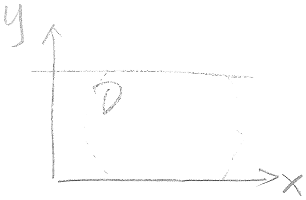
2. Classical Maximum Principles for 2nd order Parabolic Equations



2.1 Parabolic equations

Let $D \subseteq \mathbb{R}^{n+1} = \{(x,t) \mid x \in \mathbb{R}^n, t \in \mathbb{R}\}$ bounded domain

D is between the hyperplanes $t=0$ & $t=T$

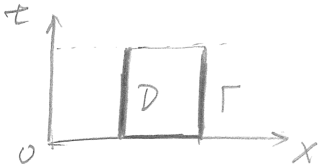
with $\bar{D} \cap \{t=0\} \neq \emptyset$, $D \cap \{t=T\} \neq \emptyset$

<u>Definition</u> Define the <u>parabolic boundary</u> of D as 抛物边界

$$\Gamma = \overline{\partial D \cap \{0 \le t < T\}}$$

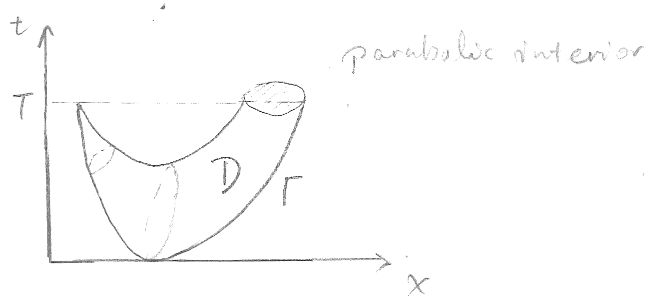
Define the <u>parabolic interior</u> of D to be $\bar{D} \setminus \Gamma$.

Ex 1 $\Omega \subseteq \mathbb{R}^n$ bounded domain $D = \Omega \times (0,T)$

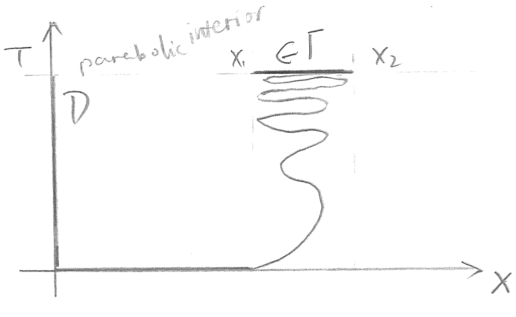


$\Gamma = \partial\Omega \times [0,T] \cup \Omega \times \{t=0\}$ $\quad \bar{D} \setminus \Gamma = \Omega \times (0,T]$

Ex 2.



Ex 3



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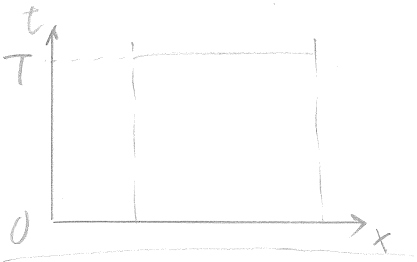
Review

$u(x,t)$ $(x,t) \in D \subseteq \mathbb{R}^{n+1}$ domain

e.g. $D = \Omega \times (0,T)$ $\quad \Gamma = (\Omega \times \{t=0\}) \cup (\partial\Omega \times [0,T])$

parabolic boundary

$\bar{D} \setminus \Gamma$ — parabolic interior



Fact: $(x_0, T) \in$ parabolic interior i.e. $\bar{D}\setminus\Gamma$.

$$\Rightarrow \exists\, \delta > 0 \text{ s.t. } B_\delta((x_0,T)) \cap D = \underbrace{B_\delta((x_0,T)) \cap \{t<T\}}_{=:B_\delta^-}$$

Proof: Since $(x_0,T) \notin \Gamma$ & Γ is compact.

$\exists\, \delta > 0$ small s.t. $B_\delta(x_0,T) \cap \Gamma = \emptyset$

$$B_\delta^- = \underbrace{(B_\delta^- \cap D)}_{\text{open}} \cup (B_\delta^- \cap D^c)$$

$$= B_\delta^- \cap \bar{D}^c \cup (\partial D \cap \{t=T\})$$

$$= (B_\delta^- \cap \bar{D}^c) \cup (B_\delta^- \cap \Gamma) \cup (B_\delta^- \cap \underbrace{(\partial D \cap \{t=T\})}_{\emptyset})$$

$$= \underbrace{B_\delta^- \cap \bar{D}^c}_{\text{open}}$$

$$(B_\delta^- \cap D) \cap (B_\delta^- \cap \bar{D}^c) = \emptyset$$

$\Rightarrow$ $B_\delta^- \cap \bar{D}^c = \emptyset$

$\Rightarrow$ $B_\delta^- \cap D = B_\delta^-$

$\Rightarrow$ $B_\delta^- \subset D$

Consider operator

$$Lu = \frac{\partial u}{\partial t} - [a_{ij}(x,t)\, u_{x_i x_j} + b_i(x,t)\, u_{x_i} + c(x,t)\, u]$$

e.g. $Lu = \dfrac{\partial u}{\partial t} - a^2 \Delta u = f(x,t)$ $\begin{cases} >0 & \text{oven} \\ <0 & \text{fridge} \end{cases}$

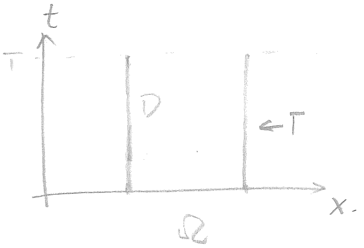
Definition We say that L is strictly parabolic in D if $\exists$ constant $\lambda_0 > 0$ s.t. $(a_{ij}(x,t))_{n\times n} \geq \lambda_0 I_{n\times n}$, $\forall (x,t) \in \bar{D}\setminus\Gamma$

i.e. $\sum_{i,j=1}^{n} a_{ij}(x,t)\,\xi_i\xi_j \geq \lambda_0 |\xi|^2$, $\forall (x,t)\in \bar{D}\setminus\Gamma$, $\forall \xi \in \mathbb{R}^n$

$\xi = (\xi_1, \dots, \xi_n)$

Weak maximum principle

Intuition



Expect: $\max_{\bar{D}} u = \max_{\Gamma} u$

$D = \Omega \times (0,T)$

$u_t - \Delta u = f(x,t) \leq 0$

Weak maximum principle ($C \equiv 0$) need only ($A \geq 0$ degenerate parabolic) (WMP)

Suppose L is strictly parabolic (positive semidefinite) with $C \equiv 0$ in a bounded domain, $D \subset \mathbb{R}^{n+1}$ $u \in C^{2,1}(\bar{D}\setminus\Gamma) \cap C^0(D)$ satisfies $Lu \leq 0$ in $\bar{D}\setminus\Gamma$. Then $\max_{\bar{D}} u = \max_{\Gamma} u$.

Proof: First consider the special case $Lu < 0$ in $\bar{D}\setminus\Gamma$

Let $(x_0,t_0) \in \bar{D}$ such that $u(x_0,t_0) = \max_{\bar{D}} u$

Case 1 $(x_0,t_0) \in \Gamma$ Done.

Case 2. $(x_0,t_0) \in \bar{D}\setminus\Gamma$

negative semidefinite

At (x_0,t_0), $u_t \geq 0$. $\nabla_x u = 0$ $(u_{x_ix_j})_{n\times n} \leq 0$

$$Lu = \underbrace{u_t}_{\geq 0} - [\underbrace{a_{ij}u_{x_ix_j}}_{\leq 0} + \underbrace{b_i u_{x_i}}_{=0}]$$

need only $A \geq 0$

$Lu \geq 0$ Contradiction.

General case $Lu \le 0$ in $\bar{D} \setminus \Gamma$

Let $v_\varepsilon = u - \varepsilon t$ $\quad \varepsilon > 0$ small

$L v_\varepsilon = Lu - \varepsilon L(t) = Lu - \varepsilon < 0$ in $\bar{D} \setminus \Gamma$

Recall in the elliptic case. $v_\varepsilon = u + \varepsilon e^{\alpha x_1}$

$L v_\varepsilon \ge \varepsilon e^{\alpha x_1} [\underbrace{\lambda_0}_{\text{strictly elliptic}} \alpha^2 - M\alpha]$

Special case $\Longrightarrow$ $\max_{\bar{D}} v_\varepsilon = \max_{\Gamma} v_\varepsilon$

Let $\underset{\text{uniformly convergence}}{\varepsilon \to 0} \Rightarrow \max_{\bar{D}} u = \max_{\Gamma} u$ $\quad \square$

Remark: In the proof, we only need $\underline{(a_{ij}(x,t))_{n\times n} \ge 0}$ in $\bar{D} \setminus \Gamma$
(different from the elliptic case) $\quad$ (degenerate parabolic condition)

In particular, when $a_{ij}(x,t) \equiv 0$ $\forall i, j$.

$$Lu = \frac{\partial u}{\partial t} - b_i(x,t) u_{x_i} \quad \text{transport equation.}$$

we have WMP ($c \equiv 0$).

WMP ($c \le 0$) Let L be degenerately parabolic with $c \le 0$ in a bounded domain. $D \subseteq \mathbb{R}^{n+1}$. Suppose $u \in C^{2,1}(\bar{D} \setminus \Gamma) \cap C^0(\bar{D})$ satisfies $Lu \le 0$ in $\bar{D} \setminus \Gamma$. Then $\max_{\bar{D}} u \le \max_{\Gamma} u^+$

Proof Case 1. $\max_{\bar{D}} u \le 0$ Trivial

Case 2. $\max_{\bar{D}} u > 0$ The proof is the same as WMP($c \equiv 0$) $\quad \square$

(In fact, $\max_{\bar{D}} u = \max_{\Gamma} u$ when $\max_{\bar{D}} u > 0$)

<u>Theorem</u>. Let L be degenerately parabolic with $c(x,t) \le M$ in a bounded domain. $D \subset \mathbb{R}^{n+1}$. Suppose $u \in C^0(\bar{D}) \cap C^{2,1}(\bar{D} \setminus \Gamma)$ satisfies $Lu \le 0$ in $\bar{D} \setminus \Gamma$, $u|_\Gamma \le 0$. Then $u \le 0$ in $\bar{D}$.

Proof: Let $v(x,t) = u(x,t)e^{-Mt}$. WTS $v \le 0$ in $\bar{D}$.

$$Lv = v_t - [a_{ij}v_{x_ix_j} + b_i v_i + cv]$$
$$= u_t e^{-Mt} - Mv - e^{-Mt}[a_{ij}u_{x_ix_j} + b_i u_{x_i} + cu]$$

$$\underbrace{Lv + Mv}_{=: \tilde{L}v} = v_t - [a_{ij}v_{x_ix_j} + b_i v_i + (c-M)v]$$
$$= e^{-Mt}\left(u_t - [a_{ij}u_{x_ix_j} + b_i u_{x_i} + cu]\right)$$
$$\le 0$$

Apply WMP ($c \le 0$) to $\tilde{L}v \le 0 \Rightarrow \max_{\bar{D}} v \le \max_{\Gamma} v^+ = 0$

$\Rightarrow \quad v \le 0$ in $\bar{D}$

$\Rightarrow \quad u \le 0$ in $\bar{D}$ $\square$

Comparison Principle Same assumptions on L, $c(x)$ and D as in the previous Theorem.

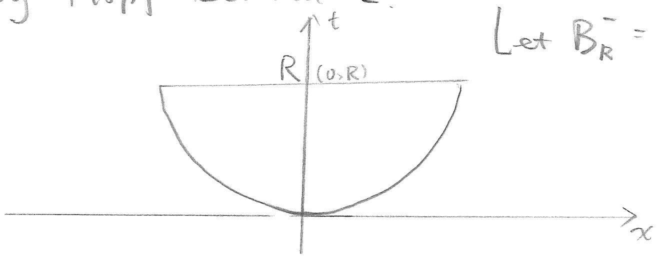
Suppose $u, v \in C^{2,1}(\bar{D}\setminus\Gamma) \cap C^0(\bar{D})$ satisfy

$$\begin{cases} Lu \ge Lv & \text{in } \bar{D}\setminus\Gamma \\ u|_\Gamma \ge v|_\Gamma \end{cases} \quad \text{Then } u \ge v \text{ in } \bar{D}$$

Proof Concider $v - u$ $\square$

2.3 Hopf boundary point lemma.

Baby Hopf Lemma 1.



Let $B_R^- = \{(x,t) \in \mathbb{R}^n \times \mathbb{R} \mid |x|^2 + (t-R)^2 < R^2, \ 0 < t \le R\}$

Suppose L is strictly parabolic in B_R^- and a_{ij}, b_i and c are bounded in B_R^-. Let $u \in C^{2,1}(B_R^-) \cap C^0(\overline{B_R^-})$ satisfy

$$Lu \leq 0 \quad \text{on } B_R^- \text{ (fridge)}$$

Assume there exists a $p_0 = (x_0, t_0) \in \partial B_R^- \cap \{0 < t < R\}$,

$$u(x_0, t_0) > u(x, t) \quad \text{for all } (x, t) \in \overline{B_R^-} \setminus \{p_0\}$$

Then for any outward pointing vector $\vec{\eta}$ at p_0, we have $\frac{\partial u}{\partial \vec{\eta}}(p_0) > 0$ if it exists.

provided the following holds

(i) $c \equiv 0$ (ii) $c \leq 0$ in B_R^- and $u(p_0) \geq 0$ (iii) $u(p_0) = 0$.

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Review $B_R^- = \{(x,t) \mid |x|^2 + (t-R)^2 < R^2,\ 0 < t \leq R\}$

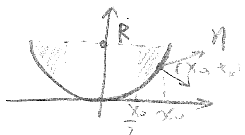
- L strictly parabolic $\|a_{ij}, b_i, c\|_{L^\infty} \leq M$
- $u \in C^{2,1}(B_R^-) \cap C^0(\overline{B_R^-})$, $Lu \leq 0$ in B_R^-

Assume $\exists\ p_0 = (x_0, t_0) \in \partial B_R^- \cap \{0 < t < R\}$, $u(p_0) > u(x,t)$ $\forall (x,t) \in \overline{B_R^-} \setminus \{p_0\}$

Then $\frac{\partial u}{\partial \vec{\eta}}(p_0) > 0$ if it exists. $\vec{\eta}$ is any outward pointing vector at p_0, provided one of the three cases holds

(i) $c(x) \equiv 0$. (ii) $c(x) \leq 0$ $u(p_0) \geq 0$; (iii) $u(p_0) = 0$.

Proof of Hopf Lemma 1.



To prove (i) & (ii), we will construct $0 \leq v \in C^\infty(\mathbb{R}^{n+1})$, s.t.

(a) $Lv < 0$ in $\Sigma = \{(x,t) \in B_R^- \mid |x| > \frac{|x_0|}{2}\}$

(b) $v|_{\Gamma_{B_R^-}} = 0$ $\leftarrow$ parabolic boundary

(c) $\frac{\partial v}{\partial \vec{\eta}}(p_0) < 0$

Then define $w(x,t) = u(x,t) - u(p_0) + \varepsilon v(x,t)$, $\varepsilon > 0$ bounded

Now apply WMP ($c \le 0$) on Σ to $w(x,t)$

- $Lw = Lu + c(x,t)u(p_0) + \varepsilon Lv < 0$ in Σ
- $w|_{\Gamma_\Sigma} \le 0$ ($\{|x| = \frac{|x_0|}{2}\} \cap \bar{B}_R^-$ is cpt, ε sufficiently small)

$\Rightarrow \max_{\bar{\Sigma}} w \le \max_{\Gamma_\Sigma} w^+ = 0 \Rightarrow \left.\begin{array}{l} w \le 0 \text{ in } \bar{\Sigma} \\ \text{note } w(p_0) = 0 \end{array}\right\} \Rightarrow \frac{\partial w}{\partial \eta}(p_0) \ge 0$

$$\Rightarrow \quad \frac{\partial w}{\partial \eta}(p_0) = \frac{\partial u}{\partial \eta}(p_0) + \varepsilon \frac{\partial v}{\partial \eta}(p_0) \ge 0$$

$$\Rightarrow \quad \frac{\partial u}{\partial \eta}(p_0) > 0.$$

If (iii) holds, then $u < 0$ on $\bar{B}_R^- \setminus \{p_0\}$

$$0 \ge Lu = \underbrace{u_t - [a_{ij}u_{x_ix_j} + b_i u_{x_i} + c^- u}_{=: \tilde{L}u \le 0} + c^+ u]$$

define $c^- = \min\{c(x,t), 0\} \le 0$

$c^+ = \max\{c(x,t), 0\} \ge 0$

$c(x,t) = c^- + c^+$.

Apply (ii) to $\tilde{L} \Rightarrow \frac{\partial u}{\partial \eta}(p_0) > 0$.

Now construct v if $c \le 0$ in B_R^-

$$v(x,t) = \underbrace{e^{-\alpha(|x|^2 + (t-R)^2)}}_{=: E(x,t)} - e^{-\alpha R^2} \quad (\alpha > 0 \text{ to be chosen})$$

$\Rightarrow$ (b) holds.

(c): $\frac{\partial v}{\partial \eta}(p_0) = \nabla v \cdot \vec{\eta}$

$= E(-2\alpha)\underbrace{(x, t-R) \cdot \vec{\eta}}_{\text{angle} < 90^\circ} > 0$.

< 0.

$v_{x_i} = E \cdot (-2\alpha x_i)$

$v_t = E(-2\alpha(t-R))$

$v_{x_i x_j} = E(-2\alpha x_j)(-2\alpha x_i) + E(-2\alpha)\delta_{ij}$

(a): $Lv \le 0$ in Σ

$$Lv = E\{-2\alpha(t-R) - \underbrace{a_{ij}(4\alpha^2)x_i x_j}_{4\alpha^2\lambda_0|x|^2 \text{ (strict parabolic)}} + 2\alpha a_{ij}\delta_{ij} + 2\alpha b_i x_i - c\}$$

$$Lv \le \qquad + \underbrace{\cancel{C e^{-\alpha R^2}}}_{\le 0}$$

$$Lv \le E\{2\alpha R - 4\alpha^2\lambda_0|x|^2 + 2\alpha\sum_{i=1}^{n} a_{ii} + 2\alpha|b||x| + M\}$$

Cauchy-Schwartz

$$\le E\{2\alpha R - 4\alpha\lambda_0\frac{|x_0|^2}{4} + 2\alpha nM + 2\alpha\sqrt{n}MR + M\}$$

$$= E\{-\lambda|x_0|^2R^2 + (2R + 2nM + 2\sqrt{n}MR)\alpha + M\}$$

< 0 provided $\alpha > 0$ sufficiently large $\square$

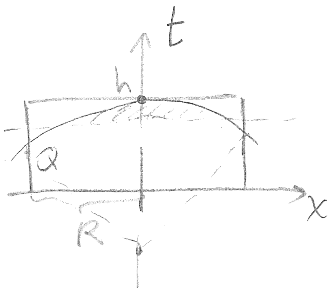
Remarks:

1. This lemma is true if $\bar{B_R}$ is replaced by B_R, provided p_0 is neither the South pole nor the North pole. (产生矛盾)

2. True if $\bar{B_R}$ is shifted but not rotated.

Baby Hopf Lemma 2

Let $Q(R,h) = \{(x,t) \in \mathbb{R}^n \times \mathbb{R} \mid |x| < R,\ 0 \le t \le h\}$



Suppose L is strictly parabolic on $Q(R,h)$ and $\|a_{ij}, b_i, C\|_{L^\infty(Q(R,h))}$

Assume $u \in C^{2,1}(Q(R,h)) \cap C^0(\overline{Q(R,h)})$

$Lu \le 0$ in $Q(R,h)$

$u(0,h) > u(x,t) \quad \forall (x,t)\ |x| \le R,\ t \in (0,h)$

Then $\frac{\partial u}{\partial t}(0,h) > 0$ provided one of the followings hold.

(i) $C \equiv 0$ (ii) $C \le 0$ in $Q(R,h)$ & $u(0,h) \ge 0$ (iii) $u(0,h) = 0$

Remark. However, at $(0,h)$

$$0 \geq Lu = u_t - (\underbrace{a_{ij}u_{x_ix_j}}_{\leq 0} + \underbrace{bu_{x_i}}_{=0} + \underbrace{cu}_{\leq 0})$$

$\Rightarrow u_t(0,h) > 0$ will give a contradiction

Corollary. The conditions in Baby Hopf Lem 2 cannot hold at the same time. In particular, $\nexists u$ s.t. $Lu \leq 0$ in $Q(R,h)$, $u(0,h) > u(x,t)$ $\forall |x| < R$, $t \in (0,h)$ with (i) or (ii) or (iii)

Pf of Baby Hopf Lemma 2.

To prove (i) & (ii), take a $\underline{\text{large } \rho < 0}$ & small $\delta > 0$ such that $N = B_\rho(0,h-\rho) \cap \{t > h-\delta\} \subset Q(R,h)$

Shall construct $0 \leq v \in C^\infty(\mathbb{R}^{n+1})$ s.t.

(a) $Lv < 0$ in N

(b) $v|_{\partial B_\rho(0,h-\rho)} = 0$

(c) $\frac{\partial v}{\partial t}(0,h) < 0$.

Construct:

$$v(x,t) = \rho^2 - |x|^2 - (t-(h-\rho))^2$$

$$= \rho^2 - [|x|^2 + (t-(h-\rho))^2] \Rightarrow (b)$$

$$\leq -2(t-h)\rho \leq 2\delta\rho \quad \text{in } N.$$

$$\frac{\partial v}{\partial t}\Big|_{(0,h)} = -2(t-(h-\rho)) = -2\rho < 0 \Rightarrow (c)$$

(a) $Lv = v_t - [a_{ij} v_{x_i x_j} + b_i v_{x_i} + cv]$

$$= -2(t-(h-\rho)) + 2a_{ij}\delta_{ij} + 2b_i x_i - cv$$

$$\le 2\rho(-1+\delta M) + 2\delta + 2M(n+\sqrt{n}R) < 0$$

if ρ is big, δ is small

Let $w = u - u(0,h) + \varepsilon v$.

- $Lw < 0$ in N if (i) or (ii) holds
- $w|_{\Gamma_N} \le 0$ ∂N, parabolic boundary

By WMP ($c \le 0$)

$\Rightarrow \max_{\bar{N}} w \le \max_{\Gamma_N} w^+ = 0.$

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Review $D \subseteq \mathbb{R}^{n+1}$ bounded domain

$$Lu = u_t - [a_{ij}(x,t)\, u_{x_i x_j} + b_i(x,t)\, u_{x_i} + c(x,t)\, u]$$

(A) L is strictly parabolic in $\bar{D}\setminus\Gamma$, $\|a_{ij}, b_i, c\|_{L^\infty(D)} < M$

$u \in C^{2,1}(\bar{D}\setminus\Gamma) \cap C^0(\bar{D})$, $Lu \le 0$ (fridge) in $\bar{D}\setminus\Gamma$.

Baby Hopf Lemma 1 $\bar{D}\setminus\Gamma = B_R^- = \{(x,t) \in \mathbb{R}^n\times\mathbb{R} \,/\, |x| + (t-R)^2 < R^2$ $0<t\le R\}$

Assume (A) holds, if $\exists\, p_0 = (x_0,t_0) \in \partial B_R^- \cap \{0<t<R\}$

s.t. $u(p_0) > u(x,t)$, $\forall (x,t) \in \bar{B}_R^- \setminus \{p_0\}$

$\Rightarrow$ $\frac{\partial u}{\partial \vec{\eta}}(p_0) > 0$, $\vec{\eta}$ any outward pointing vector

provided (i) $c \equiv 0$ in $\bar{D}\setminus\Gamma$ or (ii) $c \le 0$ in $\bar{D}\setminus\Gamma$ and $u(p_0) \ge 0$

or (iii) $u(p_0) = 0$

Remarks
- Works for B_R & $p_0 = (x_0,t_0) \in \partial B_R \cap \{0<t<2R\}$
- Can shift B_R^- & B_R

Baby Hopf Lemma 2

$$\bar{D}\setminus T = Q(R,h) = \{(x,t)\in\mathbb{R}^n\times\mathbb{R} \mid |x|<R,\ 0<t\le h\}$$

Assume (A) holds. If $p = (0,h)$ $u(p_0) > u(x,t)$ $\forall (x,t)\in Q(R,h) \cap \{0\le t<h\}$

$\Longrightarrow$ $\frac{\partial u}{\partial t}(0,h) > 0$ provided (i) or (ii) or (iii) holds

However at p_0 (maximal point)

$$Lu = \underbrace{u_t}_{>0} - [\underbrace{a_{ij}\,u_{x_ix_j}}_{\substack{\le 0 \\ \text{strictly parab.}}} + \underbrace{b_i u_{x_i}}_{\substack{=0 \\ \text{maximal}}} + \underbrace{cu}_{\substack{\le 0 \\ \text{(i),(ii) or (iii)}}}\} > 0$$

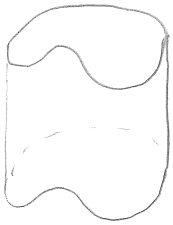
contradict with $Lu\le 0$ in $\bar{D}\setminus T$

Hopf Boundary Point Lemma (grown-up version)

Assume (A) holds. Suppose $\exists\, p_0\in\partial D$ satisfies interior sphere condition at p_0 with $\overrightarrow{OP_0} \not\parallel$ t-axis (i.e. p is neither the south pole nor the north pole) and $u(p_0) > u(p)$, $\forall p\in D$. Then $\frac{\partial u}{\partial \vec\eta}(p_0) > 0$ (if it exists) for any outward pointing vector $\vec\eta$ at p_0, provided

(i) $c\equiv 0$ or (ii) $c\le 0$ in $\bar{D}\setminus T$, $u(p_0)\ge 0$ or (iii) $u(p_0)=0$.

e.g.



$D = \Omega\times(0,T)$

$\partial\Omega$ is C^2

Proof: It follows from Baby Hopf 1 with $B_{\bar R}$ replaced by B_R

$\square$

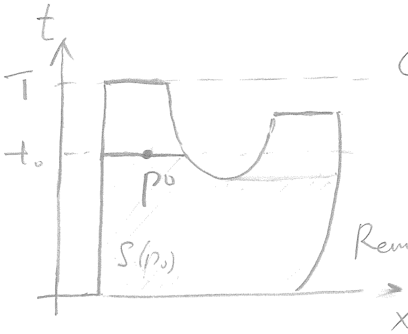
2.4 Strong Maximum Principle

Notation: $\forall\, p_0 \in \bar{D}\setminus\Gamma$ define

$S(p_0) = \{ Q \in D \mid \exists$ continuous path γ connecting p_0 & Q s.t. $\gamma \subset D$, and when traveling along γ from Q to p_0, its t-coordinate is <u>non-decreasing</u> $\}$

$\Gamma = \partial D \cap \{0 \le t < T\}$

e.g



$C(p_0) = S(p_0) \cap \{t = t_0\}$

Remark: $S(p_0)$ & $C(p_0)$ are both connected (since they are both path-connected)

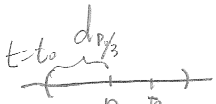
Strong Maximum Principle:

Suppose that (A) holds. If there exists a $p_0 = (x, t_0) \in \bar{D}\setminus\Gamma$ such that $u(p_0) = \max_{\bar{D}} u$, then $u \equiv u(p_0) (=: M)$ on $S(p_0)$, provided (i) $C \equiv 0$ in D or (ii) $C \le 0$ in D $u(p_0) \ge 0$ or (iii) $u(p_0) = 0$

Proof: Let $F = \{(x,t) \in \bar{D} \mid u(x,t) = M\}$ WTS $S(p_0) \subset F$

Let $d_{p_0} = \mathrm{dist}(p_0, \Gamma) > 0$ (since Γ is compact & $p_0 \notin \Gamma$)

Claim 1. $B_{\frac{d_{p_0}}{3}}(p_0) = \{(x,t) \mid |x - x_0| \le \frac{d_{p_0}}{3}\} \subset F$



Otherwise, $\exists\, \bar{p}(\bar{x}, t_0) \in B_{\frac{d_{p_0}}{3}}(p_0)$ s.t. $u(\bar{p}) < M$.

Let $\delta = \frac{1}{2}\mathrm{dist}(\bar{p}, F) > 0$ (since F is compact and $\bar{p} \notin F$)

Define a semi-ellipsoid

$$E_\sigma = \left\{(x,t) \;\middle|\; \frac{|x-\bar{x}|^2}{(\sigma\delta)^2} + \frac{|t-t_0|^2}{\delta^2} < 1,\ t \le t_0 \right\}, \quad \sigma > 0$$

Observations: • If $0 < \sigma < 1$, then $E_\sigma \cap F = \emptyset$ (since $\mathrm{dist}(\bar{p}, F) = 2\delta$)

• If $\sigma\delta \ge \frac{d_{p_0}}{3}$ then $\left.\begin{array}{l} p_0 \in \bar{E}_\sigma \\ p_0 \in F \end{array}\right\} \Rightarrow \bar{E}_\sigma \cap F \ne \emptyset$

Then increasing σ, we have that E_σ touches F at some $\tilde{p} \in \bar{D}\setminus\Gamma$ before touching Γ. Why? $\text{dist}(\bar{p},\Gamma) = |\bar{p}' - \bar{p}|$

$$\geq |\bar{p}' - p_0| - |\bar{p} - p_0|$$

$$\geq \text{dist}(p_0, \Gamma) - \text{dist}(p_0, \bar{p})$$

$$\geq d_{p_0} - \frac{d_{p_0}}{3} = \frac{2}{3} d_{p_0} > \frac{1}{3} d_{p_0}$$

$$\geq \text{dist}(p_0, \bar{p})$$

$$\geq \text{dist}(\bar{p}, F) = 2\delta$$

$\tilde{p}$ cannot be the south pole of E_σ. Otherwise,

$$|\bar{p} - \tilde{p}| = \delta = \frac{1}{2}\text{dist}(\bar{p}, F), \text{ but } \tilde{p} \in F.$$

So $|\bar{p} - \tilde{p}| \geq \text{dist}(\bar{p}, F) = 2\delta$, contradiction

Now we can construct a ball B inscribed in $\bar{E}_\sigma$, tangent to F at $\tilde{p}$. $\tilde{p} \neq$ south pole of B. Then by Baby Hopf

$1 \Rightarrow \exists\, \vec{\eta}$ s.t. $\frac{\partial u}{\partial \eta}(\tilde{p}) > 0$.

But $\tilde{p} \in \bar{D}\setminus\Gamma$, $\tilde{p} \in F$ (so $u(\tilde{p}) = \max_{\bar{D}} u$) $\Rightarrow \nabla_{(x,t)} u(\tilde{p}) = 0$

($\nabla_x u(\tilde{p}) = 0$ obvious, if $\tilde{t} < T$ $\frac{\partial u}{\partial t}(\tilde{p}) = 0$ ✓;

if $\tilde{p}(\tilde{x},\tilde{t})$, $\tilde{t} = T$ & $\frac{\partial u}{\partial t}(\tilde{p}) > 0$ contradicts $Lu \leq 0$)

$\frac{\partial u}{\partial \eta}(\tilde{p}) = \nabla_{(x,t)} u(\tilde{p})\, \vec{\eta} = 0$ contradicts $\frac{\partial u}{\partial \eta}(\tilde{p}) > 0$

Claim 1 holds.

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Review

$Lu = u_t - [a_{ij} u_{x_i x_j} + b_i u_{x_i} + cu]$ in $\bar{D} \setminus \Gamma$

$\|a_{ij}, b_i, c\|_{L^\infty(D)} < M$, L strictly parabolic

$Lu \le 0$ in $\bar{D} \setminus \Gamma$. If $\exists\, p_0 = (x_0, t_0) \in \bar{D} \setminus \Gamma$ s.t.

$u \equiv u(p_0) \triangleq M$ on $S(p_0)$, provided (i) $c \equiv 0$ in D or

(ii) $c \le 0$ in D & $M \ge 0$

(iii) $M = 0$

Pf. $F \triangleq \{(x,t) \in \bar{D} \mid u(x,t) = M\}$

WTS $S(P_0) \subseteq F$ $\quad d_{p_0} = \mathrm{dist}(p_0, \Gamma) > 0$

Claim 1 $B_{d_{p_0}/3} = \{(x, t_0) \mid |x - x_0| < d_{p_0}/3\} \subset F$

claim 2 $C(P_0) \subset F$

By Claim 1, $F \cap C(P_0)$ is relatively open in $C(P_0)$ and

$F \cap C(P_0) = \emptyset$

Also, $F \cap C(P_0)$ is relatively closed in $C(P_0)$. Since $C(P_0)$ is connected. $F \cap C(P_0) = C(P_0) \Rightarrow$ Claim 2.

Claim 3 $u \equiv M$ on $S(P_0)$

Otherwise $\exists\, Q = (x_0, t_0) \in S(P_0)$ s.t. $u(Q) < M \Rightarrow Q \notin F$.

Let $p_1 = (x_1, t_1)$ be the first intersection of γ (a path connecting Q with p_0) with F (when going upwards) Then on arc $\widehat{QP_1}$ $u < u(P_0)$

By claim 2 $u < u(P_0)$ on $S(P_0) \cap \{t_2 \le t < t_1\}$

so we can construct a cylinder

$$Q_{P_1}(R,h) = \{(x,t) \mid |x - x_1| < R,\ t-h < t \le t_1\}$$

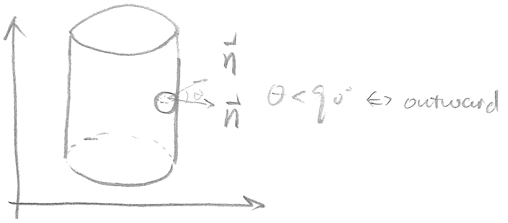
By Corollary of Baby Hopf 2, this is impossible $\Rightarrow$ Claim 3 $\square$

Application (Comparison principle (CP))

$\Omega \subseteq \mathbb{R}^n$ bounded domain with $\partial\Omega \in C^2$, $D = \Omega \times (0,T)$.

$S = \partial\Omega \times [0,T]$. Let $Lu = u_t - a_{ij}(x,t) + b_i(x,t)u_i + f(x,t,u)$
符号无所谓
$(x,t) \in \bar{D} \setminus \Gamma$

$Bu = \frac{\partial u}{\partial \vec{\eta}}(x,t) + \beta(x,t)u$, $(x,t) \in S$, $\beta \ge 0$ on S. $\vec{\eta}$ outward pointing vector field on S.



- L is strictly parabolic on D & a_{ij}, b_i bounded on $\bar{D} \setminus \Gamma$
- $\forall (x,t) \in \bar{D} \setminus \Gamma$, $|f_u(x,t,u)| \le M$ $\forall (x,t) \in \bar{D} \setminus \Gamma$, $|u| \le R$

Assume $u, v \in C^{2,1}(\bar{D})$ s.t. $\begin{cases} Lu \ge Lv & \text{in } \bar{D} \setminus \Gamma \\ Bu \ge Bv & \text{on } S \\ u|_{t=0} \ge v|_{t=0} & \text{on } \Omega \end{cases}$

Then $u \ge v$ on $\bar{D}$. Moreover, ~~if any one of the three "$\ge$" is strict at~~ some point, ~~then $u > v$ if $x \in \bar{\Omega}, t > 0$~~ ~~in particular~~, if $u|_{t=0} \not\equiv v|_{t=0}$, then $u > v$ if $x \in \bar{\Omega}, t > 0$

Remarks $\beta \equiv 0$ on S is allowed. (c.f. the elliptic eq case $\beta \ge 0, \not\equiv 0$ on $\partial\Omega$)

This also holds if $Bu = u$ on S. ("$x \in \bar{\Omega}$" replaced by "$x \in \Omega$") (WMP $\Rightarrow u \ge v$ in $\bar{D}$).

Proof: Let $w = v - u$, WTS $w \le 0$ in D.

$$w_t - a_{ij}(x,t)w_{x_ix_j} + b_i(x,t)w_{x_i} + \underbrace{f(x,t,v) - f(x,t,u)}_{\substack{= f_u(x,t,\xi)(v-u) \triangleq C(x,t)w \\ \text{mean value theorem} \\ (\xi = \xi(x,t) \text{ between } v \text{ and } u)}} \ge 0$$

semilinear

Let $\tilde{w} = e^{-Mt} w$, $\tilde{w}_t = e^{-Mt} w_t - Me^{-Mt} w$,

where $|f_u(x,t,u)| \le M$ (u, v are bounded, so is ξ)

$$\tilde{w}_t - \{a_{ij}\tilde{w}_{x_ix_j} - b_i\tilde{w}_{x_i} - \underbrace{[c(x,t)+M]}_{\le 0 \text{ in } \bar{D}\setminus\Gamma}\tilde{w}\} \le 0 \text{ in } \bar{D}\setminus\Gamma$$

Case 1 $\max_{\bar{D}} \tilde{w} < 0$, done

Case 2. $\max_{\bar{D}} \tilde{w} \ge 0$
- (a) $\max_{\bar{D}} \tilde{w}$ is attained in some $(x_0,t_0) \in \bar{D}\setminus\Gamma$
- (b) $\max_{\bar{D}} \tilde{w}$ is not achieved in $\bar{D}\setminus\Gamma$ but at some $(x_0,t_0) \in \Gamma$.

(a). By SMP ($c \le 0$ Case (ii) & $\max \ge 0$) $\tilde{w} \equiv$ const for $0 \le t \le t_0$

But $\tilde{w}(x,0) = w(x,0) = v(x,0) - u(x,0) = 0$.

$\Rightarrow \tilde{w} \equiv 0$ for $0 \le t \le t_0$

$\Rightarrow \max_{\bar{D}} \tilde{w} = \tilde{w}(x,t_0) = 0$

$\Rightarrow \tilde{w} \le 0$ and hence $w \le 0$ in $\bar{D}$

If $u \ge v$ at $t=0$ is strict at some point, then Case 2(a) cannot happen.

(b) (b_1) If $(x_0,t_0) \in S$. Baby Hopf 1 $\Rightarrow \frac{\partial \tilde{w}}{\partial \eta}(x_0,t_0) > 0$

But $B\tilde{w} = \underbrace{\frac{\partial \tilde{w}}{\partial \eta}}_{>0} + \underbrace{\beta\tilde{w}}_{\ge 0} = e^{-Mt} Bw \le 0$ on S.

at (x_0,t_0) (by $Bu \ge Bv$ on S)

Contradiction

(b2) $(x_0, t_0) \notin S \Rightarrow t_0 = 0$

Then $\max_{\bar{D}} \tilde{w} = \tilde{w}(x_0, 0) \leq 0 \Rightarrow \max_{\bar{D}} \tilde{w} = 0$

$(\tilde{w}(x, 0) \leq 0) \Rightarrow \tilde{w}$ & $w \leq 0$ in $\bar{D}$

Since $\max_{\bar{D}} \tilde{w} = 0$ is attained only at the bottom of $\bar{D}$

i.e. $t = 0$. We see that $\tilde{w}$ & $w \leq 0$ for $x \in \bar{\Omega}$, $t > 0$ $\square$

For the remark: In the case $Bu = u$ on S (Dirichlet B.C.) we only need to modify the Pf in (b) as follows:

$$\max_{\bar{D}} \tilde{w} = \max_{\Gamma} w \leq 0 \Rightarrow \tilde{w} \leq 0 \text{ in } \bar{D}$$

Since $\max_{\bar{D}} \tilde{w}$ is not attained in $\bar{D} \setminus \Gamma$, we have $\tilde{w} < 0$ in $\bar{D} \setminus \Gamma = \Omega \times (0, T]$.