Partial Differential Equations I (Handwrite Notes)

1. Classical Maximum principles for 2nd order Elliptic Equations

1.1 Elliptic equations

$\Omega \subseteq \mathbb{R}^n$ a domain (open, connected, not necessary simply connected. $x = (x_1, \cdots, x_n) \in \Omega$. $u = u(x)$

$$Lu := \sum_{i,j=1}^{n} a_{ij}(x) \frac{\partial u}{\partial x_i \partial x_j} + \sum_{i=1}^{n} b_i(x) \frac{\partial u}{\partial x_i} + c(x) u$$

Elliptic operator, linear about u.

a_{ij}, b_i just depend on x.

Notation $\frac{\partial u}{\partial x_i} = u_{x_i} = D_{x_i} u$ $\quad \frac{\partial^2 u}{\partial x_i \partial x_j} = u_{x_i x_j} = D_{ij} u$.

$\underset{\text{gradient}}{\nabla u} = (u_{x_1}, u_{x_2}, \cdots u_{x_n}) = Du$ $\quad \underset{\text{Hessian matrix.}}{D^2 u} = [D_{ij} u]_{1 \le i,j \le n}$

<u>Definition</u>: L is called elliptic in Ω if $(a_{ij}(x))_{n \times n}$ is symmetric and positive definite for all $x \in \Omega$.

<u>Definition</u>: L is called <u>strictly elliptic</u> if it is elliptic in Ω and there is a constant $\lambda_0 > 0$ such that:

$$\underbrace{\sum_{i,j=1}^{n} a_{ij}(x) \xi_i \xi_j}_{\parallel} \ge \lambda_0 |\xi|^2, \text{ for all } \xi = (\xi_1, \cdots, \xi_n) \in \mathbb{R}^n$$

and for all $x \in \Omega$.

$\xi A(x) \xi^T$. $A(x) = (a_{ij}(x))_{1 \le i,j \le n}$.

二次型

(i.e. $A(x) \ge \lambda I_{n \times n}$)

最小特征值不小于λ_0，正定

Linear 2nd order elliptic equations

$$Lu = f(x)$$

where $u(x)$ is unknown, $f(x)$ is given

Semi-linear $\quad Su = \sum_{i,j=1}^{n} a_{ij}(x)\, u_{x_i x_j} + b(x, u, Du) = 0$

(最高) 二阶导项为线性，低阶求不线性

Quasi-linear $\quad Qu = \sum_{i,j=1}^{n} a_{ij}(x, u, Du)\, u_{x_i x_j} + b(x, u, Du)$

a_{ij} 不能为二阶导数项的函数.

Examples: ① Let $f(z) = u(x,y) + i v(x,y)$

$z = x + iy$

be an analytic function

$\Leftrightarrow$ Cauchy-Riemann equations $\quad u_x = v_y \quad v_x = -u_y$

$\Rightarrow$ $\left.\begin{aligned} &\Delta u = u_{xx} + u_{yy} = (v_y)_x - (v_x)_y = 0 \\ &\Delta v = 0. \end{aligned}\right\}$ u, v are harmonic on Ω.

$\Delta = \sum_{i=1}^{n} \frac{\partial^2}{\partial x_i^2}$ is the Laplace operator

Formula: $\Delta(fg) = g\Delta f + 2(\nabla f)\cdot(\nabla g) + f(\Delta g)$ (Hw)

$$\Delta(u^2+v^2) = \Delta(u^2) + \Delta(v^2) = 2\nabla u \cdot \nabla u = 2\left[|\nabla u|^2 + |\nabla v|^2\right]$$
$$\geq 0$$

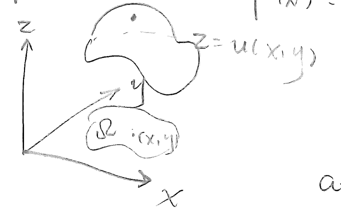
$\Rightarrow$ $u^2 + v^2$ is a lower or sub-harmonic function

($\Delta f \leq 0 \Rightarrow f$ is an upper or super-harmonic function)

② Electrostatics $\quad -\Delta u(x) = f(x)$ (the Poisson equations)

$u(x)$: electric potential $\quad f(x)$: density of charges

③ Mean curvature



mean curvature of the surface $z=u(x,y)$ at $(x, y, u(x,y))$

Notation

$F(x) = (F_1(x), \dots, F_n(x))$

$\nabla \cdot F = \operatorname{div} F$

$= \sum_{i=1}^{n} \frac{\partial F_i}{\partial x_i}$

$$H(x,y) = \frac{1}{2} \nabla \cdot \left(\frac{\nabla u}{\sqrt{1+|\nabla u|^2}} \right)$$

minimal surface:

$$H(x,y) \equiv 0 \quad \text{in } \Omega.$$

$$\Rightarrow \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}\right) \cdot \left(\frac{u_x}{\sqrt{1+|\nabla u|^2}}, \frac{u_y}{\sqrt{1+|\nabla u|^2}} \right) = 0$$

$$\Rightarrow [1+(u_y)^2]\, u_{xx} + [1+(u_x)^2]\, u_{yy} - 2 u_x u_y u_{xy} = 0$$

quasi-linear equation of 2nd order (HW)

$a_{11} = 1+u_y^2$, $a_{22} = 1+u_x^2$ $\quad a_{12} = a_{21} = -u_x u_y$

Question. Is $A(x,y) = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$ positive definite?

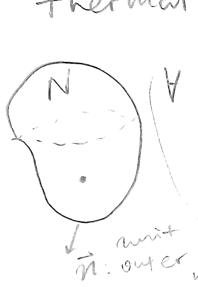
$$\sum_{i,j=1}^{2} a_{ij} \xi_i \xi_j \geq \boxed{\lambda_0} |\xi|^2 \ ?$$

$$(1+u_y^2)\,\xi_1^2 - 2u_x u_y \xi_1 \xi_2 + (1+u_x^2)\,\xi_2^2$$

$$= |\xi|^2 + (u_y \xi_1 - u_x \xi_2)^2 \geq |\xi|^2. \quad \text{Take } \lambda_0 = 1$$

④ Heat Equation: $u(x,t)$ temperature at location $x \in \Omega$, time $t > 0$. $\quad C$: specific heat

thermal energy density function. $\quad \rho$: density of medium



$\forall N \subset \Omega$ sub domain $\quad \bar{E}(x,t) = C\rho\, u(x,t)$

$$\underbrace{\frac{d}{dt}\int_N \bar{E}(x,t)\,dx}_{\text{net rate of change of thermal energy}} = \overbrace{-\int_{\partial N} \vec{J}(x,t)\cdot\vec{n}\,dS}^{\text{net rate of change due to the flowing heat}} + \overbrace{\int_N F(x,t)\,dx}^{\text{net rate of change due to reaction - degradation of thermal energy}}$$

Fourier's Law $\vec{J}(x,t)$ and $(-\nabla u)$ make an angle $< \frac{\pi}{2}$ (90°)

$\Rightarrow \vec{J}(x,t) = A_{n\times n}(x,t)\,(-\nabla u)$ $A_{n\times n}$ sym pos-def

$(\vec{J}\cdot(-\nabla u) = (-\nabla u)A(-\nabla u)^T \geq 0)$

isotropic medium $A = k\,I_{n\times n}$ $k>0$ constant

(thermal conductivity of the medium)

$$\frac{d}{dt}\int_N E(x,t)\,dx = \underbrace{\int_{\partial N}(A\nabla u)\cdot\vec{n}\,dS}_{\int_N \operatorname{div}(A\nabla u)\,dx \text{ (Hw)}} + \int_N F(x,t)\,dx$$

$N \subset \Omega$ arbitrary $\Rightarrow c\rho u_t - \operatorname{div}(A\nabla u) = F(x,t)$

$\forall x \in \Omega,\ t>0$

(general heat equation)

zero-flux on $\partial\Omega$ (insulated)

绝热

$$0 = \vec{J}\cdot\vec{n} = A(-\nabla u)\cdot\vec{n} = \frac{\partial u}{\partial \vec{n}_A}$$

(Neumann Conormal type boundary condition)

Heat equation in isotropic medium

$$u_t - a^2\Delta u = f(x,t)$$

where $a^2 = \frac{k^2}{\rho c}$ is called heat diffusion of the heat medium. $f = F/\rho c$

Suppose $f(x,t) = f(x)$.

$\Rightarrow$ steady state (equilibrium) $u = u(x)$

$$-a^2 \Delta u = f(x) \quad \text{(Poisson equation)}$$

$$\left(\Delta u = \hat{f}(x) \begin{cases} > 0 & \text{heat degraded (sink e.g fridge)} \\ < 0 & \text{heat created (source e.g. oven)} \end{cases} \right)$$

20200909

Review: Divergence theorem: $\Omega \subseteq \mathbb{R}^n$ bounded open $\partial\Omega$ is C^1

$\vec{n}(x)$: unit normal vector outward pointing

well-defined on $\partial\Omega$ and is continuous.

Gauss-Green Theorem If $u \in C^1(\bar{\Omega})$ then

$$\int_\Omega u_{x_i} \, dx = \int_{\partial\Omega} u \cdot \vec{n}_i \, dS. \quad (i = 1, 2, \cdots, n)$$

Corollary $\vec{u}(x) = (u_1(x), \cdots, u_n(x))$, $x \in \Omega \subseteq \mathbb{R}^n$.

$$\int_\Omega \operatorname{div}(\vec{u}) \, dx = \int_\Omega \sum_i \frac{\partial u_i}{\partial x_i} \, dx = \int_{\partial\Omega} \sum_i u_i \, \vec{n}_i \, dS$$

$$= \int_{\partial\Omega} \vec{u} \cdot \vec{n} \, dS. \quad \left(\begin{array}{c}\text{divergence} \\ \text{theorem}\end{array}\right)$$

⑤ Wave equation $u_{tt} - k^2 \Delta u = f(x, t)$.

$$\text{when } \left. \begin{array}{l} f(x,t) = f(x) \\ u(x,t) = u(x) \end{array} \right\} \Rightarrow -k^2 \Delta u = f(x)$$

⑥ Irrotational and incompressible fluid.

$\vec{V}(x)$: velocity vector field of a fluid in $\Omega \subseteq \mathbb{R}^3$.

irrotational: $\operatorname{curl} \vec{V} = \nabla \times \vec{V} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \partial_x & \partial_y & \partial_z \\ V_1 & V_2 & V_3 \end{vmatrix}$

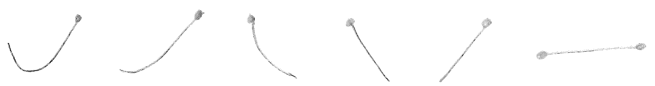
$\downarrow$

$\operatorname{curl} \vec{V} = 0 \Longrightarrow \vec{V} = \nabla \varphi$ for some $\varphi : \Omega \to \mathbb{R}$

$\Rightarrow \operatorname{div}(\nabla \varphi) = 0 \Rightarrow \Delta \varphi = 0.$

1.2. Weak Maximum Principle (WMP).

Baby example: Suppose $u \in C^2((a,b)) \cap C^0([a,b])$ $u''(x) \geq 0$ in $[a,b]$



$$Lu = \sum_{i,j=1}^{n} a_{ij}(x) D_{ij} u + \sum_{i=1}^{n} b_i(x) D_i u + c(x) u$$

Question:

$Lu \geq 0$ in $\Omega \Rightarrow \max_{\bar{\Omega}} u = \max_{\partial\Omega} u$?

Weak maximum principle (WMP) with $c \equiv 0$. $\xi A \xi^T \geq \lambda_0 |\xi|^2$ $\forall \xi \in \mathbb{R}^n$ $\Updownarrow$

Suppose that (A1) Ω <u>bounded</u> domain in $\mathbb{R}^n$

(A2) L is strictly elliptic on Ω. ($A \geq \lambda_0 I_{n\times n}$, $\lambda_0 > 0$)

(A3) $|b_i(x)| \leq M$ $(i = 1, 2, \dots, n)$ $\forall x \in \Omega$

(uniformly bounded)

If $u \in C^2(\Omega) \cap C^0(\bar{\Omega})$ <u>$Lu \geq 0$</u> in Ω, then

$$\max_{\bar{\Omega}} u = \max_{\partial\Omega} u. \quad (*1)$$

Remark • $Lu \geq 0$. u is called a lower solution or a subsolution of $Lu = 0$
(upper)
($\leq$)
(sup-).

• $u(x,y) = 2020x^2 - y^2$ $\Delta u > 0$ but

$\Rightarrow$ $\Delta u > 0 \not\Rightarrow$ concave up.

Proof of WMP with $c \equiv 0$:

Step 1 Suppose $Lu > 0$ in Ω.

$u \in C^0(\bar{\Omega}) \Rightarrow \exists\, x_0 \in \bar{\Omega}$ s.t. $u(x_0) = \max_{\bar{\Omega}} u$

If $x_0 \in \partial\Omega$, we are done.

Now suppose $x_0 \in \Omega \Rightarrow \nabla u(x_0) = 0$

Hessian $D^2 u(x_0) = (D_{ij} u(x_0))_{1 \le i,j \le n}$

≤ 0. (non-positive definite) 半负定

$A = (a_{ij}(x))$ positive definite

$\exists$ orthogonal matrix $P_{n\times n}$, $PP^T = P^TP = I_{n\times n}$ s.t.

$$P^TAP = \begin{bmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{bmatrix}$$ λ_i's are eigenvalues of A

$(\lambda_i \ge \lambda_0 > 0)$

$$\operatorname{trace}(AH) = \operatorname{trace}(P^TAHP)$$

$$= \operatorname{trace}(\underbrace{P^TAP}_{\parallel}\underbrace{P^THP}_{\le 0})$$

$$\begin{bmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{bmatrix} \begin{bmatrix} \tilde{h}_{11} & \tilde{h}_{12} & \cdots & \tilde{h}_{1n} \\ \tilde{h}_{n1} & \tilde{h}_{n2} & \cdots & \tilde{h}_{nn} \end{bmatrix}$$

$$\le 0.$$

$$= \lambda_1 \tilde{h}_{11} + \cdots + \lambda_n \tilde{h}_{nn}$$

$$\le 0$$

(Here only need $\lambda_i \ge 0$ i.e. $A \ge 0$).

Step 2. $Lu \ge 0$ in Ω $\forall \varepsilon > 0$, define $v(x) = u(x) + \varepsilon e^{\alpha x_1}$

where x_1 is the 1st component of x $\alpha > 0$ to be chosen

$Lv > 0$ in Ω $Lv = Lu + \varepsilon L(e^{\alpha x_1})$.

$$L(e^{\alpha x_1}) = \alpha^2 a_{11}(x) e^{\alpha x_1} + \alpha b_1(x) e^{\alpha x_1}$$

$$\geq \underbrace{\alpha^2 e^{\alpha x_1} \lambda_0}_{\substack{\text{strictly elliptic} \\ \text{take } \xi = (1,0,\cdots 0)^T}} - \underbrace{\alpha M e^{\alpha x_1}}_{\substack{b_i(x) \text{ uniformly} \\ \text{bounded}}}$$

$$\geq (\alpha^2 \lambda_0 - \alpha M) e^{\alpha x_1}$$

> 0 if α is sufficiently large

Now apply Step 1 to v:

$$\begin{array}{ccc} \max\limits_{\bar{\Omega}} v & = & \max\limits_{\partial\Omega} v \\ \Big\downarrow {\scriptstyle \varepsilon \to 0} & & \Big\downarrow {\scriptstyle \varepsilon \to 0} \\ \max\limits_{\bar{\Omega}} u & = & \max\limits_{\partial\Omega} u \end{array}$$

($\because e^{\alpha x_1}$ is bounded $\Rightarrow v \to u$ uniformly on Ω as $\varepsilon \to 0$)

Another perspective

or $\max\limits_{\bar{\Omega}} u \leq \max\limits_{\bar{\Omega}} v = \max\limits_{\partial\Omega} v$

where N is a bound of $\varepsilon^{\alpha x_1}$ $\quad \leq \max\limits_{\partial\Omega} u + \varepsilon N \quad \forall \varepsilon > 0$

$$\Rightarrow \max_{\bar{\Omega}} u \leq \max_{\partial\Omega} u$$

$\geq$ obvious

$\square$

Remarks: 1. It is clear from the proof that the theorem holds under weaker conditions

e.g. $b_i(x)$ is bounded from below for some $i = 1, 2, \cdots, n$

(See [G-T pp 32-33] for more discussions)

2. If "$Lu \geq 0$" is replaced by "$Lu \leq 0$" then "weak minimum
($\Leftrightarrow L(-u) \geq 0$)
principle" holds.

$\min\limits_{\bar{\Omega}} u = \min\limits_{\partial\Omega} u$ $\quad$ ($\because \max(-u) = -\min(u)$)

3. If "$Lu \geq 0$" is replaced by $Lu = 0$ then

$$\max_{\bar{\Omega}} |u| = \max_{\partial\Omega} |u|.$$

4. Physical meaning of $Lu \geq 0$ in Ω

$$u_t - Lu = f(x) \begin{cases} > 0 & \text{source} \\ < 0 & \text{sink} \end{cases}$$

$-Lu = f(x) \quad Lu = -f(x) \Rightarrow f(x) \leq 0$ cold is created in Ω.

Question: What if $c(x) \not\equiv 0$.

Bad news. $Lu = u'' + u, \quad x \in (0, \pi) \quad c(x) \equiv 1.$

$L(\sin x) = 0.$

$$\max_{\bar{\Omega}} \sin x \neq \max_{\partial\Omega} \sin x$$

WMP with $c(x) \leq 0$ in Ω.

Assume (A_1)–(A_3) and $u \in C^2(\Omega) \cap C^0(\bar{\Omega})$ satisfies $Lu \geq 0$ in Ω. Then $\max_{\bar{\Omega}} u \leq \max_{\partial\Omega} u^+ \quad (*2)$

where $u^+(x) = \max\{0, u(x)\} \geq 0$

$u^-(x) = \min\{0, u(x)\} \leq 0.$

Remarks: 1. This is not true if "$c \leq 0$" is not satisfied.

Recall the above example,

$$\max_{\bar{\Omega}} \sin x = 1 \qquad \max_{\partial\Omega} (\sin x)^+ = 0.$$

2. Cannot replace "u^+" by "u" and "$\leq$" cannot be replaced by "$=$"

$Lu = u'' - u$, $\Omega = (-1,1)$, $u = -(x^2+100)$

$Lu = -2 + x^2 + 100 > 0$ $\quad \max_{\bar\Omega} u = 100$ $\quad \max_{\partial\Omega} u = -101$

$\max_{\partial\Omega} u^+ = 0$

20200914

Review: WMP : $Lu = a_{ij}D_{ij}u + b_i D_i u + c(x)u$ (Einstein Summation)

Assumptions (A1) $\Omega \subseteq \mathbb{R}^n$ bounded domain

(A2) Strict ellipticity $A(x) \geq \lambda_0 I_{n\times n}$ $\lambda > 0$ $\forall x \in \Omega$

$c(x) \equiv 0$: $u \in C^2(\Omega) \cap C^0(\bar\Omega)$ $Lu \geq 0$ in Ω $\Rightarrow \max_{\bar\Omega} u = \max_{\partial\Omega} u$

$c(x) \leq 0$ (Same conditions) $\Rightarrow \max_{\bar\Omega} u \leq \max_{\partial\Omega} u^+$

(provided $\max_{\bar\Omega} u \geq 0$ $\Leftarrow \max_{\bar\Omega} u^+ = \max_{\partial\Omega} u$)

$u^+(x) = \max\{u(x), 0\} \geq 0$

Remarks 3 If "$Lu \geq 0$" is replaced by "$Lu \leq 0$", then Let

$v = -u \Rightarrow Lv \geq 0 \xRightarrow{\text{WMP}} \max_{\bar\Omega} v \leq \max_{\partial\Omega} v^+ = \max_{\partial\Omega} (-u)^+$

$\max_{\bar\Omega} v = \max_{\bar\Omega}(-u) = -\min_{\bar\Omega} u$, $\quad \max_{\partial\Omega}(-u)^+ = -\min_{\partial\Omega}(u^-)$

$\Rightarrow \min_{\bar\Omega} u \geq \min_{\partial\Omega} u^-$

4. If $Lu = 0$ in Ω with $c \leq 0$, then

$\max_{\bar\Omega} |u| = \max_{\partial\Omega} |u|$ (Homework 1 Exercise 3)

Proof of WMP with $c \leq 0$

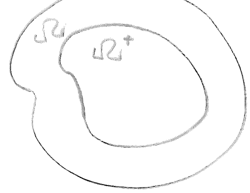
Let $\Omega^+ = \{x \in \Omega \mid u(x) > 0\} \overset{\text{open.}}{\subseteq} \Omega$

- $\Omega^+ = \emptyset \Rightarrow u \le 0$ on Ω, done
- $\Omega^+ \ne \emptyset$ Observe that on Ω^+

$$0 \le Lu = \underbrace{a_{ij} u_{x_i x_j} + b_i u_i}_{L_0 u} + \underbrace{C(x) u}_{\le 0}.$$

$\Rightarrow L_0 u \ge 0$ with " $C(x)$ of $L_0 u$ equal to 0.

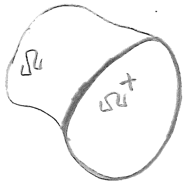
$$\overset{\text{WMP}}{\underset{(C \equiv 0)}{\Longrightarrow}} \quad \max_{\overline{\Omega^+}} u = \max_{\partial \Omega^+} u \qquad (*4)$$



Case 1 $\underline{\partial \Omega^+ \subset \Omega \Rightarrow u|_{\partial \Omega^+} \equiv 0}$

$$\Rightarrow \max_{\overline{\Omega^+}} u \overset{(*4)}{=} \max_{\partial \Omega^+} u = 0$$ a contradiction since $u|_{\Omega^+} > 0$

Case 2. $\partial \Omega^+ \cap \partial \Omega \ne \emptyset \Rightarrow \max_{\partial \Omega^+} u = \max_{\partial \Omega} u = \underline{\max_{\partial \Omega} u^+}$



$$\max_{\partial\Omega^+} u \overset{(*4)}{=} \max_{\overline{\Omega^+}} u = \underline{\max_{\overline{\Omega}} u}$$

$\square$

Comparison Principle (CP) with $c \le 0$.

Assume $(A_1)-(A_3)$ hold and $u, v \in C^2(\Omega) \cap C^0(\overline{\Omega})$ satisfy

$$\begin{cases} Lu \le Lv \\ u|_{\partial \Omega} \ge v|_{\partial \Omega} \end{cases}$$ then $u \ge v$ in Ω.

Proof: $Lu \le Lv \Rightarrow L(v-u) \ge 0$ in Ω

By WMP $(c \le 0)$ $\max (v-u) \le \max_{\partial \Omega} (v-u)^+ = 0$

$\Rightarrow v \le u$ in Ω.

$\square$

<u>Corollary</u>: Under the same conditions as in CP, the Dirichlet BVP $\begin{cases} Lu = f(x) \text{ in } \Omega \\ u|_{\partial\Omega} = \varphi(x) \end{cases}$ has at most one solution in $C^2(\Omega) \cap C^0(\bar{\Omega})$

Proof: Suppose u_1, u_2 are two solutions

$$\begin{cases} Lu_1 = Lu_2 \text{ in } \Omega \\ u_1|_{\partial\Omega} = u_2|_{\partial\Omega} \end{cases}$$

By CP $u_1 \equiv u_2$ in Ω. $\square$

More applications of WMP $(c \le 0)$

1 (DBVP) $\begin{cases} \Delta u + f(u) = 0 & \text{in } \Omega \subseteq \mathbb{R}^n \text{ bounded domain} \\ u|_{\partial\Omega} = 0 & (u: \Omega \to \mathbb{R}^n) \end{cases}$

where $f \in C^1(\mathbb{R})$ and f decrease, i.e. $f'(s) \le 0$ for all $s \in \mathbb{R}$.

Then (DBVP) has at most one solution in $C^2(\Omega) \cap C^0(\bar{\Omega})$

(Homework 1, Exercise 4)

2. Assume (A_1)–(A_3) hold and $c(x) \le 0$ in Ω. Suppose $u \in C^2(\Omega) \cap C^0(\bar{\Omega})$ and $Lu = f(x)$ in Ω. Then

$$\max_{\bar{\Omega}} |u| \le \max_{\partial\Omega} |u| + k \sup_{\Omega} |f|, \quad k = k(d_0, M) \quad (*5)$$

Comments: $(*5)$ means structural stability:

$f_a(x)$ = approximation of $f(x)$ in Ω.

φ_a = approximation of $\varphi(x) = u|_{\partial\Omega}$

$u_a(x)$ is a solution of $\begin{cases} Lu_a = f_a(x) \\ u_a(x)\big|_{\partial\Omega} = \varphi_a(x) \end{cases}$

$$\Rightarrow \begin{cases} L(u-u_a) = f(x) - f_a(x) & \text{in } \Omega \\ u - u_a = \varphi(x) - \varphi_a(x) & \text{on } \partial\Omega \end{cases}$$

$$\xRightarrow{(*5)} \max_{\bar\Omega} |u-u_a| \le \max_{\partial\Omega} |\varphi(x) - \varphi_a(x)| + k \sup_{\Omega} |f - f_a|$$

Pf. If $\sup_{\Omega} |f(x)| = \infty$, nothing needs to prove.

So we assume $\|f\|_\infty = \sup_{\Omega} |f| < \infty$. Define:

$$\bar u(x) = \max_{\partial\Omega} |u| + \left(e^{2\alpha d} - e^{\alpha(x_1+d)}\right)\|f\|_\infty, \quad d = \max_{x\in\bar\Omega} |x|$$

$$L\bar u = -\alpha^2 a_{11}(x) e^{\alpha(x_1+d)} \|f\|_\infty - \alpha b_1(x) e^{\alpha(x_1+d)} \|f\|_\infty + \underbrace{c(x)\bar u(x)}_{\le 0}$$

$\xi^T A \xi \ge \lambda_0 |\xi|^2$

$\xi = (1, 0, \dots, 0)$

$a_{11} \ge \lambda_0$

$|b_1(x)| \le M$

$$\le -\underbrace{\left[\lambda_0\alpha^2 - \alpha M\right]}_{\ge 1 \text{ if } \alpha > 0 \text{ sufficiently large}} \underbrace{e^{\alpha(x_1+d)}}_{\ge 1} \|f\|_\infty \quad (\alpha \text{ to be determined})$$

$$\le -\|f\|_\infty$$

$$\Rightarrow \begin{cases} L(\pm u) = \pm f(x) \ge -\|f\|_\infty \ge L\bar u & \text{in } \Omega \\ \pm u\big|_{\partial\Omega} \le \max_{\partial\Omega} |u| \le \bar u(x)\big|_{\partial\Omega} \end{cases}$$

$$\xRightarrow[\text{with } c \le 0]{CP} \pm u(x) \le \bar u(x) \quad \forall x \in \Omega \Rightarrow \max_{\bar\Omega} |u| \le \max_{\bar\Omega} \bar u(x)$$

$$\le \max_{\partial\Omega} |u| + \underbrace{e^{2\alpha d}}_{\triangleq k = k(\lambda_0, M)} \|f\|_\infty$$

$\square$

3 Question: What if Ω is unbounded?

Bad news: $u(x,y) = (1+x^2)y \Rightarrow \begin{cases} \Delta u = 2y & \Omega = \{(x,y) \mid y>0\} \\ u|_{\partial\Omega} = 0 \end{cases}$

$\max_{\bar\Omega} = \infty$, $\max_{\partial\Omega} u^+ = 0$

No WMP even with $c \equiv 0$

However, this can be solved if ∞ is included in $\bar\Omega$

and $\lim_{\substack{x\to\infty \\ x\in\Omega}} u(x)$ exists (possibly $\pm\infty$).

<u>Theorem</u> Suppose Ω is unbounded and any sufficiently large $R>0$. L is strictly elliptic on $\Omega \cap B_R(0)$ and b_i's are bounded on $\Omega \cap B_R(0)$.

Assume $u \in C^2(\Omega) \cap C^0(\bar\Omega)$ satisfies $Lu \geq 0$ and

$u(\infty) := \lim_{\substack{x\to\infty \\ x\in\Omega}} u(x)$ exists (possibly $\pm\infty$)

Then (i) $\sup_{\bar\Omega} u = \max(\sup_{\partial\Omega} u, u(\infty))$ if $c \equiv 0$ in Ω

(ii) $\sup_{\bar\Omega} u \leq \max(\sup_{\partial\Omega} u^+, u(\infty))$ if $c \leq 0$ in Ω

Pf: (i) Apply WMP $(c\equiv 0)$ to L on $\Omega \cap B_R(0)$

$$\Rightarrow \max_{\overline{\Omega\cap B_R(0)}} u = \max_{\partial(\Omega\cap B_R(0))} u = \max\Big(\underbrace{\max_{\partial\Omega\cap \overline{B_R(0)}} u}_{\Gamma_1}, \underbrace{\max_{(\partial B_R)\cap\bar\Omega} u}_{\Gamma_2}\Big)$$

Let $R\to\infty$ $\downarrow$

$$\sup_{\bar\Omega} u = \max\Big(\sup_{\partial\Omega} u, u(\infty)\Big)$$

(ii) Apply WMP ($C \leq 0$)

$$\max_{\overline{\Omega \cap B_R(0)}} u \leq \max_{\partial(\Omega \cap B_R(0))} u^+ = \max \left(\max_{\partial\Omega \cap \overline{B_R(0)}} u^+ , \max_{\partial B_R \cap \Omega} u^+ \right)$$

$$\downarrow \qquad\qquad \downarrow \qquad\qquad \downarrow$$

$$\sup_{\Omega} u \qquad \sup_{\partial\Omega} u^+ \qquad u(\infty)^+$$

omitted

20200921

Review: WMP ($C \equiv 0$) $Lu = a_{ij}(x) u_{x_i x_j} + b_i(x) u_{x_i} + C(x) u$

unbounded $\Omega \subseteq \mathbb{R}^n$: $\forall R > 0$ on $\Omega \cap B_R(0)$

- L is strictly elliptic
- $|b_i(x)| \leq M$ $(i = 1, 2, \cdots, n)$

Then, $\forall u \in C^2(\Omega) \cap C^0(\overline{\Omega})$. $\underbrace{\lim_{\substack{x \in \Omega \\ x \to \infty}} u}_{=: u(\infty)}$ exists (possibly $\pm\infty$)

& $Lu \geq 0$ in Ω

$\Rightarrow$ (i) $C \equiv 0$ in Ω $\quad \sup_{\Omega} u = \max \{ \sup_{\partial\Omega} u , u(\infty) \}$

(ii) $C \leq 0$ in Ω $\quad \sup_{\Omega} u \leq \max \{ \sup_{\partial\Omega} u^+ , u(\infty) \}$

Example: Let Ω be a bounded domain in $\mathbb{R}^n$ ($n \geq 3$) Suppose $u \in C^2(\Omega^c)$ &

$$\begin{cases} \Delta u = 0 \text{ in } \Omega^c \\ u(\infty) \overset{def}{=} \lim_{\substack{x \in \Omega^c \\ |x| \to \infty}} u(x) = 0 \end{cases}$$

Then $|u(x)| \leq \frac{k}{|x|^{n-2}}$ $\forall x \in \Omega^c$ for some constant $k > 0$.

Recall: Fundamental solution: $\Gamma(x) = \begin{cases} -\frac{1}{2\pi} \ln|x| & n = 2 \\ \frac{1}{4\pi|x|} & n = 3 \\ \frac{1}{(n-2) n \alpha_n |x|} & n \geq 3 \end{cases}$

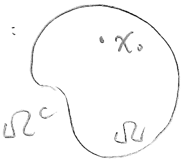
α_n = volume of a unit ball in $\mathbb{R}^n$

Dirac measure giving unit mass to $x = 0$

Check: $\Delta \Gamma(x) = 0$, $\forall x \in \mathbb{R}^n$, $x \neq 0$ (HW) $\Delta \Gamma(x) = \underline{\delta_0(x)}$

$u(x) \triangleq (\Gamma * f)(x) = \int_{\mathbb{R}^n} \Gamma(x-y) f(y)\,dy \quad f \in C_0^2(\mathbb{R}^n)$

$\Longrightarrow -\Delta u(x) = f(x),\ \forall x \in \mathbb{R}^n$ (will prove it later)

Proof:  Take $x_0 \in \Omega$ & $M > 0$ large such that

$$\underline{M\Gamma(x-x_0) \geq |u|(x),\ \forall x \in \partial\Omega}$$

$\Gamma(x-x_0)$ 大于一个最小值，
u 小于等于一个最大值

Let $v(x) = M\Gamma(x-x_0) \pm u(x)$

Then $\begin{cases} \Delta v(x) = 0 & \text{in } \Omega^c \\ v|_{\partial\Omega^c} \geq 0, & v(\infty) = 0 \end{cases}$

$\overset{\text{WMP on } \Omega^c}{\Longrightarrow}$ with $c(x) \equiv 0$

$$\inf_{\Omega^c} v = \min\{\underbrace{\inf_{\partial\Omega^c} v}_{\geq 0},\ \underbrace{v(\infty)}_{= 0}\}$$

$$= 0.$$

$\Longrightarrow v(x) \geq 0$ in Ω^c i.e. $|u| \leq \underline{M\Gamma(x-x_0)} = \frac{\tilde{M}}{|x-x_0|^{n-2}}$

$$\leq \underline{\frac{k}{|x|^{n-2}}} \quad \forall x \in \Omega^c$$

for some $k > 0$

$\square$

Remark: If in addition $u > 0$ on $\partial\Omega$, we can take $M' > 0$ small such that $\underline{M'\Gamma(x-x_0) \leq u(x) \quad x \in \partial\Omega}$

$u(x) \geq \frac{k'}{|x|^{n-2}}$ for $|x|$ large for some $k' > 0$
(on Ω^c)

Thus, if $u > 0$ on $\partial\Omega$, we have $\frac{k'}{|x|^{n-2}} \leq u(x) \leq \frac{k}{|x|^{n-2}}$ for x large. i.e. u has the same decay rate as $\Gamma(x)$ as $x \to \infty$.

1.3 Strong Maximum Principle

Baby example $Lu = u'' \geq 0$ in (a,b)

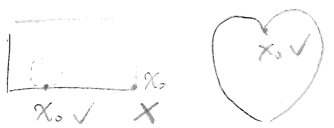
If $\max\limits_{[a,b]} u = u(x_0)$ $x_0 \in (a,b) \Rightarrow u(x) \equiv$ constant $u(x_0)$ on $[a,b]$

Q. $Lu \geq 0$ on Ω, $\max\limits_{\bar{\Omega}} u = u(x_0)$, $x_0 \in \Omega \Rightarrow u(x) \equiv u(x_0)$ on $\bar{\Omega}$?

Smoothness assumption on $\partial\Omega$.

<u>Definition</u> We say that Ω satisfies <u>interior sphere condition</u> at $x_0 \in \partial\Omega$, if there exists an open ball $B \subset \Omega$ such that $x_0 \in \partial B$

Example.



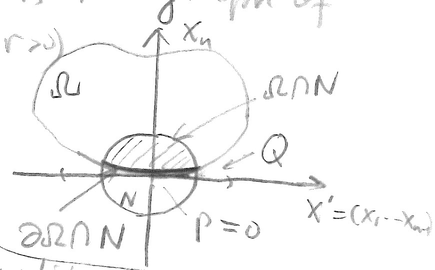
<u>Fact</u> If $\partial\Omega$ is C^2-smooth, then Ω satisfies interior sphere condition at every point on $\partial\Omega$

<u>Definition</u> We say $\partial\Omega$ is <u>C^m-smooth</u> $(m \geq 0)$ if $\forall p \in \partial\Omega$. $\exists$ a neighborhood N of p and a shift and rotation of $(x_1, \cdots, x_n)$ coordinate system s.t. p is the origin and

(i) $\exists$ a function φ, C^m-smooth in a neighborhood Q of O' in in $x' = (x_1, \cdots, x_{n-1})$ sphere s.t. $\partial\Omega \cap N$ is the graph of $x_n = \varphi(x')$ $x' \in Q$

(often take $N = B_r(p)$ for some $r > 0$)

(ii) $\Omega \cap N = \{(x', x_n) \in N \mid x_n > \varphi(x')\}$



Proof of the Fact: Without loss of generality, assume x_n axis is the inner normal direction of $\partial\Omega$ at $p = 0$

$\Longrightarrow$ • $\varphi(0')=0$ (p 在原点)

• $\nabla_{x'}\varphi(0')=0$: $f(x_1,\cdots,x_n)=\varphi(x')-x_n=0$ on $\partial\Omega\cap N$

$$\nabla_x f=(\nabla_{x'}\varphi,-1)\,//\,(0,1)$$

at $x=0$

Taylor expansion at $0'$: $\varphi\in C^2$

$$\varphi(x')=\overset{\to 0}{\varphi(0')}+\overset{\to 0}{\nabla_{x'}\varphi(0)}\,x'+\frac{1}{2}(x')^T\underbrace{[D^2_{x'}\varphi(0)]}_{\text{Hessian symmetric}}x+o(|x|^2)$$

Eigenvalues of $D^2_x\varphi(0)$ are called principal curvatures of $\partial\Omega$ at p. Now, fix a constant $C>$ max of eigenvalues of $D^2_{x'}\varphi(0)$ $\Longrightarrow$ $(x')^T D_{x'}\varphi(0)\,x'<C|x'|^2\ \ \forall\, x'\in\mathbb{R}^{n-1}$

中间用正交矩阵对角化

$$\Longrightarrow\ \varphi(x')=\frac{1}{2}(x')^T D^2_x\varphi(0)+o(|x'|^2)\le C|x'|^2\ \ (x'\approx 0)$$

($y=cx^2$)

Let $B=\{(x',x_n)\mid |x'|^2+(x_n-R)^2<R^2\}$

$$\Updownarrow$$

$$|x'|^2+x_n^2-2Rx_n+R^2<R^2\Longrightarrow x_n>\frac{|x'|^2}{2R}$$

$$|x'|^2+x_n^2<2Rx_n$$

If R is small, then

• $B\subset N$

• If $(x',x_n)\in B$, then $|x_n|>\dfrac{|x'|^2}{2R}\ge C|x'|^2\ge\varphi(x')$

$\Longrightarrow$ B stays above the function of φ.

$\Longrightarrow B\subset\Omega$

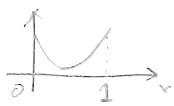
Remark. $\partial\Omega$ - C^1 smooth $\not\Longrightarrow$ interior sphere condition
$\not\Longleftarrow$

(See G-T P35)

Hopf Boundary Point Lemma

Ex. $Lu = u'' \geq 0$ in $(0,1)$ strict local max at $x=1$ & $x=0$

$\Rightarrow u'(1) = 0,\ u'(0) = 0$



Hopf boundary point lemma

Let $\Omega \subseteq \mathbb{R}^n$ be a domain (not necessarily bounded)

$|a_{ij}(x)|,\ |b_i(x)|,\ |c(x)| \leq M$ on Ω. $u \in C^2(\Omega)$ satisfies

- $Lu \geq 0$ in Ω
- u is continuous at $x_0 \in \partial\Omega$, where the interior sphere condition is satisfied $\left(\lim\limits_{\substack{x \to x_0 \\ x \in \partial\Omega}} u(x) = u(x_0)\right)$
- x_0 is strict local maximum point of u ($\exists$ nbhd of x_0, say N, s.t. $u(x) < u(x_0)$, $\forall x \in N \cap \Omega$)

Then for any outward pointing vector $\vec{v}$ at x_0, i.e. $\vec{v} \cdot (x_0 - y) > 0$ we have $\frac{\partial u}{\partial \vec{v}}(x_0) > 0$ (if it exists) provided one of the following holds: $\frac{\partial u}{\partial \vec{v}}(x_0) = \lim\limits_{h \to 0^+} \frac{u(x_0) - u(x_0 - h\vec{v})}{h}$

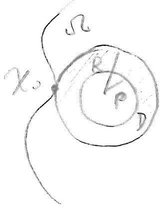
(i) $c \equiv 0$ in Ω; (ii) $c \leq 0$ in Ω. $u(x_0) \geq 0$ (iii) $u(x_0) = 0$ regardless the sign of $c(x)$.

Remarks 1 If $Lu \leq 0$ in Ω & has strict local minimum point at $x_0 \in \partial\Omega$

$\Rightarrow \frac{\partial u}{\partial \vec{v}}(x_0) < 0$ (In (ii) replace "$u(x_0) \geq 0$" by "$u(x_0) \leq 0$")

2 Only need $Lu \geq 0$ (i) and (ii) in a small neighborhood of x_0

Proof



Let $D = B_R(y) \setminus \overline{B_\rho(y)}$, we will construct $v \in C^\infty(\mathbb{R})$ under "$c(x) \leq 0$ in Ω", s.t.

(a) $Lv > 0$ in D (b) $v|_{\partial B_R} = 0$ (c) $\frac{\partial v}{\partial \vec{v}}(x_0) < 0$

Then define $w(x) = u(x) - u(x_0) + \varepsilon v(x)$, $\varepsilon > 0$ small, to be determined

$$Lw = Lu - Lu(x_0) + \varepsilon Lv$$
$$\geq -c(x) u(x_0) \geq 0 \quad \text{in } D$$

$w|_{\partial B_R(y)} \leq 0$, $w|_{\partial B_\rho(y)} \leq 0$ provided $\varepsilon > 0$ small ($\because u(x)|_{\partial B_\rho(y)} \leq u(x_0) - \delta$ for some $\delta > 0$)

$\xrightarrow[(c \leq 0)]{WMP}$ $w \leq 0$ in D.
Also $w(x_0) = 0$ $\Big\}$ $\Rightarrow \dfrac{\partial w}{\partial \vec{\nu}}(x_0) \geq 0$

$$\left(\lim_{h \to 0} \frac{w(x_0) - w(x_0 - h\vec{\nu})}{h} \right)$$

$$\frac{\partial u}{\partial \vec{\nu}}(x_0) + \frac{\partial v}{\partial \vec{\nu}}(x_0) \geq 0$$

$$\Longrightarrow \frac{\partial u}{\partial \vec{\nu}}(x_0) \geq -\varepsilon \frac{\partial v}{\partial \vec{\nu}}(x_0) > 0$$

Suppose case (iii) occurs, i.e. $u(x_0) = 0 \Rightarrow u(x) < 0$ in $B_R(y)$. Recall $0 \leq Lu = \underbrace{a_{ij}(x) u_{x_i x_j} + b_i(x) u_i + c^-(x) u}_{=: L_0 u} + c^+(x) u$, $c^+(x) u \leq 0$

$\Rightarrow L_0 u \geq 0$ in $B_R(y)$

Apply (ii) to L_0, u in $B_R(y)$ $\Rightarrow \dfrac{\partial u}{\partial \vec{\nu}}(x_0) > 0$

Construction of v

$$v(x) = e^{-\alpha|x-y|^2} - e^{\alpha R^2} \quad \alpha > 0 \text{ to be determined}$$

$\forall x \in B_R(y)$, $v(x) \geq 0$ & $v(x)|_{\partial B_R(y)} = 0$

$$v_{x_i} = e^{-\alpha|x-y|^2}(-2\alpha)(x_i - y_i), \quad \frac{\partial v}{\partial \vec{\nu}}(x_0) = (\nabla v \cdot \vec{\nu})(x_0)$$
$$= e^{-\alpha|x-y|^2}(-2\alpha)(x-y) \cdot \vec{\nu}$$

$$v_{x_i x_j} = (-2\alpha)\left[e^{-\alpha|x-y|^2}(-2\alpha)(x_i - x_j)(x_i - y_i) + e^{-\alpha|x-y|^2}\delta_{ij}\right]$$

$$= \left[4\alpha^2 (x_i - y_i)(x_j - y_j) - 2\alpha\,\delta_{ij}\right] e^{-\alpha|x-y|^2}$$

$$Lv = \sum a_{ij} v_{x_i x_j} + \sum b_i x_i + c(x)\, v$$

$$= 4\alpha^2 e^{-\alpha|x-y|^2} \overbrace{\sum_{i,j} a_{ij}(x_i - y_i)(x_j - y_j)}^{=(x-y)A(x-y)^T \geq \lambda_0 |x-y|^2} \underbrace{- 2\alpha e^{-\alpha|x-y|^2} \sum_{i=1}^{n} a_{ii}}_{\geq -2\alpha e^{-\alpha|x-y|^2} nM}$$

$$-2\alpha e^{-\alpha|x-y|^2} \underbrace{\sum b_i (x_i - y_i)}_{\text{Cauchy-Schwarz} \leq |b||x-y| \leq \sqrt{n} M |x-y|} + c(x) e^{-\alpha|x-y|^2} \underbrace{- c(x) e^{-\alpha R^2}}_{\geq 0}$$

$$\geq 4\lambda_0 e^{-\alpha|x-y|^2} |x-y|^2 \alpha^2 - 2\alpha e^{-\alpha|x-y|^2} nM - 2\alpha e^{-|x-y|^2} \sqrt{n} MR$$

$$- M e^{-\alpha|x-y|^2} + 0 \quad \text{drop off}$$

(Note that $|x-y| \geq \rho$)

$$\geq \left[4\lambda_0 \rho^2 \alpha^2 - 2\alpha(nM + \sqrt{n} MR) - M\right] e^{-\alpha|x-y|^2}$$

> 0 in D when α is taken sufficiently large. $\square$

Strong Maximum Principle Suppose L is strictly elliptic on $\Omega \subseteq \mathbb{R}^n$ (possibly unbounded) $|a_{ij}|, |b_i|, |c| \leq M$; $u \in C^2(\Omega) \cap C^0(\bar{\Omega})$ satisfies $Lu \geq 0$ in Ω and $\max_{\bar{\Omega}} u$ is attained at $x_0 \in \Omega$. Then $u \equiv \text{const} = u(x_0)$ in Ω, provided one of the following holds:
(i) $c \equiv 0$ in Ω, (ii) $c \leq 0$ in Ω & $u(x_0) \geq 0$ (iii) $u(x_0) = 0$

Proof: Let $\Omega^- = \{x \in \Omega \mid u(x) < u(x_0)\}$ open

If $\Omega^- = \emptyset$, we are done.

Assume $\Omega^- \neq \emptyset$. Claim $\partial\Omega^- \cap \Omega \neq \emptyset$.

Suppose rather $\partial\Omega^- \cap \Omega = \emptyset$. Let $\Omega_0 = \Omega \setminus \overline{\Omega^-}$

$\Rightarrow$ • Ω_0 open ; • $\Omega_0 \neq \emptyset$ ($\Leftarrow x_0 \in \Omega$)

Now $\Omega = \Omega_0 \cup (\Omega \cap \overline{\Omega^-})$

$$= \Omega_0 \cup (\underbrace{(\Omega \cap \Omega^-)}_{= \Omega^-} \cup \underbrace{(\Omega \cap \partial\Omega^-)}_{= \emptyset})$$

$= \Omega_0 \cup \Omega^-$ 竟然将连通的集合分成两个非空不相交开集的并，矛盾。

with $\Omega_0 \cap \Omega^- = \emptyset$ contradicts that Ω is connected.

Let x_1 be a point in Ω^- that is closer to $\partial\Omega^-$ than to $\partial\Omega$. Consider the largest open ball $B \subset \Omega$ having x_1 as center. Then $u(x_2) = u(x_0)$ for $x_2 \in \partial B \cap \partial\Omega^-$ while $u(x) < u(x_0)$ in B.

Apply Hopf boundary point Lemma to u on B

$\Rightarrow \nabla u(x_0) \neq 0$.

However, u attain its maximum in Ω at the interior point $x_2 \in \Omega$. Hence we must have $\nabla u(x_2) = 0$, which leads to a contradiction.

$\Rightarrow \Omega^- = \emptyset \Rightarrow u(x) \equiv u(x_0) \quad \forall x \in \Omega$

$\square$

20200928

Review

- L strictly elliptic & $|a_{ij}|, |b_i|, |c| \leq M$ on the domain $\Omega \subseteq \mathbb{R}^n$
- $u \in C^2(\Omega) \cap C^0(\overline{\Omega})$, $\max_{\overline{\Omega}} u = u(x_0)$, $x_0 \in \Omega$, $Lu \geq 0$ in Ω (fridges)
- Either one holds (i) $c(x) \equiv 0$ in Ω, (ii) $c(x) \leq 0$ in Ω, $u(x) \geq 0$ or (iii) $u(x_0) = 0$.

Then (a) (Hopf) $x_0 \in \partial\Omega$, strictly local max, interior sphere at x_0. $\Rightarrow$ $\frac{\partial u}{\partial \nu}(x_0) > 0$ ($\vec{\nu}$ outward pointing vector)

(b) (SMP) $x_0 \in \Omega \Rightarrow u(x) \equiv u(x_0)$ in Ω.

Applications:

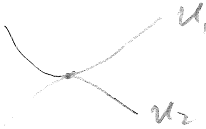
1. <u>Separation of solutions</u>: $\Omega \in \mathbb{R}$ bounded domain

$$Lu = a_{ij}(x) u_{x_i x_j}, \text{ strictly elliptic}$$

a_{ij}, b_i bounded on Ω. Suppose $u_1, u_2 \in C^2(\Omega)$ satisfy

$Lu_i = f(x, u_i)$, where for M fixed, $f_s(x,s)$ is bounded for all $x \in \Omega$ and $s \in [-M, M]$ (e.g. $f(x,s) = c(x)s^2$, $f_s(x,s) = 2s\,c(x)$)

If $u_1 \geq u_2$ in Ω and $u_1(x_0) = u_2(x_0)$ for some $x_0 \in \Omega$, then $u_1 \equiv u_2$ in Ω.



Proof: Let $\Omega_\varepsilon = \{x \in \Omega \mid \text{dist}(x, \partial\Omega) > \varepsilon\}$, $\varepsilon > 0$ small s.t. $x_0 \in \Omega_\varepsilon$. Only need to show $u_1 \equiv u_2$ on each Ω_ε.

Since $\bar{\Omega}_\varepsilon$ is compact, u_i is continuous on $\bar{\Omega}_\varepsilon$

$\Rightarrow \exists M > 0$, s.t. $|u_i(x)| \leq M$ for $i = 1, 2$, $\forall x \in \bar{\Omega}_\varepsilon$

Let $g(x,t) = f(x, tu_1 + (1-t)u_2)$, $w = u_1 - u_2$

Then $Lw = Lu_1 - Lu_2 = f(x, u_1) - f(x, u_2) = g(x,1) - g(x,0)$

$$= \int_0^1 \frac{\partial g}{\partial t}(x,t)\,dt$$

$$= \int_0^1 \frac{\partial f}{\partial s}(x, tu_1 + (1-t)u_2)(u_1 - u_2)\,dt$$

$$= \underbrace{\int_0^1 \frac{\partial f}{\partial s}(x, tu_1 + (1-t)u_2)\,dt}_{=:\,c(x)} \cdot w(x)$$

$C(x)$ is bounded on Ω_ε.

$$\left.\begin{array}{l} \Rightarrow Lw - C(x)\, w = 0 \text{ in } \Omega_\varepsilon \\ w \geq 0 \text{ in } \Omega_\varepsilon \\ w(x_0) = 0 \end{array}\right\} \xRightarrow{\text{SMP(iii)}} w = 0 \text{ in } \Omega_\varepsilon$$

2. Suppose $\Omega \subseteq \mathbb{R}^n$ bounded domain satisfies interior sphere condition at every point $p \in \partial\Omega$. Consider

$$\begin{cases} \Delta u = 0 & \text{in } \Omega \\ \dfrac{\partial u}{\partial \vec{\nu}} = 0 & \text{on } \partial\Omega \end{cases}$$

$\vec{\nu}$ - any outward pointing vector field on $\partial\Omega$.

Then every solution $u \in C^1(\bar{\Omega}) \cap C^2(\Omega)$ is a constant.

Proof. Let $x_0 \in \bar{\Omega}$ s.t. $\max\limits_{\bar{\Omega}} u = u(x_0)$

Case 1: $x_0 \in \Omega$, SMP implies that $u \equiv u(x_0)$

Case 2 $\max\limits_{\bar{\Omega}} u$ is not achieved in Ω, $x_0 \in \partial\Omega$

$\Rightarrow x_0$ strict local max $\xRightarrow{(i)} \dfrac{\partial u}{\partial \nu}(x_0) > 0$ contradiction □

3. Comparison Principle (i) Ω bounded (ii) Interior sphere condition at every point on $\partial\Omega$. (iii) L is strictly elliptic on Ω. (iv) a_{ij}, b_i, c bounded on Ω. (v) $c \leq 0$ on Ω.

Suppose $u, v \in C^2(\Omega) \cap C^1(\bar{\Omega})$ such that

$$\begin{cases} Lu \geq Lv \text{ in } \Omega \\ \dfrac{\partial u}{\partial \vec{\nu}} + \beta(x)\, u \leq \dfrac{\partial u}{\partial \vec{\nu}} + \beta(x) v \end{cases}$$

$\beta(x) \geq 0$, not $\equiv 0$ on $\partial\Omega$

$\vec{\nu}$ outward pointing vector

Then $u \leq v$ on Ω.

Remeark 1. Recall "old CP" from WMP: (i), (iii), (iv)' $\|b_i\| \le M$ & (v)

$$\begin{cases} Lu \ge Lv & \text{in } \Omega \\ u \le v & \text{on } \partial\Omega \end{cases} \Rightarrow u \le v \text{ in } \Omega \quad (\text{i.e. } \beta(x) = \infty \text{ on } \partial\Omega)$$

Remark 2. (Robin BVP)

$$\begin{cases} Lu = f(x) \\ \frac{\partial u}{\partial \vec{\nu}} + \beta(x) u = g(x), \quad x \in \partial\Omega \end{cases}$$ has at most one solution in $C^2(\Omega) \cap C^1(\bar{\Omega})$

Proof: Let $w = u - v$. Then $$\begin{cases} Lw = Lu - Lv \ge 0 \text{ in } \Omega \\ \frac{\partial w}{\partial \vec{\nu}} + \beta(x) w \le 0 \text{ on } \partial\Omega \end{cases}$$

Let $M = \max_{\bar{\Omega}} w$

• $M \le 0$ done. • $M > 0$

Case 1 $x_0 \in \Omega \xRightarrow{SMP(ii)} w(x) \equiv w(x_0)$, $x \in \Omega$

$\Rightarrow$ Boundary condition cannot hold.

Case 2. M is achieved only on $\partial\Omega$.

$\Rightarrow x_0 \in \partial\Omega$ & $w(x_0)$ is strictly local max

$\xRightarrow{Hopf(ii)} \frac{\partial w}{\partial \vec{\nu}}(x_0) > 0$.

$\Rightarrow$ Boundary condition will hold at x_0

$\Rightarrow \max_{\bar{\Omega}} w > 0$ is wrong $\square$