H)$.*

We postpone for a moment the proof of Proposition 9.12 and look instead at the following:

Proposition 8.13 (Proposition 4.1.9, [BO])**.** *The map $E \colon C_c(\Gamma, A) \to A$ given by*

$$E\left(\sum_{s\in\Gamma} a_s \lambda_s\right) = a_e$$

extends to a faithful conditional expectation from $A \rtimes_{\alpha,r} \Gamma$ onto A. In particular,

$$\max_{s\in\Gamma} \|a_s\| \leq \left\| \sum_{s\in\Gamma} a_s\lambda_s \right\|_{A\rtimes_{\alpha,r}\Gamma}$$

for all $\sum_{s\in\Gamma} a_s\lambda_s \in C_c(\Gamma, A)$.

Proof. By taking $g = h = e$ in (6), we get

$$(1 \otimes P_e)((1 \otimes \lambda) \times \pi) \left(\sum_{t\in\Gamma} a_t t \right) (1 \otimes P_e) = a_e \otimes P_e, \quad \sum_{t\in\Gamma} a_t t \in C_c(\Gamma, A).$$

Hence

$$E(x) \otimes P_e = (1 \otimes P_e)((1 \otimes \lambda) \times \pi)(x)(1 \otimes P_e), \quad x \in C_c(\Gamma, A).$$

Therefore E is a contraction on $C_c(\Gamma, A)$ and hence it can be extended to a contraction $E\colon A \rtimes_{\alpha,r} \Gamma \to A$. It is clearly a projection onto A, so by Tomiyama's theorem, E is a conditional expectation of $A \rtimes_{\alpha,r} \Gamma$ onto A.

It remains to show that E is faithful. For this, we'll give a different proof than the one in the book. By taking $g = h$ in (6), we get

$$(1 \otimes P_g)((1 \otimes \lambda) \times \pi) \left(\sum_{t\in\Gamma} a_t t \right) (1 \otimes P_g) = \alpha_{g^{-1}}(a_e) \otimes P_g,$$

so we have

$$(1 \otimes P_g)((1 \otimes \lambda) \times \pi)(x)(1 \otimes P_g) = \alpha_{g^{-1}}(E(x)) \otimes P_g, \quad x \in C_c(\Gamma, A). \tag{7}$$

Now we need the following result.

Lemma 8.14. *Suppose that $T \in B(K)_+$ and $T \neq 0$ (where K is a Hilbert space) and that there exist projections $E_n \in B(K)$ such that $\sum_n E_n = I_K$. Then there exists n such that $E_n T E_n \neq 0$.*

Proof. Choose $x \in K$ such that $\langle T^{1/2}x, x\rangle \neq 0$ and set $x_n := E_n(x)$ for all n. Then $x = \sum_n x_n$, where the sum is norm-convergent, and so

$$0 \neq \langle T^{1/2}x, x\rangle = \sum_n \sum_m \langle T^{1/2}x_n, x_m\rangle.$$

Hence there are n, m such that $\langle T^{1/2}x_n, x_m\rangle \neq 0$, so we must have $T^{1/2}x_n \neq 0$. Thus

$$\langle E_n T E_n x, x\rangle = \langle T x_n, x_n\rangle = \|T^{1/2}x_n\|^2 \neq 0,$$

proving the claim. □

We are now ready to prove faithfulness of E. We show that if $x \in A \rtimes_{\alpha,r} \Gamma$, $x \geq 0$, $x \neq 0$, then $E(x) \neq 0$. Suppose by contradiction that $E(x) = 0$. Then by (7) we get

$$(1 \otimes P_g)((1 \otimes \lambda) \times \pi)(x)(1 \otimes P_g) = \alpha_{g^{-1}}(E(x)) \otimes P_g = 0, \quad g \in \Gamma.$$

By Lemma 9.14, we deduce that $((1 \otimes \lambda) \times \pi)(x) = 0$. But we have proved that $(1 \otimes \lambda) \times \pi$ is injective. Hence $x = 0$, a contradiction! Finally, note that for all $s \in \Gamma$ we have $a_s = E(z\lambda_s^*)$, where $z = \sum_{t\in\Gamma} a_t t$. This implies the desired inequality, so the proof is complete. □

Remark 8.15. The map E above extends also to a conditional expectation of $A \rtimes_\alpha \Gamma$ onto A, but in general this is *not* faithful (unless $A \rtimes_\alpha \Gamma = A \rtimes_{\alpha,r} \Gamma$ which happens, for example, if Γ is amenable – see Theorem 4.2.6 – or, more generally, if Γ acts amenably on A – see Theorem 4.3.4). These considerations follow from the existence of a contractive surjection $j\colon A \rtimes_\alpha \Gamma \to A \rtimes_{\alpha,r} \Gamma$ (since $C_c(\Gamma, A)$ is dense in both $A \rtimes_\alpha \Gamma$ and $A \rtimes_{\alpha,r} \Gamma$ and $\|\cdot\|_u \geq \|\cdot\|_{\alpha,r}$), so that $E \circ j\colon A \rtimes_\alpha \Gamma \to A$ is the desired map. If $\ker j \neq 0$, then $\ker(E \circ j) \neq 0$ (as $\ker j \subseteq \ker(E \circ j)$), but $\ker(E \circ j)$ is an ideal in $A \rtimes_\alpha \Gamma$ and every ideal contains positive elements. In this case, $E \circ j$ is *not* faithful.

Proof of Proposition 9.12. We start with some calculations. For a finite set $F \subseteq \Gamma$, let $P_F = P \in B(\ell^2(\Gamma))$ be the canonical projection onto $\operatorname{span}\{\delta_g : g \in F\}$. Let $(e_{p,q})_{p,q\in F}$ be the canonical matrix units in $PB(\ell^2(\Gamma))P \cong M_F(\mathbb{C})$ (note that the isomorphism is an isometry). Now let $A \subseteq B(H)$ be faithfully represented and let $\pi\colon A \to B(H \otimes \ell^2(\Gamma))$ be a regular representation. Then for all $a \in A$, we have

$$(1 \otimes P)\pi(a) = (1 \otimes P)\pi(a)(1 \otimes P) = \sum_{q\in F} \alpha_{q^{-1}}(a) \otimes e_{q,q},$$

using that $\pi(a)$ is a diagonal matrix so that it commutes with $1 \otimes P$. Hence for all $s \in \Gamma$,

$$\begin{aligned}(1 \otimes P)\pi(a)(1 \otimes \lambda_s)(1 \otimes P) &= [(1 \otimes P)\pi(a)]\,[(1 \otimes P)(1 \otimes \lambda_s)(1 \otimes P)]\\ &= \left[\sum_{q\in F} \alpha_{q^{-1}}(a) \otimes e_{q,q}\right](1 \otimes P\lambda_s P)\\ &= \left[\sum_{q\in F} \alpha_{q^{-1}}(a) \otimes e_{q,q}\right]\left[\sum_{p\in F\cap sF} 1 \otimes e_{p,s^{-1}p}\right]\\ &= \sum_{p\in F\cap sF} \alpha_{p^{-1}}(a) \otimes e_{p,s^{-1}p} \in A \otimes M_F(\mathbb{C}).\end{aligned}$$

Hence for all $x = \sum_{s\in\Gamma} a_s\lambda_s \in C_c(\Gamma, A) \subseteq B(H \otimes \ell^2(\Gamma))$, we have

$$(1 \otimes P)\pi(x)(1 \otimes P) = \sum_{s\in\Gamma}\sum_{p\in F\cap sF} \alpha_{p^{-1}}(a_s) \otimes e_{p,s^{-1}p} \in A \otimes M_F(\mathbb{C}). \tag{9}$$

But $A \otimes M_F(\mathbb{C}) \cong M_F(A) \subseteq M_F(B(H))$, where the inclusion is a *-homomorphism and hence isometric. This means that the norm of a matrix in $M_F(A)$ only depends on the norms of its entries (which are elements of A), and not on the specific embedding of A into $B(H)$. By (9), $\|(1 \otimes P)\pi(x)(1 \otimes P)\|$ only depends on the norm on A, and since

$$\|\pi(x)\| = \sup\{\|(1 \otimes P_F)\pi(x)(1 \otimes P_F)\| : F \subseteq \Gamma \text{ finite}\},$$

the proof is complete. ☐

Lecture 10, GOADyn
October 12, 2021

Section 5.1: Exact groups

Definition 10.1 (Definition 5.1.1, [BO])**.** A discrete group Γ is *exact* if $C^*_\lambda(\Gamma)$ is exact.

Theorem 10.2 (Guentner, Higson and Weinberger, 2005, Theorem 5.1.2, [BO])**.** *Let F be a field. Then any subgroup Γ of $\mathrm{GL}(n, F)$ is exact (as a discrete group).*

Proof. See reference [73], [BO]. □

Corollary 10.3. $\mathrm{GL}(n, \mathbb{Z})$*,* $\mathrm{SL}(n, \mathbb{Z})$*,* $n \geq 2$*, are all exact groups.*

Proof. They are subgroups of $\mathrm{GL}(n, \mathbb{Q})$. □

Remark 10.4. Using Proposition 2.5.9, [BO] and the fact that exactness passes to subalgebras (cf. Exercise 2.3.2, [BO]), we deduce that subgroups of exact groups are exact.

Definition 10.5.

(1) Let $E \subset \Gamma$ be a finite subset. The *tube* of width E is the set

$$\mathrm{Tube}(E) = \{(s, t) \in \Gamma \times \Gamma \,:\, st^{-1} \in E\}.$$

(2) The *uniform Roe algebra* $C^*_u(\Gamma)$ (named after John Roe) is the C^*-subalgebra of $B(\ell^2(\Gamma))$ generated by $C^*_\lambda(\Gamma)$ and $\ell^\infty(\Gamma)$.

Proposition 10.6 (Proposition 5.1.3, [BO])**.** *Let $\alpha\colon \Gamma \to \mathrm{Aut}(\ell^\infty(\Gamma))$ be the left translation action $\alpha_s(f) = s.f$, for all $s \in \Gamma$ and $f \in \ell^\infty(\Gamma)$. Then*

$$C^*_u(\Gamma) \cong \ell^\infty(\Gamma) \rtimes_{\alpha,r} \Gamma.$$

Proof. By Definition 4.1.4 and Proposition 4.1.5 [BO], we can realize the reduced crossed product $\ell^\infty(\Gamma) \rtimes_{\alpha,r} \Gamma$ as the C^*-subalgebra of $B(\ell^2(\Gamma) \otimes \ell^2(\Gamma)$ generated by $\{\pi(f),\ 1 \otimes \lambda_s \,:\, f \in \ell^\infty(\Gamma),\ s \in \Gamma\}$, where

$$\pi(f)(v \otimes \delta_t) = \alpha_{t^{-1}}(f)v \otimes \delta_t = (t^{-1}.f)v \otimes \delta_t\,, \quad f \in \ell^\infty(\Gamma), v \in \ell^2(\Gamma), t \in \Gamma.$$

(As usual, we consider $g \in \ell^\infty(\Gamma)$ as a multiplication operator on $\ell^2(\Gamma)$.)

Since $(x, y) \mapsto (x, yx)$ is a bijection of $\Gamma \times \Gamma$ onto itself (Check that!), we can define a unitary operator U on $\ell^2(\Gamma) \otimes \ell^2(\Gamma) = \ell^2(\Gamma \otimes \Gamma)$ by

$$U(\delta_x \otimes \delta_y) = \delta_x \otimes \delta_{yx}, \quad x, y \in \Gamma.$$

For $f \in \ell^\infty(\Gamma)$ and $s, t \in \Gamma$ we now have

$$\begin{aligned} U\pi(f)(\delta_s \otimes \delta_t) = U(\alpha_t^{-1}(f)\delta_s \otimes \delta_t) = U(f(ts)\delta_s \otimes \delta_t) &= f(ts)\delta_s \otimes \delta_{ts} \\ &= \delta_s \otimes f(ts)\delta_{ts} \\ &= \delta_s \otimes f\delta_{ts} \\ &= (1 \otimes f)(\delta_s \otimes \delta_{ts}) \\ &= (1 \otimes f)U(\delta_s \otimes \delta_t). \end{aligned}$$

Hence $U\pi(f) = (1 \otimes f)U$, which implies

$$U\pi(f)U^* = 1 \otimes f. \tag{1}$$

Moreover, for $s, t, u \in \Gamma$,

$$U(1 \otimes \lambda_s)(\delta_u \otimes \delta_t) = U(\delta_u \otimes \delta_{st}) = \delta_u \otimes \delta_{stu} = (1 \otimes \lambda_s)(\delta_u \otimes \delta_{tu}) = (1 \otimes \lambda_s)U(\delta_u \otimes \delta_t).$$

Hence U commutes with $1 \otimes \lambda_s$, and therefore,

$$U(1 \otimes \lambda_s)U^* = 1 \otimes \lambda_s. \tag{2}$$

By (1) and (2), the map $\varsigma\colon x \mapsto UxU^*$ is a *-isomorphism of $\ell^\infty(\Gamma) \rtimes_{\alpha,r} \Gamma$ onto $1 \otimes C_u^*(\Gamma)$ which, in turn, is *-isomorphic to $C_u^*(\Gamma)$. □

Corollary 10.7 (to the proof of Proposition 10.6)**.** *There is a (unique)* **-isomorphism* ρ *of* $\ell^\infty(\Gamma) \rtimes_{\alpha,r} \Gamma$ *onto* $C_u^*(\Gamma)$ *such that*

$$\rho(\pi(f)) = f, \quad f \in \ell^\infty(\Gamma),$$
$$\rho(1 \otimes \lambda_s) = \lambda_s, \quad s \in \Gamma.$$

Definition 10.8 (Definition 5.1.4, [BO])**.** A bounded function $k\colon \Gamma \times \Gamma \to \mathbb{C}$ is called a *positive definite kernel* if for every finite subset $F \subset \Gamma$

$$[k(s,t)]_{s,t \in F} \in M_F(\mathbb{C})_+. \tag{3}$$

Note that condition (3) implies that

(i) $k(s,s) \geq 0$ for all $s \in \Gamma$,
(ii) $k(t,s) = \overline{k(s,t)}$ for all $s, t \in \Gamma$ and
(iii) $|k(s,t)|^2 \leq k(s,s)k(t,t)$ for all $s, t \in \Gamma$,

where the last condition follows from the fact that applying (3) to $F = \{s,t\}$ in the case $s \neq t$, we get

$$\det \begin{pmatrix} k(s,s) & k(s,t) \\ k(t,s) & k(t,t) \end{pmatrix} \geq 0.$$

Hence, if $k\colon \Gamma \times \Gamma \to \mathbb{C}$ satisfies (3) and $\sup_{s \in \Gamma} k(s,s) < \infty$, then k is a bounded function on $\Gamma \times \Gamma$.

Note for example that if $T \in B(\ell^2(\Gamma))$, $T \geq 0$, then the kernel associated to T

$$k_T(s,t) = \langle T\delta_t, \delta_s \rangle, \quad s, t \in \Gamma$$

is a positive definite kernel.

Now set

$$\mathcal{A}_0(\Gamma) = \operatorname{span}\left(\bigcup_{s \in \Gamma} \ell^\infty(\Gamma)\lambda_s \right). \tag{4}$$

Since $\lambda_s f \lambda_s^{-1} = s.f$, for all $f \in \ell^\infty(\Gamma)$, $s \in \Gamma$, $\mathcal{A}_0(\Gamma)$ is a dense *-subalgebra of $C_u^*(\Gamma)$ and $\mathcal{A}_0(\Gamma)$ is the smallest *-algebra generated by $\ell^\infty(G) \cup \{\lambda_s : s \in \Gamma\}$.

Remark 10.9 (Remark 5.1.5, [BO])**.**

(a) If $T \in \mathcal{A}_0(\Gamma)_+$, then k_T is a positive definite kernel with support in some $\mathrm{Tube}(F)$, where $F \subset \Gamma$ is finite.

(b) Conversely, if k is a positive definite kernel with support in some $\mathrm{Tube}(F)$, where $F \subset \Gamma$ finite, then $k = k_T$ for a unique operator $T \in \mathcal{A}_0(\Gamma)_+$.

Proof. (a) Let $f \in \ell^\infty(\Gamma)$, $u \in \Gamma$. Then for $s, t \in \Gamma$,

$$k_{f\lambda_u}(s,t) = \langle f\lambda_u\delta_t, \delta_s\rangle = f(ut)\langle \delta_{ut}, \delta_s\rangle = \begin{cases} f(ut), & st^{-1} = u \\ 0, & st^{-1} \neq u. \end{cases}$$

By (4), every $T \in \mathcal{A}_0(\Gamma)$ is of the form $T = \sum_{u\in F} f_u\lambda_u$, where $F \subset \Gamma$ is finite and $f_u \in \ell^\infty(\Gamma)$, for all $u \in F$. Hence, by the above computation, the kernel

$$k_T = \sum_{u\in F} k_{f_u\lambda_u}$$

has support in $\{(s,t) \in \Gamma \times \Gamma : st^{-1} \in F\} = \mathrm{Tube}(F)$, and if $T \in \mathcal{A}_0(\Gamma)_+$, then k_T is also a positive definite kernel.

(b) Let k be a (bounded) positive definite kernel on $\Gamma \times \Gamma$ with $\mathrm{supp}(k) \subset \mathrm{Tube}(F)$, for a finite set $F \subset \Gamma$. For $u \in F$, set

$$f_u(x) = k(x, u^{-1}x), \quad u \in \Gamma,\ x \in \Gamma.$$

Then $f_u \in \ell^\infty(\Gamma)$, for all $u \in F$ and

$$\begin{aligned} k_{f_u\lambda_u}(s,t) &= \begin{cases} f_u(ut), & st^{-1} = u \\ 0, & st^{-1} \neq u. \end{cases} \\ &= \begin{cases} k(s,t), & st^{-1} = u \\ 0, & st^{-1} \neq u. \end{cases} \end{aligned}$$

Set $T = \sum_{u\in F} f_u\lambda_u \in \mathcal{A}_0(\Gamma)$. The above computation shows that k and k_T coincide on $\mathrm{Tube}(F)$, and since k and k_T vanish outside $\mathrm{Tube}(F)$, we have $k = k_T$. Moreover, since k is positive definite, T is positive, i.e., $T \in \mathcal{A}_0(\Gamma)_+$, which proves (b). □

Theorem 10.10 (Theorem 5.1.6, [BO])**.** *Let Γ be a discrete group. Then the following are equivalent:*

(1) *Γ is exact.*

(2) *For every finite set $E \subset \Gamma$ and $\varepsilon > 0$, there exists a positive definite kernel k with $\mathrm{supp}(k) \subset \mathrm{Tube}(F)$ for some finite set F, such that, moreover,*

$$|k(s,t) - 1| < \varepsilon, \quad (s,t) \in \mathrm{Tube}(E).$$

(3) *For every finite set $E \subset \Gamma$ and $\varepsilon > 0$, there exists a finite set $F \subset \Gamma$ and $\varsigma\colon \Gamma \to \ell^2(\Gamma)$ such that*
   - *$\|\varsigma_t\|_2 = 1$ for all $t \in \Gamma$,*
   - *$\mathrm{supp}(\varsigma_t) \subset Ft$ for all $t \in \Gamma$ and*
   - *$\|\varsigma_s - \varsigma_t\|_2 < \varepsilon$ for all $(s,t) \in \mathrm{Tube}(E)$.*

(4) *For every finite set $E \subset \Gamma$ and $\varepsilon > 0$, there exists a finite set $F \subset \Gamma$ and $\mu\colon \Gamma \to \mathrm{Prob}(\Gamma)$ such that*
   - *$\mathrm{supp}(\mu_t) \subset Ft$ for all $t \in \Gamma$ and*
   - *$\|\mu_s - \mu_t\|_1 < \varepsilon$ for all $(s,t) \in \mathrm{Tube}(E)$.*

(5) *$C^*_u(\Gamma)$ is nuclear.*

Before proving the theorem, let's look at the following interesting application of it:

Example 10.11 (= Proposition 5.1.8, [BO). , with a different proof] The free groups $(\mathbb{F}_n)_{2\le n\le\infty}$ are exact.

Proof. Since $\mathbb{F}_n$ $(3 \le n \le \infty)$ can be embedded in $\mathbb{F}_2$, by Remark 10.4 it is enough to show that $\mathbb{F}_2$ is exact. Let a, b be the generators of $\mathbb{F}_2$ and let $|x|$ be the length of a reduced word $x \in \mathbb{F}_2$. Define

$$d_r(s,t) = |st^{-1}|, \quad s,t \in \mathbb{F}_2.$$

Then d_r is a right invariant metric on $\mathbb{F}_2$:

$$d_r(su,tu) = d_r(s,t), \quad s,t,u \in \mathbb{F}_2.$$

The (right) Cayley graph G of $\mathbb{F}_2$ is the graph obtained by letting $\mathbb{F}_2$ be the set of vertices and connecting $s, t \in \mathbb{F}_2$ with an edge if and only if $d_r(s,t) = 1$.

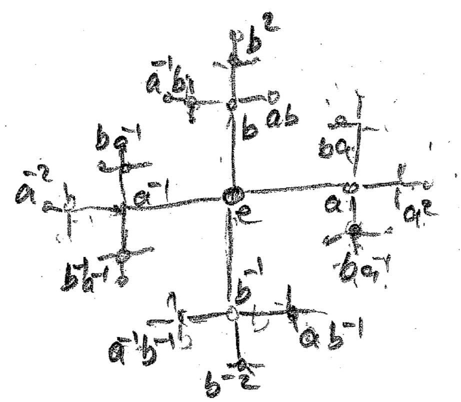


Figure 1. (Right) Cayley graph G of $\mathbb{F}_2$

The Cayley graph of $\mathbb{F}_2$ is a homogeneous tree of degree 4. Consider the infinite path $P(e) = \{e, a, a^2, a^3, \ldots\}$. For every $x \in G$, we can construct an infinite path $P(x)$ that eventually merges into $P(e)$ by setting

$$P(x) = \{x, \gamma(x), \gamma^2(x), \ldots\},$$

where $\gamma(e) = a$ and for $x \neq e$ with reduced word $x = s_1 \cdots s_n$, $s_j \in \{a, a^{-1}, b, b^{-1}\}$ we define

$$\gamma(x) = \begin{cases} ax & \text{if } s_1 = s_2 = \cdots = s_n = a \\ s_2 \cdots s_n & \text{otherwise.} \end{cases}$$

For instance,

$$P(aba) = \{aba, ba, a, a^2, a^3, \ldots\}.$$

Note that $x, \gamma(x), \gamma^2(x), \ldots$ is a list of distinct elements from $\mathbb{F}_2$ and $d_r(\gamma^k(x), \gamma^{k+1}(x)) = 1$, for all $k \ge 0$. Also, $P(x) \cap P(e) \supset \{a^j, a^{j+1}, \ldots\}$, for some $j \ge 0$, and hence

$$P(x) \cap P(y) \neq \emptyset, \quad x, y \in \mathbb{F}_2.$$

Fix now $x, y \in \mathbb{F}_2$ and let $k \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}$ be the smallest number such that $\gamma^k(x) \in P(y)$. Then $\gamma^k(x) = \gamma^\ell(y)$, for some $\ell \in \mathbb{N}_0$, and hence

$$\gamma^{k+i}(x) = \gamma^{\ell+i}(y), \quad i \ge 0,$$

while

$$\gamma^p(x) \neq \gamma^q(y) \quad \text{when} \begin{cases} 0 \leq p \leq k-1 \\ 0 \leq q \leq \ell - 1 \end{cases}$$

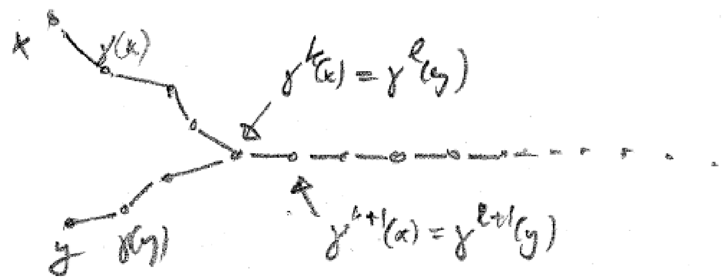


FIGURE 2. $P(x)$ and $P(y)$

Since the Cayley graph is a tree, there is a unique shortest path from x to y, and this path has length $d_r(x, y)$. Hence from Figure 2, we have

$$d_r(x, y) = k + \ell.$$

We will prove that $\mathbb{F}_2$ satisfies condition (4) in Theorem 10.10. Fix $n \in \mathbb{N}$. For $z \in \mathbb{F}_2$, define

$$\mu_z^{(n)} = \frac{1}{n}(\delta_z + \delta_{\gamma(z)} + \cdots + \delta_{\gamma^{n-1}(z)}) \in \text{Prob}(\mathbb{F}_2).$$

Since $d_r(z, \gamma^j(z)) = j$, we have $\text{supp}(\mu_z^{(n)}) \subset F_n z$ where $F_n = \{s \in \mathbb{F}_2 : |s| \leq n\}$. Let $E \subset \mathbb{F}_2$ be a finite set and let $\varepsilon > 0$. Set $m = \max\{|s| : s \in E\}$. Then

$$\text{Tube}(E) \subset \{(s,t) \in \mathbb{F}_2 \times \mathbb{F}_2 : d_r(s,t) \leq m\}.$$

Let now $(x, y) \in \text{Tube}(E)$ and define k, ℓ (depending on the pair (x, y)) as above. Then for all $n \geq m$,

$$k + \ell = d_r(x, y) \leq m \leq n.$$

Hence, using Figure 2, it is not hard to see that

$$\|\mu_x^{(n)} - \mu_y^{(n)}\|_1 = \frac{1}{n}(k + \ell + |k - \ell|) \leq \frac{2m}{n}.$$

Thus for $n > (2m)/\varepsilon$,

$$\|\mu_x^{(n)} - \mu_y^{(n)}\|_1 < \varepsilon, \quad (x, y) \in \text{Tube}(E),$$

which shows that $\mathbb{F}_2$ satisfies condition (4) in Theorem 10.10. Therefore $\mathbb{F}_2$ is exact. □

The proof of (1) ⇒ (2) in Theorem 10.10 uses the following:

Exercise 10.12 (Exercise 3.9.5, [BO], slightly reformulated)**.** Let $A \subset B(H)$ be an exact unital C^*-algebra and let $(P_i)_{i \in I}$ be an increasing net of projections in $B(H)$, such that $P_i \to 1$ strongly. Let $E \subset A$ be a finite set and let $\varepsilon > 0$. Then there exists $P \in \{P_i : i \in I\}$ and a u.c.p. map $\theta\colon PB(H)P \to B(H)$ such that

$$\|\theta(PaP) - a\| < \varepsilon, \quad a \in E.$$

Solution of exercise. By the definition of exact C^*-algebras and Exercise 2.1.6, [BO], we know that the inclusion map $i\colon A \hookrightarrow B(H)$ is nuclear. Hence with $E \subset A$ finite and $\varepsilon > 0$, there exists $k \in \mathbb{N}$, $\varphi\colon A \to M_k(\mathbb{C})$ and $\psi\colon M_k(\mathbb{C}) \to B(H)$ such that φ and ψ are u.c.p. maps and

$$\|(\psi \circ \varphi)(a) - a\| < \frac{\varepsilon}{3}, \quad a \in E.$$

(We have used Proposition 4.11 (Proposition 2.2.6., [BO]) therein.)

Use now Arveson's extension theorem (Theorem 3.8 (Theorem 1.6.1, [BO])) to extend φ to a u.c.p. map $\widetilde{\varphi}\colon B(H) \to M_k(\mathbb{C})$. By Corollary 1.6.3, [BO], there exists a net $(\varphi_\lambda)_{\lambda\in\Lambda}$ of ultraweakly continuous u.c.p. maps from $B(H)$ to $M_k(\mathbb{C})$ that converges point-norm to $\widetilde{\varphi}$. So there exists $\lambda_0 \in \Lambda$ such that for $\lambda \geq \lambda_0$,

$$\|(\psi \circ \varphi_\lambda)(a) - (\psi \circ \widetilde{\varphi})(a)\| < \frac{\varepsilon}{3}, \quad a \in E.$$

Since $\widetilde{\varphi}(a) = \varphi(a)$, for $a \in E$, we deduce for all $\lambda \geq \lambda_0$ that

$$\|(\psi \circ \varphi_\lambda)(a) - a\| < \frac{2\varepsilon}{3}, \quad a \in E.$$

Set $\varphi' = \varphi_{\lambda_0}\colon B(H) \to M_k(\mathbb{C})$. Then

$$\|(\psi \circ \varphi')(a) - a\| < \frac{2\varepsilon}{3}, \quad a \in E.$$

Since φ' is ultraweakly continuous and $\lim_i P_i = I_{B(H)}$ SOT, we conclude that $\varphi'(P_i a P_i) \to \varphi'(a)$, for all $a \in B(H)$. But the ultraweak topology coincides with the norm topology on $M_k(\mathbb{C})$ (Why?), hence

$$\lim_i \|\varphi'(P_i a P_i) - \varphi'(a)\| = 0, \quad a \in B(H).$$

In particular, there exists $i \in I$ such that with $P = P_i$, we have

$$\|\varphi'(PaP) - \varphi'(a)\| < \frac{\varepsilon}{3}, \quad a \in E.$$

So altogether we deduce (using $\|\psi\| \leq 1$) that for all $a \in E$,

$$\begin{aligned}\|(\psi \circ \varphi')(PaP) - a\| &\leq \|(\psi \circ \varphi')(PaP) - (\psi \circ \varphi')(a)\| + \|(\psi \circ \varphi')(a) - a\| \\ &< \frac{\varepsilon}{3} + \frac{2\varepsilon}{3} = \varepsilon.\end{aligned}$$

Set now $\theta = (\psi \circ \varphi')|_{PB(H)P}$. Then the desired conclusion holds. □

Proof of Theorem 10.10. **(1)⇒(2):** Assume that Γ is exact, i.e., that $C^*_\lambda(\Gamma)$ is exact. Let $E \subset \Gamma$ be finite and $\varepsilon > 0$ be given. It follows from Exercise 10.12 that there exists a finite set $F_0 \subset \Gamma$ such that with P being the orthogonal projection of $\ell^2(\Gamma)$ onto $\ell^2(F_0)$ there is a u.c.p. map $\psi\colon PB(\ell^2(\Gamma))P \to B(\ell^2(\Gamma)$ such that

$$\|\psi(P\lambda(s)P) - \lambda(s)\| < \varepsilon, \quad s \in E.$$

Set now $\theta = \psi \circ \varphi$, where $\varphi(x) = PxP$, $x \in C^*_\lambda(\Gamma)$. Then

$$\|\theta(\lambda(s)) - \lambda(s)\| < \varepsilon, \quad s \in E.$$

Define a kernel $k\colon \Gamma \times \Gamma \to \mathbb{C}$ by

$$k(s,t) = \langle \theta(\lambda(st^{-1}))\delta_t, \delta_s\rangle, \quad s,t \in \Gamma.$$

For all $s_1, \dots, s_n \in \Gamma$, $\alpha_1, \dots, \alpha_n \in \mathbb{C}$, we have

$$\begin{aligned}\sum_{i,j} k(s_i, s_j)\overline{\alpha_i}\alpha_j &= \sum_{i,j} \langle \theta(\lambda(s_i s_j^{-1}))\alpha_j \delta_{s_j}, \alpha_i \delta_{s_i}\rangle \\ &= \left\langle \underbrace{\theta^{(n)}\left([\lambda(s_i s_j^{-1})]_{i,j}\right)}_{\geq 0 \text{ in } M_n(C^*_\lambda(\Gamma))} \begin{bmatrix}\alpha_1\delta_{s_1}\\ \vdots \\ \alpha_n \delta_{s_n}\end{bmatrix}, \begin{bmatrix}\alpha_1\delta_{s_1}\\ \vdots \\ \alpha_n \delta_{s_n}\end{bmatrix} \right\rangle \geq 0.\end{aligned}$$

This shows that k is positive definite.

Note further that $\varphi(\lambda(s)) = P\lambda(s)P$ is zero precisely when $0 = \langle \lambda(s)\delta_t, \delta_u\rangle = \langle \delta_{st}, \delta_u\rangle$, i.e., when $st \neq u$ or $s \neq ut^{-1}$ for $t, u \in F_0$. Hence $\varphi(\lambda(s)) = 0$ if and only if $s \notin F_0 \cdot F_0^{-1}$ (which is a finite set). Then $\theta(\lambda(s)) = 0$ when $s \notin F_0 \cdot F_0^{-1}$ and hence $k(s,t) = 0$ when $(s,t) \notin \mathrm{Tube}(F_0 \cdot F_0^{-1})$. Moreover if $st^{-1} \in \mathrm{Tube}(E)$, then

$$\|\theta(\lambda(st^{-1})) - \lambda(st^{-1})\| < \varepsilon.$$

Hence

$$|k(s,t) - \underbrace{\langle \lambda(st^{-1})\delta_t, \delta_s\rangle}_{\langle \delta_s, \delta_s\rangle = 1}| < \varepsilon,$$

i.e., $|k(s,t) - 1| < \varepsilon$ for $(s,t) \in \mathrm{Tube}(E)$. Hence (1) $\Rightarrow$ (2) holds (with $F = F_0 \cdot F_0^{-1}$).

(2) $\Rightarrow$ (3): Let $E \subset \Gamma$ be finite and $\varepsilon > 0$ be given. Let $0 < \varepsilon^* < \frac{1}{2}$ (to be specified later). By (2) there exists $k\colon \Gamma \times \Gamma \to \mathbb{C}$ positive definite kernel and a finite set $F^* \subset \Gamma$ such that

$$|k(s,t) - 1| < \varepsilon^*, \quad (s,t) \in \mathrm{Tube}(E^*) \tag{i}$$

where $E^* = E \cup \{e\}$ and

$$k(s,t) = 0, \quad (s,t) \notin \mathrm{Tube}(F^*). \tag{ii}$$

By Remark 10.9, $k(s,t) = \langle a\delta_t, \delta_s\rangle$, $s, t \in \Gamma$, for an element $a \in \mathcal{A}_0(\Gamma)_+ \subset C^*_u(\Gamma)_+$. Since $\mathcal{A}_0(\Gamma)$ is dense in $C^*_u(\Gamma)$, we can find $b_n \in \mathcal{A}_0(\Gamma)$ such that $b_n \to a^{1/2}$ in norm and thus $b_n^* b_n \to a$ in norm. Hence there exists $b \in \mathcal{A}_0(\Gamma)$ such that

$$\|b^* b - a\| < \varepsilon^*. \tag{iii}$$

By (i) we have $|k(t,t) - 1| < \varepsilon^*$ for all $t \in \Gamma$. Hence by (iii),

$$\begin{aligned}\big| \|b\delta_t\|^2 - 1\big| &\leq \big|\|b\delta_t\|^2 - k(t,t)\big| + |k(t,t) - 1| \\ &= \langle (b^*b - a)\delta_t, \delta_t\rangle + |k(t,t) - 1| < 2\varepsilon^*.\end{aligned}$$

Thus

$$1 - 2\varepsilon^* < \|b\delta_t\|^2 < 1 + 2\varepsilon^*. \tag{iv}$$

Since $0 < \varepsilon^* < \frac{1}{2}$ we have $b\delta_t \neq 0$ for all $t \in \Gamma$. For $t \in \Gamma$ we put

$$\begin{cases} \widehat{\varsigma}_t = b\delta_t \\ \varsigma_t = \frac{1}{\|b\delta_t\|} b\delta_t. \end{cases} \tag{v}$$

Note that $\|\varsigma_t\| = 1$ for $t \in \Gamma$. Moreover for $(s,t) \in \mathrm{Tube}(E)$

$$|\langle \widehat{\varsigma}_t, \widehat{\varsigma}_s\rangle - 1| = |\langle b^* b\delta_t, \delta_s\rangle - 1| < |\langle a\delta_t, \delta_t\rangle - 1| + \varepsilon^* = |k(t,t) - 1| + \varepsilon^* < 2\varepsilon^*.$$

Hence $\mathrm{Re}\langle \widehat{\varsigma}_t, \widehat{\varsigma}_s\rangle > 1 - 2\varepsilon^*$, so by (iv),

$$\mathrm{Re}\langle \varsigma_t, \varsigma_s\rangle > \frac{1-2\varepsilon^*}{\|b\delta_t\|\|b\delta_s\|} > \frac{1-2\varepsilon^*}{1+2\varepsilon^*}.$$

Therefore for all $(s,t) \in \mathrm{Tube}(E)$,

$$\|\varsigma_s - \varsigma_t\|^2 = \|\varsigma_s\|^2 + \|\varsigma_t\|^2 - 2\mathrm{Re}\langle \varsigma_t, \varsigma_s\rangle < 2 - 2\cdot\frac{1-2\varepsilon^*}{1+2\varepsilon^*} = \frac{4\varepsilon^*}{1+2\varepsilon^*} < 4\varepsilon^*.$$

Thus, setting $\varepsilon^* = \min\{\frac{\varepsilon^2}{4}, \frac{1}{3}\}$, we have $0 < \varepsilon^* < \frac{1}{2}$ as required and

$$\|\varsigma_s - \varsigma_t\|^2 < 4\varepsilon^* < \varepsilon^2, \quad (s,t) \in \mathrm{Tube}(E).$$

Since $b \in \mathcal{A}_0(\Gamma)$ there exists a finite set $F \subset \Gamma$ and elements $(b_s)_{s\in F}$ in $\ell^\infty(\Gamma)$, such that $b = \sum_{s\in F} b_s\lambda_s$. Then for $t, u \in \Gamma$,

$$\langle \widehat{\varsigma}_t, \delta_u\rangle = \langle b\delta_t, \delta_u\rangle = \sum_{s\in F} b_s(st)\langle \delta_{st}\delta_u\rangle = 0$$

if $u \notin Ft$. Hence

$$\mathrm{supp}(\varsigma_t) = \mathrm{supp}(\widehat{\varsigma}_t) \subset Ft$$

which shows (3).

(3) ⇒ (4): Let E, ε, F and ς be as in (3) and set

$$\mu_t(p) = |\varsigma_t(p)|^2, \quad t, p \in \Gamma.$$

Then $\mu_t \in \mathrm{Prob}(\Gamma)$, for all $t \in \Gamma$, and

$$\mathrm{supp}(\mu_t) = \mathrm{supp}(\varsigma_t) \subset Ft.$$

Moreover

$$|\mu_s - \mu_t| = (|\varsigma_s| - |\varsigma_t|)(|\varsigma_s| + |\varsigma_t|) \leq |\varsigma_s - \varsigma_t||\varsigma_s| + |\varsigma_s - \varsigma_t||\varsigma_t|.$$

Hence, by Hölder's inequality,

$$\|\mu_s - \mu_t\|_1 < \|\varsigma_s - \varsigma_t\|_2\|\varsigma_s\|_2 + \|\varsigma_s - \varsigma_t\|_2\|\varsigma_t\|_2 = 2\|\varsigma_s - \varsigma_t\|_2.$$

Thus for $(s,t) \in \mathrm{Tube}(E)$, $\|\mu_s - \mu_t\| < 2\varepsilon$. Since $\varepsilon > 0$ was arbitrary, we have proved (4).

(4) ⇒ (5): The proof of this implication in Brown-Ozawa relies on Theorem 4.4.3, [BO]. Below is a self-contained proof based on Section 4.1 only. (The construction of the map ψ' below is similar to the proof of Lemma 4.3.3, [BO].) We will prove (4) ⇒ (3) ⇒ (5):

(4) ⇒ (3): Let E, ε, F and μ be as in (3). Put

$$\varsigma_t(p) = \sqrt{\mu_t(p)}, \quad t, p \in \Gamma.$$

Then $\|\varsigma_t\|_2 = 1$ for all $t \in \Gamma$ and

$$\mathrm{supp}(\varsigma_t) = \mathrm{supp}(\mu_t) \subset Ft.$$

Moreover, for $s, t \in \Gamma$,

$$|\varsigma_t - \varsigma_s|^2 = |\mu_t^{1/2} - \mu_s^{1/2}|^2 \leq |\mu_t^{1/2} - \mu_s^{1/2}||\mu_t^{1/2} + \mu_s^{1/2}| = |\mu_t - \mu_s|.$$

Hence $\|\varsigma_t - \varsigma_s\|_2^2 \leq \|\mu_t - \mu_s\|$. Therefore $\|\varsigma_t - \varsigma_s\|_2 \leq \varepsilon^{1/2}$ for all $(s,t) \in \mathrm{Tube}(E)$. Since $\varepsilon > 0$ was arbitrary, we have proved (3).

(3) ⇒ (5): Since $\ell^\infty(\Gamma)$ is an abelian C^*-algebra, it is nuclear (by Proposition 2.4.2, [BO]), and hence $M_n(\ell^\infty(\Gamma))$ is also nuclear, for all $n \in \mathbb{N}$ (cf. Corollary 2.4.4, [BO]). Thus if we can show that the identity operator $C_u^*(\Gamma)$ has an approximate factorization through $M_n(\ell^\infty(\Gamma)$,

$$\begin{array}{ccc} C_u^*(\Gamma) & \xrightarrow{\mathrm{id}_{C_u^*(\Gamma)}} & C_u^*(\Gamma) \\ & {\scriptstyle\varphi_i} \searrow \quad \nearrow {\scriptstyle\psi_i} & \\ & M_{n_i}(\ell^\infty(\Gamma)) & \end{array} \tag{⊠}$$

with u.c.p. maps φ_i and ψ_i such that

$$\|(\psi_i \circ \varphi_i)(a) - a\| \to 0, \quad a \in C_u^*(\Gamma),$$

then $C_u^*(\Gamma)$ is nuclear by Exercise 2.3.11, [BO].

Lemma 10.13. *Let F be a finite subset of Γ. Then there is a unique u.c.p. map $\varphi\colon C_u^*(\Gamma) \to M_F(\ell^\infty(\Gamma)) = \ell^\infty(\Gamma) \otimes M_F(\mathbb{C})$ such that for all $a \in \ell^\infty(\Gamma)$ and $s \in \Gamma$,*

$$\varphi(a\lambda_s) = \sum_{p \in F \cap sF} \alpha_p^{-1}(a) \otimes e_{p,s^{-1}p}$$

where $(e_{pq})_{p,q \in F}$ are the matrix units of $M_F(\mathbb{C})$.

Proof. Let $\rho\colon \ell^\infty(\Gamma) \rtimes_{\alpha,r} \Gamma \to C_u^*(\Gamma)$ be the *-isomorphism from the prof of Proposition 10.6. Then for $a \in \ell^\infty(\Gamma)$ and $s \in \Gamma$,

$$\rho^{-1}(a\lambda_s) = \pi(a)(1 \otimes \lambda_s).$$

Let $P \in B(\ell^2(\Gamma))$ be the projection of $\ell^2(\Gamma)$ onto $\ell^2(F)$. Since

$$\pi(a) = \sum_{q \in \Gamma} \alpha_q^{-1}(a) \otimes e_{qq},$$

we have (as in the proof of Proposition 4.1.5, [BO]) that

$$\begin{aligned}(1 \otimes P)\pi(a)(1 \otimes \lambda_s)(1 \otimes P) &= \left(\sum_{p \in F} \alpha_p^{-1}(a) \otimes e_{pp}\right)(1 \otimes P\lambda_s P) \\ &= \left(\sum_{p \in F} \alpha_p^{-1}(a) \otimes e_{pp}\right)\left(\sum_{p \in F \cap sF} 1 \otimes e_{p,s^{-1}p}\right) \\ &= \sum_{p \in F} \alpha_p^{-1}(a) \otimes e_{p,s^{-1}p}.\end{aligned}$$

Hence putting

$$\varphi(z) = (1 \otimes P)\rho^{-1}(z)(1 \otimes P), \quad z \in C_u^*(\Gamma),$$

we get a u.c.p. map satisfying

$$\varphi(a\lambda_s) = \sum_{p \in F \cap sF} \alpha_p^{-1}(a) \otimes e_{p,s^{-1}p}$$

for $a \in \ell^\infty(\Gamma)$ and $s \in \Gamma$ and the range of φ is contained in $M_F(\ell^\infty(\Gamma))$. □

Lemma 10.14. *Let $F \subset \Gamma$ be a finite set and let $(T(p))_{p\in F}$ be elements in $\ell^\infty(\Gamma)$ such that*

$$\sum_{p\in F} T(p)T(p)^* = 1.$$

Then the map $\psi\colon M_F(\ell^\infty(\Gamma)) \to C^*_u(\Gamma)$ defined by

$$\psi(a \otimes e_{pq}) = T(p)\lambda_p a \lambda_q^* T(q)^*$$

is a u.c.p. map satisfying

$$\|(\psi\circ\varphi)(a\lambda_s) - a\lambda_s\| \le \|a\| \left\| \sum_{p\in F} T(p)\alpha_s(T(s^{-1}p)^*) - 1 \right\|, \quad a \in \ell^\infty(\Gamma), s \in \Gamma.$$

Proof. We have

$$\psi(1) = \psi\left(\sum_{p\in F} 1 \otimes e_{pp}\right) = \sum_{p\in F} T(p)T(p)^* = 1$$

by the assumptions and ψ is completely positive because for $[a_{pq}]_{p,q\in F}$ in $M_F(\ell^\infty(\Gamma))$ and $F = \{p_1,\dots,p_k\}$ we have

$$\begin{aligned}\psi([a_{pq}]) &= \psi\left(\sum_{p,q\in F} a_{pq}\otimes e_{pq}\right)\\ &= \begin{bmatrix} T(p_1)\lambda_{p_1} & \cdots & T(p_n)\lambda_{p_n}\end{bmatrix} [a_{p_i p_j}]_{i,j} \begin{bmatrix} (T(p_1)\lambda_{p_1})^* \\ \cdots \\ (T(p_n)\lambda_{p_n})^*\end{bmatrix}.\end{aligned}$$

We next compute

$$\begin{aligned}(\psi\circ\varphi)(a\lambda_s) &= \psi\left(\sum_{p\in F\cap sF} \alpha_p^{-1}(a)\otimes e_{p,s^{-1}p}\right)\\ &= \sum_{p\in F\cap sF} T(p)\lambda_p\alpha_p^{-1}(a)\lambda_p^{-1}\lambda_s T(s^{-1}p)^*\\ &= \sum_{p\in F\cap sF} T(p)a\lambda_s T(s^{-1}p)^*\\ &= \sum_{p\in F\cap sF} T(p)a\alpha_s(T(s^{-1}p)^*)\lambda_s\\ &= \sum_{p\in F\cap sF} T(p)\alpha_s(T(s^{-1}p)^*)a\lambda_s,\end{aligned}$$

where in the last step we have used that $\ell^\infty(\Gamma)$ is abelian. Hence

$$\|(\psi\circ\varphi)(a\lambda_s) - a\lambda_s\| \le \left\| \sum_{p\in F\cap sF} T(p)\alpha_s T(s^{-1}p)^* \right\| \|a\lambda_s\|$$

which proves Lemma 10.14. ∎

End of proof of (3) ⇒ (5): Since $C_u^*(\Gamma)$ is the norm closure of

$$\mathcal{A}_0(\Gamma) = \operatorname{span}\left(\bigcup_{s\in\Gamma} \ell^\infty(\Gamma)\lambda_s\right)$$

it is clear that we can obtain an approximate factorization of $\mathrm{id}_{C_u^*(\Gamma)}$ of the form (⊠) on page 9, provided we can prove the following claim:

Claim. Assuming *(3)*, then for every finite set $E \subset \Gamma$ and every $\varepsilon > 0$ there exists a finite set $F \subset \Gamma$ and $(T(p))_{p\in F}$ in $\ell^\infty(\Gamma)$ such that

(a) $\sum_{p\in F} T(p)T(p)^* = 1$ and

(b) $\left\|\sum_{p\in F\cap sF} T(p)\alpha_s(T(s^{-1}p)^*) - 1\right\| \le \varepsilon$ for all $s \in E$.

Proof of claim. Let $E \subset \Gamma$ be a finite set and let $\varepsilon > 0$. By (3), there exists $\varsigma\colon \Gamma \to \ell^2(\Gamma)$ and a finite set $F \subset \Gamma$ such that

- $\|\varsigma_t\|_2 = 1$ for all $t \in \Gamma$,
- $\operatorname{supp}(\varsigma_t) \subset Ft$ for all $t \in \Gamma$ and
- $\|\varsigma_s - \varsigma_t\|_2 < \frac{\varepsilon}{2}$ for all $(s,t) \in \operatorname{Tube}(E)$.

Define $T(p) \in \ell^\infty(\Gamma)$ for all $p \in \Gamma$ by

$$T(p)(x) = \varsigma_x(p^{-1}x), \quad x \in \Gamma.$$

Since $\operatorname{supp}(\varsigma_x) \subset Fx$, we can check that $T(p) = 0$ for all $p \in \Gamma \setminus F$. Therefore, for all $x \in \Gamma$,

$$\left(\sum_{p\in F} T(p)T(p)^*\right)(x) = \sum_{p\in\Gamma} |T(p)|^2(x) = \sum_{p\in\Gamma} |\varsigma_x(p^{-1}x)|^2 = \|\varsigma_x\|_2^2 = 1.$$

Hence (a) in the claim holds. For all $x \in \Gamma$ and $s \in E$,

$$\begin{aligned}
\left(\sum_{p\in F\cap sF} T(p)\alpha_s(T(s^{-1}p)^*)\right)(x) &= \sum_{p\in\Gamma} T(px)\overline{\alpha_s(T(s^{-1}p))(x)} \\
&= \sum_{p\in\Gamma} T(p)(x)\overline{T(s^{-1}p)(s^{-1}x)} \\
&= \sum_{p\in\Gamma} \varsigma_x(p^{-1}x)\overline{\varsigma_{s^{-1}x}\left((s^{-1}p)^{-1}s^{-1}x\right)} \\
&= \sum_{p\in\Gamma} \varsigma_x(p^{-1}x)\overline{\varsigma_{s^{-1}x}(p^{-1}x)} \\
&= \langle \varsigma_x, \varsigma_{s^{-1}x}\rangle.
\end{aligned}$$

Hence for $x \in \Gamma$ and $s \in E$,

$$\begin{aligned}
\left|\left(\sum_{p\in F\cap sF} T(p)\alpha_s(T(s^{-1}p)^*) - 1\right)(x)\right| &= |\langle \varsigma_x, \varsigma_{s^{-1}x}\rangle - 1| \\
&= |\langle \varsigma_x, \varsigma_{s^{-1}x} - \varsigma_x\rangle| \\
&= \|\varsigma_x\|_2\|\varsigma_x - \varsigma_{s^{-1}x}\|_2 \\
&< \frac{\varepsilon}{2}
\end{aligned}$$

because $(x, s^{-1}x) \in \mathrm{Tube}(E)$ for all $s \in E$. Hence (b) in the claim also holds. □

Altogether, we have shown that $\mathrm{id}_{C_u^*(\Gamma)}$ has an approximate (point-norm) u.c.p. factorization through the nuclear C^*-algebras $M_n(\ell^\infty(\Gamma))$ $(n \in \mathbb{N})$ and hence $C_u^*(\Gamma)$ is nuclear, completing the proof of (3) ⇒ (5).

(5) ⇒ (1): Clearly $C_\lambda^*(\Gamma) \subset C_u^*(\Gamma)$. Hence

$$C_u^*(\Gamma) \text{ nuclear } \Rightarrow C_u^*(\Gamma) \text{ exact } \Rightarrow C_\lambda^*(\Gamma) \text{ exact},$$

since C^*-subalgebras of exact C^*-algebras are again exact. □

Lecture 11, GOADyn
October 14, 2021

Section 12.1: Kazhdan's property (T)

Introduction to (relative) property (T)

Let Γ be a discrete group and let $\pi\colon \Gamma \to \mathcal{U}(H)$ be a unitary representation.

Definition 11.1 (Definition 12.1.1, [BO])**.**

- A vector $\xi \in H$ is called Γ-*invariant* if $\pi(s)\xi = \xi$ for all $s \in \Gamma$.
- A net $(\xi_i)_{i\in I}$ of unit vectors in H is called *almost* Γ-*invariant* if $\|\pi(s)\xi_i - \xi_i\| \to 0$ for all $s \in \Gamma$.
- If $E \subset \Gamma$ is a set and $k > 0$, we say that a nonzero vector $\xi \in H$ is (E, k)-*invariant* if

$$\sup_{s\in E} \|\pi(s)\xi - \xi\| < k\|\xi\|.$$

Definition 11.2 (Definition 12.1.2, [BO])**.** Let $\Lambda \subset \Gamma$ be a subgroup.

- We say that the inclusion $\Lambda \subset \Gamma$ has *relative property (T)* if any unitary representation (π, H) of Γ which has almost Γ-invariant vectors, has a nonzero Λ-invariant vector.
- We say that Γ has *property (T)* if the identity inclusion $\Gamma \subset \Gamma$ has relative property (T).
- A pair (E, k) where $E \subset \Gamma$ and $k > 0$ is called a *Kazhdan pair* for the inclusion $\Lambda \subset \Gamma$ (or, for Γ, if $\Lambda = \Gamma$) if any unitary representation (π, H) of Γ which has a nonzero (E, k)-invariant vector, has a nonzero Λ-invariant vector.

The following two propositions are reformulations of Proposition 6.4.5, [BO].

Proposition 11.3. *Let Γ be a discrete group. Then the following are equivalent:*

(1) *Γ has property (T).*
(2) *There exists a Kazhdan pair (F, k) for Γ with $F \subset \Gamma$ finite and $k > 0$.*

Proof. **(1) $\Rightarrow$ (2):** We show $\neg(2) \Rightarrow \neg(1)$. Suppose that (2) does not hold. Then for all finite sets $F \subset \Gamma$ and $\varepsilon > 0$ there exists a unitary representation (π, H) without a nonzero Γ-invariant vector, but such that there exists a unit vector $\xi \in H$ with $\|\pi(t)\xi - \xi\| < \varepsilon$ for all $t \in F$. Let

$$I = \{(F, \varepsilon) \,:\, F \subset \Gamma \text{ is a finite set, } \varepsilon > 0\}.$$

If $(F_1, \varepsilon_1), (F_2, \varepsilon_2) \in I$, we say that $(F_1, \varepsilon_1) \preceq (F_2, \varepsilon_2)$ if $F_1 \subset F_2$ and $\varepsilon_2 \le \varepsilon_1$. Then $(I, \preceq)$ is a directed set. By $\neg(2)$, then for all $i = (F_i, \varepsilon_i) \in I$, there exists a unitary representation (π_i, H_i) without a nonzero Γ-invariant vector such that there exists a unit vector $\xi_i \in H_i$ satisfying

$$\|\pi_i(t)\xi_i - \xi_i\| < \varepsilon_i, \quad t \in F_i.$$

Set $\pi := \bigoplus_{i\in I} \pi_i \colon \Gamma \to B(H)$, where $H = \bigoplus_{i\in I} H_i$. Now, viewing $H_i \subset H$, we see that $\xi_i \in H$ for all $i \in I$ and that for all $t \in \Gamma$, $\|\pi(t)\xi_i - \xi_i\| \to 0$, i.e., $(\xi_i)_{i\in I}$ is a net of almost Γ-invariant unit vectors. However, we can show that π does <u>not</u> have nonzero Γ-invariant vectors, so Γ is not property (T), so $\neg(1)$ holds. Indeed, assume by contradiction that there exists a Γ-invariant $\xi \in H$, $\xi \neq 0$. Let $P_i\colon H \to H_i$, $i \in I$, be the orthogonal projection. Note that $\pi(t)P_i = P_i\pi(t)$ for all $t \in \Gamma$ and $i \in I$. Then

$$\pi_i(t)P_i\xi = P_i\pi(t)\xi = P_i\xi, \quad i \in I,\ t \in \Gamma,$$

since ξ is Γ-invariant. This shows that for all $i \in I$, $P_i\xi$ is Γ-invariant for π_i. But π_i does not have nonzero Γ-invariant vectors, and hence $P_i\xi = 0$ for all $i \in I$. Since $\xi = \sum_{i\in I} P_i\xi$, we deduce that $\xi = 0$, a contradiction. This proves $\neg(1)$.

(2) ⇒ (1): Suppose that (2) holds and that $(\xi_i)_{i\in I}$ is a net of almost Γ-invariant vectors for a given unitary representation $\pi\colon \Gamma \to B(H)$, i.e., $\|\pi(t)\xi_i - \xi_i\| \to 0$ for all $t \in \Gamma$. Then for all $t \in \Gamma$ there exists $i_t \in I$ such that $\|\pi(t)\xi_i - \xi_i\| < k$ for all $i \succeq i_t$. Since I is a directed set and F is finite, there exists $i_0 = i_0(F) \in I$ such that $i_0 \succeq i_t$ for all $t \in F$. Let $\xi_0 = \xi_{i_0}$. Then $\|\pi(t)\xi_0 - \xi_0\| < k$ for all $t \in F$, i.e., ξ_0 is (F, k)-invariant. Since (F, k) is a Kazhdan pair for Γ, it follows that π has a nonzero Γ-invariant vector, and hence (1) holds. □

Proposition 11.4. *Suppose that (E, k) is a Kazhdan pair for Γ, and that $\pi\colon \Gamma \to \mathcal{U}(H)$ is a unitary representation of Γ with the property that there exists $\xi \in H$, $\xi \neq 0$ such that*

$$\pi(t)\xi = \xi, \quad t \in E.$$

Then $\pi(t)\xi = \xi$ for all $t \in \Gamma$, i.e., ξ is Γ-invariant for π.

Proof. Set $H_0 = \{\text{all } \Gamma\text{-invariant vectors in } H\}$ and let $K = H_0^\perp$. Note that both H_0 and K are invariant under π, i.e., $\pi(t)H_0 \subset H_0$ and $\pi(t)K \subset K$ for all $t \in \Gamma$. Let $\pi_1 := \pi|_K\colon \Gamma \to B(K)$. Then π_1 has no nonzero Γ-invariant vectors. Hence for all $\eta \in K$ there exists $t \in E$ such that

$$\|\pi_1(t)\eta - \eta\| \geq k\|\eta\| \tag{$\star$}$$

Now write (uniquely) $\xi = \xi_0 + \eta$ for some $\xi_0 \in H_0$, $\eta \in K$. Then for all $t \in E$,

$$\xi = \pi(t)\xi = \pi(t)\xi_0 + \pi(t)\eta = \xi_0 + \pi(t)\eta,$$

since $\xi_0 \in H_0$. Hence $\eta = \pi(t)\eta$ for all $t \in E$. By $(\star)$, it follows that $\eta = 0$. Hence $\xi = \xi_0 \in H_0$, i.e., ξ is Γ-invariant for π. □

Next we prove the following two facts: A group with Kazhdan's property (T) is finitely generated (see Corollary 6.4.7, [BO]) – therefore it is countable – and it has finite abelianization.

Lemma 11.5. *Let Γ be a discrete group. If there exists $k > 0$ and a subset $E \subset \Gamma$ such that (E, k) is a Kazhdan pair for Γ, then E is a generating set for Γ.*

Combining this with Proposition 11.3, we deduce that if Γ has property (T), then Γ is finitely generated, and hence it is countable!

Proof. Let (E, k) be a Kazhdan pair for Γ. We must show that E is a generating set for Γ, i.e., if Γ_0 is the subgroup generated by E in Γ, then $\Gamma_0 = \Gamma$. Consider $\Gamma/\Gamma_0 = \{t\Gamma_0 : t \in \Gamma\}$ and let $\pi\colon \Gamma \to B(\ell^2(\Gamma/\Gamma_0))$ be defined by

$$\pi(t)\delta_{s\Gamma_0} = \delta_{ts\Gamma_0}, \quad t, s \in \Gamma.$$

Let $\xi = \delta_{\Gamma_0} \in \ell^2(\Gamma/\Gamma_0)$. Note that for all $t \in E$, $\pi(t)\xi = \pi(t)\delta_{\Gamma_0} = \delta_{t\Gamma_0} = \delta_{\Gamma_0} = \xi$, since $t \in E \subset \Gamma_0$. By Proposition 11.4, it follows that $\pi(t)\xi = \xi$ for all $t \in \Gamma_0$. This implies that $t\Gamma_0 = \Gamma_0$ for all $t \in \Gamma$, i.e., $\Gamma_0 = \Gamma$, as wanted. □

Lemma 11.6. *Let Γ be a discrete group, and let $\Lambda \triangleleft \Gamma$ (i.e., Λ is a normal subgroup). If Γ has property (T), then so does the quotient Γ/Λ.*

Connections with amenability

Remark 11.7. Finite groups have property (T).

This will be a consequence of Lemma 12.10 (cf. Lemma 12.1.5, [BO]) below (asserting that for any group Γ, the pair $(\Gamma, \sqrt{2})$ is Kazhdan) combined with Proposition 11.3.

Remark 11.8. If Γ is amenable with property (T), then Γ is finite.

This follows from the following:

Remark 11.9.

(1) If Γ is amenable, then the left regular representation λ has almost Γ-invariant vectors.
(2) If Γ is infinite, then the left regular representation λ has <u>no</u> nonzero Γ-invariant vectors.

Proof of Remark 11.9. (1) Has already been discussed in the proof of Theorem 7.10 (Theorem 2.6.8, [BO]). For completeness, we redo the construction. Suppose that Γ is amenable. Let $(F_i)_{i\in I}$ be a Følner net for Γ. Then for all $i \in I$, let

$$\xi_i := \frac{1}{\sqrt{|F_i|}} \sum_{t\in F_i} \delta_t \in \ell^2(\Gamma).$$

We have $\|\xi_i\| = 1$. For every $s \in \Gamma$, we have

$$\lambda(s)\xi_i - \xi_i = \frac{1}{\sqrt{|F_i|}} \left(\sum_{t\in sF_i\cap F_i} \delta_t - \sum_{t\in F_i\setminus sF_i} \delta_t \right)$$

so

$$\|\lambda(s)\xi_i - \xi_i\|^2 = \frac{1}{|F_i|} |sF_i \triangle F_i| \to 0.$$

Hence $(\xi_i)_{i\in I}$ is a net of almost Γ-invariant unit vectors for the left regular representation λ.

(2) Suppose by contradiction that there exists $\xi \in l^2(\Gamma)$, $\xi \neq 0$ such that ξ is Γ-invariant for λ. Note that for all $t \in \Gamma$, $\langle \xi, \delta_t\rangle = \langle \xi, \lambda(t)\delta_e\rangle = \langle \lambda(t^{-1})\xi, \delta_e\rangle = \langle \xi, \delta_e\rangle$, since ξ is Γ-invariant. Since

$$\|\xi\|^2 = \sum_{t\in\Gamma} |\langle \xi, \delta_t\rangle| = \sum_{t\in\Gamma} |\langle \xi, \delta_e\rangle| < \infty,$$

this contradicts the fact that Γ is infinite. □

We now show that if Γ has property (T), then Γ has finite abelianization. Let $[\Gamma\colon \Gamma]$ be the subgroup of Γ generated by $\{sts^{-1}t^{-1} : s, t \in \Gamma\}$ (the commutator subgroup of Γ). Then $[\Gamma\colon \Gamma] \triangleleft \Gamma$ and $\Gamma/[\Gamma\colon \Gamma]$ is abelian. Moreover, if $N \triangleleft G$, then Γ/N is abelian if and only if N contains $[\Gamma\colon \Gamma]$ (i.e., $[\Gamma\colon \Gamma]$ is the smallest normal subgroup of Γ with the property that the quotient is abelian). The group $\Gamma/[\Gamma\colon \Gamma]$ is called the abelianization of Γ. Now, $\Gamma/[\Gamma\colon \Gamma]$ is abelian, hence amenable. Moreover, if Γ has property (T), then by Lemma 11.6, $\Gamma/[\Gamma\colon \Gamma]$ also has property (T). By Remark 11.8, $\Gamma/[\Gamma\colon \Gamma]$ is finite.

The following are examples of discrete groups without property (T):

(1) $\mathbb{Z}^n$ is amenable, but not finite, so it does not have property (T).

(2) $\mathbb{F}_n$, $n \geq 2$ is nonamenable, but not (T) since $\mathbb{Z}^n$ is a quotient of $\mathbb{F}_n$.

Our next goal is to prove:

Lemma 11.10 (Lemma 12.1.5, [BO]). *For any group Γ, the pair $(\Gamma, \sqrt{2})$ is Kazhdan.*

The proof uses the following (see Appendix D):

Exercise 11.11 (Exercise D.1, [BO]). Let V be a bounded subset of a Hilbert space H and let

$$r_0 := \inf\{r > 0 \,:\, V \subset \overline{B}(\xi, r) \text{ for some } \xi \in H\}.$$

a) Prove that there exists a unique $\zeta \in H$, called the *circumcenter* of V, such that $V \subset \overline{B}(\zeta, r_0)$.
b) Prove that $\zeta \in \overline{\mathrm{conv}}(V)$.

Proof. a) For all $n \geq 1$, there exists $x_n \in H$ such that $V \subset \overline{B}(x_n, r_0 + \frac{1}{n})$.

Claim: $(x_n)_{n\geq 1}$ is a Cauchy sequence in H.

To prove the claim, let $1 \leq n \leq m$ and $\delta > 0$. Then $V \not\subset \overline{B}(\frac{x_n+x_m}{2}, r_0 - \delta)$, so there exists $y \in V$ so that

$$\left\| \frac{x_n + x_m}{2} - y \right\| > r_0 - \delta.$$

On the other hand, $\|x_n - y\| \leq r_0 + 1/n$, $\|x_m - y\| \leq r_0 + 1/m$. By the parallelogram identity $\|z + w\|^2 + \|z - w\|^2 = 2\|z\|^2 + 2\|w\|^2$ applied for $z = (x_n - y)/2$, $w = (x_m - y)/2$, we get

$$\begin{aligned}
\left\| \frac{x_n + x_m}{2} - y \right\|^2 + \left\| \frac{x_n - x_m}{2} \right\|^2 &= \left\| \frac{x_n - y}{2} + \frac{x_m - y}{2} \right\|^2 + \left\| \frac{x_n - y}{2} - \frac{x_m - y}{2} \right\|^2 \\
&= 2\left\| \frac{x_n - y}{2} \right\|^2 + 2\left\| \frac{x_m - y}{2} \right\|^2.
\end{aligned}$$

Hence

$$\begin{aligned}
\left\| \frac{x_n - x_m}{2} \right\|^2 &= \left\| \frac{x_n - y}{2} + \frac{x_m - y}{2} \right\|^2 + \left\| \frac{x_n - y}{2} - \frac{x_m - y}{2} \right\|^2 - \left\| \frac{x_n + x_m}{2} - y \right\|^2 \\
&\leq \frac{1}{2}\left(r_0 + \frac{1}{n}\right)^2 + \frac{1}{2}\left(r_0 + \frac{1}{m}\right)^2 - (r_0 - \delta)^2 \\
&= r_0\left(\frac{1}{n} + \frac{1}{m} + 2\delta\right) - \delta^2 \\
&< r_0\left(\frac{1}{n} + \frac{1}{m} + 2\delta\right).
\end{aligned}$$

Given $\varepsilon > 0$, let $\delta = \varepsilon^2/(12r_0)$ and $n_\varepsilon > 2/\delta$. Then for all $n, m \geq n_\varepsilon$, we get $\|x_n - x_m\|^2 < \varepsilon^2$. The claim is proved.

Hence there exists $\zeta \in H$ such that $x_n \to \zeta$ as $n \to \infty$ and it will also follow that $V \subset \overline{B}(\zeta, r_0)$. Now, suppose that there exists $\zeta' \in H$ such that $V \subset \overline{B}(\zeta', r_0)$. Let

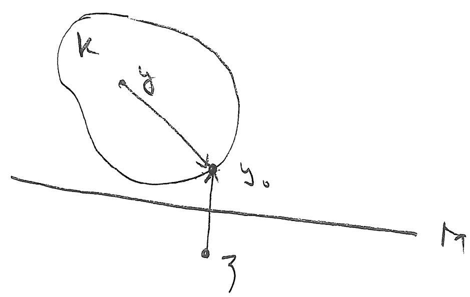
$$y_n = \begin{cases} \zeta & \text{if } n \text{ is odd} \\ \zeta' & \text{if } n \text{ is even} \end{cases}$$

Then $V \subset \overline{B}(y_n, r_0 + \frac{1}{n})$ for all $n \geq 1$. By the above proof, $(y_n)_{n\geq 1}$ is Cauchy. This implies that $\zeta = \zeta'$ and uniqueness is proved.

b) Let $K = \overline{\mathrm{conv}}(V)$. Suppose by contradiction that $\zeta \notin K$. Since K is closed and convex, there exists a unique $y_0 \in K$ such that $\|\zeta - y_0\| = \mathrm{dist}(\zeta, K)$, and moreover,

$$\mathrm{Re}\langle y_0 - \zeta, y_0 - y\rangle \leq 0, \quad y \in K.$$

Let $M = \{\zeta - y_0\}^{\perp}$. Let $y \in K$ and write (uniquely)



$$y - y_0 = \lambda(y_0 - \zeta) + z$$

for some $\lambda \in \mathbb{C}$, $z \in M$. In particular, $y - \zeta = (\lambda + 1)(y_0 - \zeta) + z$. Since

$$0 \geq \mathrm{Re}\langle y_0 - \zeta, y_0 - y\rangle = \mathrm{Re}(-\lambda\|y_0 - \zeta\|^2),$$

we deduce that $\mathrm{Re}\lambda \geq 0$. Set

$$R := \sup\{\|y - y_0\| \mid y \in K\} < \infty$$

(since K is bounded). We have $\|y - y_0\|^2 = |\lambda|^2\|y_0 - \zeta\|^2 + \|z\|^2$ and

$$\begin{aligned} \|y - \zeta\|^2 &= |\lambda + 1|^2\|y_0 - \zeta\|^2 + \|z\|^2 \\ &= (|\lambda|^2 + 1 + 2\mathrm{Re}\lambda)\|y_0 - \zeta\|^2 + \|z\|^2 \\ &\geq (|\lambda|^2 + 1)\|y_0 - \zeta\|^2 + \|z\|^2 \\ &= |\lambda|^2\|y_0 - \zeta\|^2 + \|z\|^2 + \|y_0 - \zeta\|^2 \\ &= \|y - y_0\|^2 + \|y_0 - \zeta\|^2, \end{aligned}$$

since $\mathrm{Re}\lambda \geq 0$. So $\|y - \zeta\|^2 \geq \|y - y_0\|^2 + \|y_0 - \zeta\|^2$. Hence

$$\|y - \zeta\| \geq \|y - y_0\|\left(1 + \frac{\|y_0 - \zeta\|^2}{\|y - y_0\|^2}\right)^{1/2} \geq \|y - y_0\|\left(1 + \frac{\|y_0 - \zeta\|^2}{R^2}\right)^{1/2}.$$

Set

$$\delta := \left(1 + \frac{\|y_0 - \zeta\|^2}{R^2}\right)^{-1/2} < 1.$$

Then for all $y \in K$, $\|y - y_0\| \leq \delta\|y - \zeta\|$. Hence, since $V \subset \overline{B}(\zeta, r_0)$, we deduce that $V \subset \overline{B}(y_0, \delta r_0)$, which is impossible by definition of r_0, since $\delta r_0 < r_0$. □

Proof of Lemma 11.10. Let $\pi\colon \Gamma \to \mathcal{U}(H)$ be a unitary representation and $\xi \in H$ be a nonzero $(\Gamma, \sqrt{2})$-invariant vector. We may assume that $\|\xi\| = 1$. Note that $\pi(\Gamma)\xi$ is a bounded subset of H (since

$\|\pi(t)\xi\| = \|\xi\| = 1$, for all $t \in \Gamma$, as $\pi(t)$ is unitary). Hence, by Exercise 11.11, there exists a unique $\zeta \in H$ such that $\pi(\Gamma)\xi \subset \overline{B}(\zeta, r_0)$, where

$$r_0 = \inf\{r > 0 \,:\, \pi(\Gamma)\xi \subset \overline{B}(\eta, r) \text{ for some } \eta \in H\}$$

and, moreover, $\zeta \in \overline{\text{conv}}(\pi(\Gamma)\xi)$. We claim that ζ is Γ-invariant, i.e., $\pi(t)\zeta = \zeta$ for all $t \in \Gamma$. Indeed, we know that $\pi(\Gamma)\xi \subset \overline{B}(\zeta, r_0)$. This implies that for all $t \in \Gamma$,

$$\underbrace{\pi(t)\pi(\Gamma)\xi}_{\pi(\Gamma)\xi} \subset \overline{B}(\pi(t)\zeta, r_0).$$

By uniqueness of the circumcenter, $\pi(t)\zeta = \zeta$ for all $t \in \Gamma$.

It remains to show that $\zeta \neq 0$. We now have

$$\text{Re}\langle \xi, \zeta\rangle \overset{(1)}{\geq} \inf_{s\in\Gamma} \text{Re}\langle \xi, \pi(s)\xi\rangle \overset{(2)}{=} 1 - \frac{1}{2}\sup_{s\in\Gamma} \|\xi - \pi(s)\xi\|^2 \overset{(3)}{>} 0,$$

with the following explanations:

(1) Use that $\zeta \in \overline{\text{conv}}(\pi(\Gamma)\xi)$. Suppose that $\zeta \in \text{conv}(\pi(\Gamma)\xi)$, i.e., $\zeta = \sum_{i=1}^n \alpha_i \pi(s_i)\xi$ where $\alpha_i > 0$, $\sum_{i=1}^n \alpha_i = 1$, $s_i \in \Gamma$. Then

$$\begin{aligned}\text{Re}\langle \xi, \zeta\rangle &= \sum_{i=1}^n \alpha_i \text{Re}\langle \xi, \pi(s_i)\xi\rangle \\ &\geq \sum_{i=1}^n \alpha_i \inf_{s\in\Gamma} \text{Re}\langle \xi, \pi(s)\xi\rangle \\ &= \inf_{s\in\Gamma} \text{Re}\langle \xi, \pi(s)\xi\rangle.\end{aligned}$$

The general case follows by continuity.

(2) We have $\|z - w\|^2 = \|z\|^2 + \|w\|^2 - 2\text{Re}\langle z, w\rangle$ for all $z, w \in H$. In particular, if $\|z\| = 1 = \|w\|$, then

$$\|z - w\|^2 = 2 - 2\text{Re}\langle z, w\rangle.$$

(3) ξ is $(\Gamma, \sqrt{2})$-invariant.

We get $\text{Re}\langle \xi, \zeta\rangle > 0$ which implies that $\zeta \neq 0$ and the conclusion follows. □

Lecture 12, GOADyn
October 26, 2021

Section 12.1: Kazhdan's property (T)

Equivalent characterizations of Kazhdan's property (T)

Let Γ be a discrete group and let $\pi\colon \Gamma \to B(H)$ be a unitary representation.

Definition 12.1 (See Appendix D)**.** Let $\pi_1\colon \Gamma \to B(H_1)$ be another unitary representation.

(1) We say that $\pi_1 \subset \pi$ (π_1 *is contained in* π) if there exists a projection $p \in \pi(\Gamma)' \subset B(H)$ such that $\pi_1 \sim_u \pi_p$, where $\pi_p\colon \Gamma \to B(pH)$ is defined by

$$\pi_p(t) := \pi(t)p = p\pi(t)p, \quad t \in \Gamma.$$

(Recall that if $\pi_1\colon \Gamma \to B(H_1)$, $\pi_2\colon \Gamma \to B(H_2)$ are unitary representations, then $\pi_1 \sim_u \pi_2$ if there exists a unitary $U\colon H_1 \to H_2$ such that $\pi_2(t) = U\pi_1(t)U^*$ for all $t \in \Gamma$.)

(2) We say that $\pi_1 \prec \pi$ (π_1 *is weakly contained in* π) if for all $x \in \mathbb{C}\Gamma$,

$$\|\pi_1(x)\| \leq \|\pi(x)\|,$$

or equivalently, if the map $\pi(s) \mapsto \pi_1(s)$, $s \in \Gamma$ extends to a *-homomorphism from $C^*(\pi(\Gamma))$ to $C^*(\pi_1(\Gamma))$.

Remark 12.2. $\pi_1 \subset \pi$ implies $\pi_1 \prec \pi$, but the converse is not necessarily true.

Proposition 12.3. *Let Γ be a discrete group, let $\pi\colon \Gamma \to B(H)$ be any unitary representation and let $\pi_0\colon \Gamma \to B(H)$ be the trivial representation, i.e., $\pi_0(t) = 1$, for all $t \in \Gamma$.*

a) *$\pi_0 \prec \pi$ if and only if there exists a net of almost Γ-invariant unit vectors $(\xi_i)_{i\in I} \subset H$, i.e., $\lim_i \|\pi(t)\xi_i - \xi_i\| = 0$, for all $t \in \Gamma$.*

b) *$\pi_0 \subset \pi$ if and only if there exists $\xi \in H$, $\|\xi\| = 1$ such that $\pi(t)\xi = \xi$, for all $t \in \Gamma$ (i.e., ξ is Γ-invariant).*

Consequently, Γ has property (T) if and only if for all unitary representations π of Γ, if $\pi_0 \prec \pi$, then $\pi_0 \subset \pi$.

Proof. a) "$\Leftarrow$": Let $x = \sum \alpha_t t \in \mathbb{C}\Gamma$ (a finite sum). Then $\|\pi_0(x)\| = |\sum \alpha_t|$ and

$$\|\pi(x)\xi_i\| \approx \left\|\sum \alpha_t\xi_i\right\| = \left|\sum \alpha_t\right| \|\xi_i\| = \left|\sum \alpha_t\right| = \|\pi_0(x)\|.$$

Hence $\|\pi(x)\| \geq \|\pi_0(x)\|$, i.e., $\pi_0 \prec \pi$.

"$\Rightarrow$": For the proof we will use the following:

Remark 12.4. Given $\varepsilon > 0$ and $n \in \mathbb{N}$, there exists $\delta > 0$ such that for all unit vectors $\xi_1, \dots, \xi_n \in H$ satisfying $\|\sum_{k=1}^n \xi_k\| \geq n - \delta$, we have

$$\|\xi_i - \xi_j\| \leq \varepsilon, \quad i, j \in \{1, \dots, n\}.$$

Proof of Remark 12.4. Choose $\delta > 0$ such that $\sqrt{2(2n\delta - \delta^2)} \leq \varepsilon$. Then

$$\sum_{i,j=1}^{n} \langle \xi_i, \xi_j \rangle = \left\| \sum_{k=1}^{n} \xi_k \right\|^2 \geq (n-\delta)^2 = n^2 - 2n\delta + \delta^2.$$

Hence for all $i, j \in \{1, \ldots, n\}$, $\mathrm{Re}\langle \xi_i, \xi_j \rangle \geq 1 - 2n\delta + \delta^2$. Next, recall that

$$1 - \mathrm{Re}\langle \xi_i, \xi_j \rangle = \frac{1}{2}\|\xi_i - \xi_j\|^2$$

(using that $\|z - w\|^2 = \|z\|^2 + \|w\|^2 - 2\mathrm{Re}\langle z, w\rangle$, $z, w \in H$). Hence $\|\xi_i - \xi_j\|^2 \leq 2(2n\delta - \delta^2) \leq \varepsilon^2$. □

Now, let $F \subset \Gamma$ be finite such that $e \in F$ and let $\varepsilon > 0$. Set $x = \sum_{t \in F} t$. Then $\|\pi(x)\| \geq \|\pi_0(x)\| = |F|$ (since $\pi_0 \prec \pi$). By the triangle inequality, we get $\|\pi(x)\| = |F|$. Let $\delta > 0$ be as in Remark 12.4, corresponding to $n = |F|$. Choose $\xi \in H$, $\|\xi\| = 1$ with $\|\pi(x)\xi\| \geq |F| - \delta$, i.e., $\left\|\sum_{t \in F} \pi(t)\xi\right\| \geq |F| - \delta$. By Remark 12.4, we have $\|\pi(t)\xi - \pi(s)\xi\| \leq \varepsilon$ for all $s, t \in F$. In particular (since $e \in F$), it follows that

$$\|\pi(t)\xi - \xi\| \leq \varepsilon$$

for all $t \in F$. This implies that there exists a net (ξ_i) of almost Γ-invariant unit vectors in H.

b) "$\Rightarrow$": There exists a projection $p \in \pi(\Gamma)' \subset B(H)$ such that $\pi_0 \sim_u \pi_p$. This implies that $pH \subset \mathbb{C}\xi_0$ (a 1-dimensional subspace) for some $\xi_0 \in H$. So $\pi(t)\xi_0 = \xi_0$ for all $t \in \Gamma$.

"$\Leftarrow$": Suppose that there exists $\xi_0 \in H$ such that $\pi(t)\xi_0 = \xi_0$ for all $t \in \Gamma$. Let $p \colon H \to \mathbb{C}\xi_0$ be the orthogonal projection. Check that $p \in \pi(\Gamma)' \subset B(H)$ and that $\pi_p \sim_u \pi_0$. □

Appendix D

Cocycles of unitary representations

Definition 12.5. A 1-*cocycle on* Γ with coefficients in a unitary representation (π, H) of Γ is a function $b \colon \Gamma \to H$ such that

$$b(st) = b(s) + \pi(s)b(t), \quad s, t \in \Gamma.$$

(Note that $b(e) = 0$, by setting $s = t = e$.)

Remark 12.6. If (π, H) is a unitary representation of Γ and $\xi \in H$, then

$$b(s) := \xi - \pi(s)\xi, \quad s \in \Gamma$$

defines a 1-cocycle on Γ (such a 1-cocycle is called a 1-*coboundary*).

Proof. For all $s, t \in \Gamma$, $b(s) + \pi(s)b(t) = \xi - \pi(s)\xi + \pi(s)(\xi - \pi(t)\xi) = \xi - \pi(st)\xi = b(st)$. □

Definition 12.7. Let

$$\mathrm{Aff\,Iso}(H) = \{\psi \colon H \to H : \psi(\xi) = u\xi + \xi_0\,, \xi \in H, \text{ for some } u \in \mathcal{U}(H),\ \xi_0 \in H\}.$$

Note that $\psi \in \mathrm{Aff\,Iso}(H)$ implies that ψ is an affine isometry of H. The converse, in general, is not necessarily true (e.g., if $H = \mathbb{C}$, then $\varphi(z) = \overline{z}$ is an affine isometry of H, but clearly $\varphi \notin \mathrm{Aff\,Iso}(H)$.) However, if H is a <u>real</u> Hilbert space, then $\mathrm{Aff\,Iso}(H)$ is the set of <u>all</u> affine isometries of H.

Proposition 12.8. *If $\theta\colon \Gamma \to \mathrm{Aff\,Iso}(H)$ is a group homomorphism of Γ into the group $\mathrm{Aff\,Iso}(H)$, then*

$$\theta(s)\xi = \pi(s)\xi + b(s), \quad s \in \Gamma,\ \xi \in H$$

for some unitary representation $\pi\colon \Gamma \to B(H)$ and some 1-cocycle b on $\Gamma$with coefficients in (π, H).

Conversely, if $\pi\colon \Gamma \to B(H)$ is a unitary representation of Γ and b is a 1-cocycle with coefficients in (π, H), then

$$\theta(s)\xi = \pi(s)\xi + b(s), \quad s \in \Gamma,\ \xi \in H$$

defines a group homomorphism $\theta\colon \Gamma \to \mathrm{Aff\,Iso}(H)$.

Proof. If $\theta\colon \Gamma \to \mathrm{Aff\,Iso}(H)$ is a group homomorphism, then for all $s, t \in \Gamma$, $\xi \in H$,

$$\theta(st)\xi = \underbrace{\pi(st)}_{\in \mathcal{U}(H)}\,\xi + \underbrace{b(st)}_{\in H}.$$

On the other hand, $\theta(s)\theta(t)\xi = \theta(s)(\pi(t)\xi + b(t)) = \pi(s)\pi(t)\xi + \pi(s)b(t) + b(s)$. Since θ is a group homomorphism, we have $\theta(st)\xi = \theta(s)\theta(t)\xi$. This argument shows that $\pi(st) = \pi(s)\pi(t)$ and $b(st) = \pi(s)b(t) + b(s)$ for all $s, t \in \Gamma$. The statement is proved. The converse one is similar (and more straightforward). □

Remark 12.9. If $b(s) = \xi_0 - \pi(s)\xi_0$ is a 1-coboundary and if $\theta(s)\xi = \pi(s)\xi + b(s)$ for all $s \in \Gamma$, $\xi \in H$, then

$$\theta(s)\xi = \pi(s)(\xi - \xi_0) + \xi_0, \quad s \in \Gamma,\ \xi \in H.$$

In particular $\theta(s)\xi_0 = \xi_0$ for all $s \in \Gamma$.

Conversely, if $\theta(s)\xi = \pi(s)\xi + b(s)$ for all $s \in \Gamma$, $\xi \in H$ for some 1-cocycle b and if $\theta(s)\xi_0 = \xi_0$ for all $s \in \Gamma$, for some $\xi_0 \in H$, then

$$b(s) = \theta(s)\xi_0 - \pi(s)\xi_0 = \xi_0 - \pi(s)\xi_0, \quad s \in \Gamma,$$

i.e., b is a 1-coboundary.

Lemma 12.10 (Lemma D.10, [BO])**.** *A 1-cocyle is bounded if and only if it is a 1-coboundary.*

Proof. If $b(s) = \xi - \pi(s)\xi$ for all $s \in \Gamma$ (for some $\xi \in H$), then $\|b(s)\| \le 2\|\xi\|$, for all $s \in \Gamma$, i.e., b is bounded.

Conversely, assume that b is bounded, i.e., that $b(\Gamma)$ is a bounded subset of H. Let $\zeta \in H$ be the unique circumcenter of $b(\Gamma)$, i.e., $b(\Gamma) \subset \overline{B}(\zeta, r_0)$, where $r_0 = \inf\{r > 0 : b(\Gamma) \subset \overline{B}(\eta, r), \text{for some} \eta \in H\}$ (cf. Exercise 12.11). Further, let $\theta\colon \Gamma \to \mathrm{Aff\,Iso}(H)$ be given by

$$\theta(s)\xi = \pi(s)\xi + b(s), \quad s \in \Gamma,\ \xi \in H.$$

Note that $\theta(s)b(t) = \pi(s)b(t) + b(s) = b(st)$ for $s, t \in \Gamma$. This implies that $\theta(s)b(\Gamma) = b(\Gamma)$, for all $s \in \Gamma$. Since $\theta(s) \in \mathrm{Aff\,Iso}(H)$, we deduce that

$$\underbrace{\theta(s)b(\Gamma)}_{b(\Gamma)} \subset \overline{B}(\theta(s)\zeta, r_0), \quad s \in \Gamma.$$

By uniqueness of the circumcenter, we conclude that $\theta(s)\zeta = \zeta$, for all $s \in \Gamma$. By Remark 12.9 above, b is a 1-coboundary. □

In what follows we will make use of Schoenberg's theorem (Theorem D.11, [BO]), which we now discuss. A kernel $k : \Gamma \times \Gamma \to \mathbb{R}$ is called *conditionally negative definite* if there exists a function $b\colon \Gamma \to H$, for some Hilbert space H, such that

$$k(s,t) = \|b(s) - b(t)\|^2, \quad s, t \in \Gamma.$$

It can be shown that k is conditionally negative definite if and only if the following three conditions hold:

- $k(s,t) = k(t,s)$ for all $s, t \in \Gamma$,
- $\sum_{i,j=1}^{n} \alpha_i \alpha_j k(s_i, s_j) \leq 0$ for all $s_1, \ldots, s_n \in \Gamma$ and $\alpha_1, \ldots, \alpha_n \in \mathbb{R}$ with $\sum \alpha_i = 0$,
- $k(s,s) = 0$ for all $s \in \Gamma$.

Theorem 12.11 (Schoenberg, Theorem D.11, [BO])**.** *Let k be a conditionally negative definite kernel on Γ. Then the kernel*

$$\varphi_\gamma(s,t) = e^{-\gamma k(s,t)}, \quad s, t \in \Gamma$$

is positive definite, for all $\gamma > 0$. In particular, for any 1-cocycle b on Γ and any $\gamma > 0$, the function φ_γ^b on Γ, defined by

$$\varphi_\gamma^b(s) = e^{-\gamma \|b(s)\|^2}, \quad s \in \Gamma\,,$$

is positive definite.

Remark 12.12. Let $b\colon \Gamma \to H$ be a 1-cocycle on Γ with coefficients in a unitary representation (π, H) of Γ. It follows by Schoenberg's theorem above that for all $\gamma > 0$, the function $\varphi_\gamma^b\colon \Gamma \to \mathbb{R}$ defined by

$$\varphi_\gamma^b(s) := \exp(-\gamma \|b(s)\|^2), \quad s \in \Gamma \tag{$\star$}$$

is positive definite. Consider the associated GNS triple $(\pi_\gamma^b, H_\gamma^b, \xi_\gamma^b)$ (cf. Section 2.5, Definition 2.5.7, See Lecture 6). Recall that

$$\varphi_\gamma^b(s) = \langle \pi_\gamma^b(s) \xi_\gamma^b, \xi_\gamma^b \rangle, \quad s \in \Gamma, \tag{$\star\star$}$$

where $\xi_\gamma^b = \hat{\delta}_e$ and $\pi_\gamma^b(s)\xi_\gamma^b = \hat{\delta}_s$ for all $s \in \Gamma$. Hence $\overline{\operatorname{span}}\{\pi_\gamma^b(s)\xi_\gamma^b \,:\, s \in \Gamma\} = H_\gamma^b$.

Lemma 12.13 (Lemma D.12, [BO])**.** *Let b be a 1-cocycle on Γ and $\gamma > 0$. Let $(\pi_\gamma^b, H_\gamma^b, \xi_\gamma^b)$ be the GNS triple associated to the positive definite function $\varphi_\gamma^b\colon \Gamma \to \mathbb{R}$ as in above Remark 12.12.*

Suppose that b is unbounded on a subgroup Λ of Γ. Then there are no nonzero Λ-invariant vectors for $(\pi_\gamma^b, H_\gamma^b)$.

Proof. Let $(s_n)_{n \geq 1} \subset \Lambda$ be a sequence such that $\|b(s_n)\| \to \infty$ as $n \to \infty$. We will show that for all $\zeta \in H_\gamma^b$,

$$\lim_{n\to\infty} \langle \pi_\gamma^b(s_n)\zeta, \zeta \rangle = 0. \tag{1}$$

(This gives the conclusion.) By continuity, it suffices to prove that (1) holds on a dense subset of H_γ^b. Since $\operatorname{span}\{\pi_\gamma^b(s)\xi_\gamma^b \,:\, s \in \Gamma\}$ is dense in H_γ^b, it suffices to show that (1) holds for any vector of the form $\zeta = \sum_{i=1}^N \alpha_i \pi_\gamma^b(t_i)\xi_\gamma^b$, where $t_i \in \Gamma$, $\alpha_i \in \mathbb{C}$ and $N \in \mathbb{N}$. Indeed, by $(\star\star)$,

$$\limsup_{n\to\infty} |\langle \pi_\gamma^b(s_n)\zeta, \zeta \rangle| = \limsup_{n\to\infty} \left| \sum_{i,j=1}^{N} \overline{\alpha_i} \alpha_j \varphi_\gamma^b(t_i^{-1} s_n t_j) \right|. \tag{2}$$

Now, for all $i, j \in \{1, \ldots, N\}$, by $(\star)$ we have

$$\varphi_\gamma^b(t_i^{-1} s_n t_j) = \exp(-\gamma \|b(t_i^{-1} s_n t_j)\|^2), \quad n \in \mathbb{N}. \tag{3}$$

Note that

$$b(t_i^{-1}s_nt_j) = b(t_i^{-1}) + \pi(t_i^{-1})b(s_nt_j) = b(t_i^{-1}) + \pi(t_i^{-1})b(s_n) + \pi(t_i^{-1}s_n)b(t_j).$$

By the triangle inequality,

$$\begin{aligned}\|b(t_i^{-1}s_nt_j)\| &\geq \|\pi(t_i^{-1})b(s_n)\| - (\|b(t_i^{-1})\| + \|\pi(t_i^{-1}s_n)b(t_j)\|) \\ &= \|b(s_n)\| - (\|b(t_i^{-1})\| + \|b(t_j)\|)) \to \infty, \quad \text{as } n \to \infty.\end{aligned}$$

Using this in (3), we obtain

$$\limsup_{n\to\infty} \varphi_\gamma^b(t_i^{-1}s_nt_j) = \limsup_{n\to\infty} \exp(-\gamma\|b(t_i^{-1}s_nt_j)\|^2) = 0.$$

Hence the lim sup on the left hand side of (2) is equal to zero, and the proof is complete. □

The following theorem gives a few equivalent characterizations of relative property (T).

Theorem 12.14 (Theorem 12.1.7, [BO])**.** *Let Γ be a discrete countable group and $\Lambda \subset \Gamma$ a subgroup. The following are equivalent:*

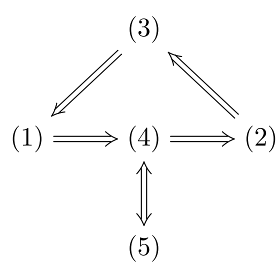
1. *The inclusion $\Lambda \subset \Gamma$ has relative property (T).*
2. *There exists a finite subset $E \subset \Gamma$ and $k > 0$ with the following property: If (π, H) is a unitary representation of Γ and P is the orthogonal projection from H onto the subspace of all Λ-invariant vectors, then*
$$\|\xi - P\xi\| \leq \frac{1}{k} \sup_{s\in E} \|\pi(s)\xi - \xi\|, \quad \xi \in H.$$
(Note that a pair (E, k) satisfying this property will then be a Kazhdan pair for $\Lambda \subset \Gamma$. Indeed, if ξ_0 is a nonzero (E, k)-invariant vector, then $\|\xi_0 - P\xi_0\| < \|\xi_0\|$, which implies that $P\xi_0 \neq 0$, i.e., $P \neq 0$, so there are nonzero Λ-invariant vectors.)
3. *Any sequence of positive definite functions on Γ that converges pointwise to the constant function 1, converges uniformly on Λ.*
4. *Every 1-cocycle $b\colon \Gamma \to H$ is bounded on Λ.*
5. *Every action of Γ on $\mathrm{Aff\,Iso}(H)$ has a Λ-fixed point.*

Moreover, if $\Lambda = \Gamma$, then the above conditions are equivalent to:

6. *The group Γ is finitely generated and for any generating subset $S \subset \Gamma$, there exists $k = k(\Gamma, S) > 0$ such that (S, k) is a Kazhdan pair.*

Note that we have already discussed in the previous lecture the equivalence of (1) and (2) when $\Lambda = \Gamma$.

Proof. We will show



(1) ⇒ (4): For this, we prove $\neg(4) \Rightarrow \neg(1)$. Suppose that there exists a 1-cocycle $b\colon \Gamma \to H$ which is unbounded on Λ. For every $n \geq 1$, consider $\varphi^b_{\frac{1}{n}} : \Gamma \to \mathbb{R}$ defined by

$$\varphi^b_{\frac{1}{n}}(s) := \exp\left(-\frac{\|b(s)\|^2}{n}\right), \quad s \in \Gamma$$

which is positive definite by Schoenberg's theorem. Let $(\pi^b_{\frac{1}{n}}, H^b_{\frac{1}{n}}, \xi^b_{\frac{1}{n}})$ be the associated GNS triple. Let

$$\pi^b := \bigoplus_{n=1}^{\infty} \pi^b_{\frac{1}{n}} : \Gamma \to B(H^b),$$

where $H^b = \bigoplus_{n=1}^{\infty} H^b_{\frac{1}{n}}$.

Claim 1. There are no nonzero Λ-invariant vectors for π^b.

Proof. Let ζ be a Λ-invariant vector for π^b. For $n \geq 1$, let $P_n\colon H^b \to H^b_{\frac{1}{n}} \subset H^b$ be the orthogonal projection. One can see that $P_n\zeta$ is a Λ-invariant vector for $\pi^b_{\frac{1}{n}}$. Then, by Lemma 12.13, we deduce that $P_n\zeta = 0$. Since $\zeta = \sum_{n=1}^{\infty} P_n\zeta$, we deduce that $\zeta = 0$. □

Claim 2. The sequence $(\xi^b_{\frac{1}{n}})_{n\geq 1}$ is almost Γ-invariant for π^b.

Proof. For all $n \geq 1$ and $s \in \Gamma$,

$$\exp\left(-\frac{\|b(s)\|^2}{n}\right) = \varphi^b_{\frac{1}{n}}(s) = \langle \pi^b_{\frac{1}{n}}(s)\xi^b_{\frac{1}{n}}, \xi^b_{\frac{1}{n}}\rangle = \langle \pi^b(s)\xi^b_{\frac{1}{n}}, \xi^b_{\frac{1}{n}}\rangle.$$

This implies $\mathrm{Re}\langle \pi^b(s)\xi^b_{\frac{1}{n}}, \xi^b_{\frac{1}{n}}\rangle \to 1$ as $n \to \infty$. Since

$$\mathrm{Re}\langle \pi^b(s)\xi^b_{\frac{1}{n}}, \xi^b_{\frac{1}{n}}\rangle = 1 - \frac{1}{2}\|\pi^b(s)\xi^b_{\frac{1}{n}} - \xi^b_{\frac{1}{n}}\|^2$$

for all $s \in \Gamma$ and $n \geq 1$, Claim 2 follows. □

It is clear that Claims 1 and 2 together imply $\neg(1)$.

(4) ⇒ (5): Let $\theta\colon \Gamma \to \mathrm{Aff\,Iso}(H)$ be a group homomorphism. Then there exists a 1-cocycle $b\colon \Gamma \to H$ associated to it (by Proposition 12.8). By (4), b is bounded on Λ. Then, by the proof of Lemma 12.10, there exists $\zeta \in H$ such that $\theta(s)\zeta = \zeta$ for all $s \in \Lambda$, i.e., θ has a Λ-fixed point.

(5) ⇒ (4): Let $b\colon \Gamma \to H$ be a 1-cocycle. Let $\theta\colon \Gamma \to \mathrm{Aff\,Iso}(H)$ be the group homomorphism associated to it (cf. Proposition 12.8). By Remark 12.9, we infer that b is a 1-coboundary on Λ. By Lemma 12.10, b is bounded on Λ.

(4) ⇒ (2): We prove $\neg(2) \Rightarrow \neg(4)$. Write $\Gamma = \bigcup_{n=1}^{\infty} E_n$, where $E_1 \subset E_2 \subset \cdots \subset \cdots$ are finite sets. By $\neg(2)$, for all $n \geq 1$ there exists a unitary representation (π_n, H_n) of Γ and $\xi_n \in H_n$ such that

$$\frac{1}{4^n}\|\xi_n - P_n\xi_n\| > \sup_{s\in E_n} \|\xi_n - \pi_n(s)\xi_n\| := \delta_n.$$

(Take $k_n = 4^{-n}$.) Here $P_n\colon H_n \to \{\text{all } \Lambda\text{-invariant vectors (in } H_n) \text{ for } \pi_n\}$ is the orthogonal projection. Let $\pi := \bigoplus_{n=1}^{\infty} \pi_n \colon \Gamma \to B(H)$, where $H = \bigoplus_{n=1}^{\infty} H_n$. Note that for all $s \in \Gamma$,

$$\left(\frac{\xi_n - \pi_n(s)\xi_n}{2^n \delta_n}\right)_{n\geq 1} \in H = \bigoplus_{n=1}^{\infty} H_n.$$

Indeed, if $s \in \Gamma$, then $s \in E_k$ for some k, and hence $s \in E_n$ for all $n \geq k$, so

$$\sum_{n=1}^{\infty} \left\| \frac{\xi_n - \pi_n(s)\xi_n}{2^n \delta_n} \right\|^2 = \sum_{n=1}^{k-1} \left\| \frac{\xi_n - \pi_n(s)\xi_n}{2^n \delta_n} \right\|^2 + \sum_{n=k}^{\infty} \left| \frac{\delta_n}{2^n \delta_n} \right|^2 < \infty.$$

Now, define a map $\sigma\colon \Gamma \to H$ by

$$\sigma(s) := \left(\frac{\xi_n - \pi_n(s)\xi_n}{2^n \delta_n}\right)_{n\geq 1} \in H, \quad s \in \Gamma.$$

We claim that σ is a 1-cocycle on Γ with coefficients in (π, H). Indeed, for all $n \geq 1$, let $\sigma_n\colon \Gamma \to H_n$ be defined by

$$\sigma_n(s) = \frac{\xi_n - \pi_n(s)\xi_n}{2^n \delta_n} = \frac{\xi_n}{2^n \delta_n} - \pi_n(s)\frac{\xi_n}{2^n \delta_n}, \quad s \in \Gamma.$$

Hence σ_n is a 1-coboundary and therefore a 1-cocycle. Then for all $s, t \in \Gamma$,

$$\sigma(st) = \bigoplus_{n=1}^{\infty} \sigma_n(st) = \bigoplus_{n=1}^{\infty} (\sigma_n(s) + \pi_n(s)\sigma_n(t)) = \sigma(s) + \pi(s)b(t).$$

So the claim above is proved.

Now, note that for all $n \geq 1$, $\pi_n(\Lambda)\xi_n$ is a bounded subset of H_n (since $\pi_n(t)$ is unitary for all $t \in \Lambda$). Hence by Exercise 12.11, there exists a unique $\zeta_n \in H_n$ such that

$$\pi_n(\Lambda)\xi_n \subset \overline{B}(\zeta_n, r_n)$$

where $r_n = \inf\{r > 0 : \pi_n(\Lambda)\xi_n \subset \overline{B}(\eta, r) \text{ for some } \eta \in H_n\}$. Moreover, $\zeta_n \in \overline{\text{conv}}(\pi_n(\Lambda)\xi_n)$. By uniqueness of ζ_n, it follows that

$$\pi_n(s)\zeta_n = \zeta_n, \quad s \in \Lambda.$$

Hence $\zeta_n \in P_n H_n$. Now, use that $\zeta_n \in \overline{\text{conv}}(\pi_n(\Lambda)\xi_n)$ to conclude that for $\varepsilon_n = \frac{1}{2}\|\xi_n - P_n\xi_n\|$ there exists $N = N(n) \in \mathbb{N}$, $\alpha_j > 0$ with $\sum_{j=1}^{N} \alpha_j = 1$, $s_j \in \Lambda$, $1 \leq j \leq N$ such that

$$\left\| \zeta_n - \sum_{j=1}^{N} \alpha_j \pi_n(s_j)\xi_n \right\| < \varepsilon_n.$$

Then

$$\|\xi_n - P_n\xi_n\| \leq \|\xi_n - \zeta_n\| \leq \left\| \xi_n - \sum_{j=1}^{N} \alpha_j \pi_n(s_j)\xi_n \right\| + \varepsilon_n \leq \sum_{j=1}^{N} \alpha_j \|\xi_n - \pi_n(s_j)\xi_n\| + \varepsilon_n.$$

Therefore there exists $j \in \{1, \ldots, N\}$ such that

$$\|\xi_n - \pi_n(s_j)\xi_n\| \geq \|\xi_n - P_n\xi_n\| - \varepsilon_n = \frac{1}{2}\|\xi_n - P_n\xi_n\|.$$

Denote s_j by s_n (note that j depends on n, after all). We have therefore proved that for all $n \geq 1$, there exists $\delta_n > 0$ and $s_n \in \Lambda$ such that

$$\left\| \frac{\xi_n - \pi_n(s_n)\xi_n}{2^n \delta_n} \right\| \geq \frac{1}{2} \frac{\|\xi_n - P_n\xi_n\|}{2^n \delta_n} > \frac{1}{2} \frac{4^n \delta_n}{2^n \delta_n} = 2^{n-1}.$$

Hence $\|\sigma(s_n)\| \geq 2^{n-1}$ for all $n \geq 1$. This implies that the 1-cocycle σ is unbounded on Λ, so $\neg(4)$ holds.

(2) ⇒ (3): Assume that (2) holds, and let (E, k) be as in condition (2). We first show that for all positive definite functions $\varphi\colon \Gamma \to \mathbb{C}$ with $\varphi(e) = 1$ we have

$$\sup_{t\in\Lambda} |1 - \varphi(t)| \leq \frac{2}{k} \max_{s\in E} \left(2\mathrm{Re}(1 - \varphi(s))\right)^{1/2}. \qquad (\square)$$

Note that if φ is positive definite, then $|\varphi(t)| \leq |\varphi(e)| = 1$ for all $t \in \Gamma$, so $\mathrm{Re}\varphi(t) \leq 1$ for all $t \in \Gamma$.

Let $(\pi_\varphi, H_\varphi, \xi_\varphi)$ be the GNS triple associated to φ. Then

$$\varphi(t) = \langle \pi_\varphi(t)\xi_\varphi, \xi_\varphi\rangle, \quad t \in \Gamma$$

and $\|\xi_\varphi\| = 1$. Let P be the projection from H_φ onto the space of Λ-invariant vectors. We now have

$$\begin{aligned}
\sup_{t\in\Lambda} |1 - \varphi(t)| &= \sup_{t\in\Lambda} |\langle \xi_\varphi - \pi_\varphi(t)\xi_\varphi, \xi_\varphi\rangle| \\
&\overset{(a)}{\leq} \sup_{t\in\Lambda} \|\xi_\varphi - \pi_\varphi(t)\xi_\varphi\| \underbrace{\|\xi_\varphi\|}_{=1} \\
&\overset{(b)}{=} \sup_{t\in\Lambda} \|\pi_\varphi(t)P^\perp\xi_\varphi - P^\perp\xi_\varphi\| \\
&\leq 2\sup_{t\in\Lambda} \|P^\perp\xi_\varphi\| = 2\|P^\perp\xi_\varphi\| \\
&= 2\|\xi_\varphi - P\xi_\varphi\| \\
&\leq \frac{2}{k} \sup_{s\in E} \|\pi(s)\xi_\varphi - \xi_\varphi\| \\
&\overset{(c)}{=} \frac{2}{k} \max_{s\in E} \left(2\mathrm{Re}(1 - \varphi(s))\right)^{1/2},
\end{aligned}$$

with the following explanations:

a) This is just the Cauchy-Schwarz inequality.
b) Write $I = P + P^\perp$. Then

$$\begin{aligned}
\xi_\varphi - \pi_\varphi(t)\xi_\varphi &= -(P + P^\perp)(\pi_\varphi(t)\xi_\varphi) + (P + P^\perp)\xi_\varphi \\
&= \pi_\varphi(t)(P + P^\perp)\xi_\varphi - (P + P^\perp)\xi_\varphi \\
&= \pi_\varphi(t)P^\perp\xi_\varphi - P^\perp\xi_\varphi
\end{aligned}$$

since $\pi_\varphi(t)P\xi_\varphi = P\xi_\varphi$.
c) E is finite, and $\|z - w\|^2 = \|z\|^2 + \|w\|^2 - 2\mathrm{Re}\langle z, w\rangle$ for all $z, w \in H$.

Now, let $(\varphi_n)_{n\geq 1}$ be a sequence of positive definite functions on Γ converging pointwise to 1 on Γ. First, note that we may assume that $\varphi_n(e) = 1$ for all $n \geq 1$. (Otherwise, set $\psi_n := \frac{\varphi_n}{\varphi_n(e)}$. Then ψ_n is positive definite on Γ with $\psi_n(e) = 1$. Since $\varphi_n(e) \to 1$ as $n \to \infty$, $\sup_{t\in\Gamma} |\psi_n(t) - \varphi_n(t)| \to 0$, as $n \to \infty$. So if we show that $(\psi_n)_{n\geq 1}$ converges uniformly to 1 on Λ, it will follow that $(\varphi_n)_{n\geq 1}$ converges uniformly to 1 on Λ.)

Now apply to each φ_n the inequality $(\square)$ proved above. For $n \geq 1$,

$$\sup_{t\in\Lambda} |1 - \varphi_n(t)| \leq \frac{2}{k} \max_{s\in E} \left(2\mathrm{Re}(1 - \varphi_n(s))\right)^{1/2} \to 0$$

as $n \to \infty$. Hence $\varphi_n \to 1$ uniformly on Λ.

(3) ⇒ (1): Let (π, H) be a unitary representation of Γ which contains almost Γ-invariant unit vectors $(\xi_n)_{n\geq 1}$. For all $n \geq 1$, let $\varphi_n : \Gamma \to \mathbb{C}$ be defined by

$$\varphi_n(s) := \langle \pi(s)\xi_n, \xi_n \rangle, \quad s \in \Gamma.$$

Then φ_n is positive definite with $\varphi_n(e) = 1$. By the Cauchy-Schwarz inequality, we have for all $s \in \Gamma$ and $n \geq 1$ that

$$|1 - \varphi_n(s)| \leq \|\pi(s)\xi_n - \xi_n\| \to 0, \quad \text{as } n \to \infty.$$

Therefore $(\varphi_n)_{n\geq 1}$ converges pointwise to the constant function 1. By hypothesis, φ_n will converge uniformly on Λ to the constant function 1. Hence there exists $N \in \mathbb{N}$ such that for all $n \geq N$,

$$\sup_{s\in\Lambda} |1 - \varphi_n(s)| < \frac{1}{2}.$$

Note that $\|\pi(s)\xi_n - \xi_n\|^2 = 2 - 2\mathrm{Re}\varphi(s) = 2\mathrm{Re}(1 - \varphi(s)) \leq 2|1 - \varphi(s)|$. Hence, for all $n \geq N$ we have $\sup_{s\in\Lambda} \|\pi(s)\xi_n - \xi_n\| < 1$, i.e., ξ_n is a $(\Lambda, 1)$-invariant vector, hence $(\Lambda, \sqrt{2})$-invariant vector. By Lemma 12.10 (cf. Lemma 12.1.5, [BO]), there exists a nonzero Λ-invariant vector, i.e., condition (1) holds. □

Further examples

Definition 12.15. Let Γ be a locally compact group. A *lattice* in Γ is a discrete subgroup Λ of Γ such that Γ/Λ carries a finite Γ-invariant regular Borel measure. (Such a measure is unique up to a constant.)

Theorem 12.16. *Let Γ be locally compact and let $\Lambda \subset \Gamma$ be a discrete subgroup. If Λ is a lattice in Γ, then Γ has property (T) if and only if Λ has property (T).*

Examples 12.17.

(1) $\mathrm{SL}(2, \mathbb{R})$ does not have property (T) (because $\mathbb{F}_2 \hookrightarrow \mathrm{SL}(2, \mathbb{R})$ as a lattice). Also $\mathrm{SL}(2, \mathbb{Z})$ does not have property (T).

(2) $\mathrm{SL}(n, \mathbb{Z})$ is a lattice in $\mathrm{SL}(n, \mathbb{R})$ for all $n \geq 3$. Both have property (T).

(3) $\mathrm{SL}(n, \mathbb{Z}) \ltimes \mathbb{Z}^n$ is a lattice in $\mathrm{SL}(n, \mathbb{R}) \ltimes \mathbb{R}^n$ for all $n \geq 3$. Both have property (T).