3. Distributions

Ω domain in $\mathbb{R}^n$. (bounded or not)

$\mathcal{D}(\Omega) = C_0^\infty(\Omega) = \{u \in C^\infty(\Omega) \mid \overline{\operatorname{supp} u} \text{ is compact in } \Omega\}$

$\operatorname{supp} u = \{x \in \Omega \mid u(x) \neq 0\}$ $\Leftrightarrow$ supp u $\Leftrightarrow$ $\exists$ cpt K, $\operatorname{supp} u \subset K \subset \Omega$

If $u \in \mathcal{D}(\Omega)$, u is compactly supported.

Ex. 1 The standard molifier

$$j(x) = \begin{cases} c\, e^{\frac{1}{|x|^2-1}} & x \in \mathbb{R}^n,\ |x| < 1 \\ 0 & x \in \mathbb{R}^n,\ |x| \geq 1 \end{cases}$$

where c is chosen s.t. $\int_{\mathbb{R}^n} j(x)\,dx = 1$.

Claim. $j(x) \in \mathcal{D}(\mathbb{R}^n) = C_0^\infty(\mathbb{R}^n)$

Pf. $\operatorname{supp} j(x) = B_1(0)$ (open) $\Rightarrow j(x)$ is compactly supported.

Consider $f(t) = \begin{cases} c\, e^{\frac{1}{t-1}} & 0 < t < 1 \\ 0 & t \geq 1 \end{cases} \Rightarrow j(x) = f(|x|^2)$

To show $j(x) \in C^\infty(\mathbb{R}^n)$, just need to show $f \in C^\infty([0,\infty))$

How about $t = 1$? $f'(1^+) = 0$ $\quad f'(1^-) = \lim_{t \to 1^-} \frac{f(t) - f(1)}{t-1}$

$$= \lim_{t \to 1} \frac{c\, e^{\frac{1}{t-1}}}{t-1}$$

$\Rightarrow f'(1)$ exists and $f'(1) = 0$ $\quad = \lim_{s \to -\infty} c e^s \cdot s$

$f''(1^+) = 0$

$$f''(1^-) = \lim_{t \to 1^-} \frac{f'(t) - f'(1)}{t-1} = \lim_{t \to 1^-} \frac{c e^{\frac{1}{t-1}} (-1)(t-1)^{-2}}{t-1} = \lim_{t \to 1^-} \frac{-c e^{\frac{1}{t-1}}}{(t-1)^3}$$

$$= \lim_{s \to -\infty} -c e^s \cdot s^3 = 0$$

Similarly $f^{(k)}(1) = 0 \quad \forall k \geq 1 \Rightarrow f \in C^\infty([0,\infty))$ $\square$

($C^\infty(\Omega) \neq C^\omega(\Omega)$) The molifier is not analytic in Ω

Is $\mathcal{D}(\Omega) \neq \emptyset$?

Fix a point $a \in \Omega$. take $\varepsilon > 0$ small such that $\overline{B_\varepsilon(a)} \subset \Omega$

Let $j_{\varepsilon,a}(x) = \frac{1}{\varepsilon^n} j\left(\frac{x-a}{\varepsilon}\right)$ $\quad \left|\frac{x-a}{\varepsilon}\right| < 1 \iff |x-a| < \varepsilon$

$\Rightarrow j_{\varepsilon,a} \in \mathcal{D}(\Omega)$. $\operatorname{supp} j_{\varepsilon,a} = B_\varepsilon(a)$ (open)

& $\int_\Omega j_{\varepsilon,a}(x)\,dx \overset{y=\frac{x-a}{\varepsilon}}{\underset{dy=\left(\frac{1}{\varepsilon}\right)^n dx}{=\!=}} \int_{\Omega \to \mathbb{R}^n} \frac{1}{\varepsilon^n} j\left(\frac{x-a}{\varepsilon}\right) dx$

$$= \int_{\mathbb{R}^n} \frac{1}{\varepsilon^n} j(y)\, \varepsilon^n\, dy = 1 \qquad \square$$

<u>Definition</u>. A distribution on $\mathcal{D}(\Omega)$ is a linear functional

$$f : \mathcal{D}(\Omega) \longrightarrow \mathbb{R}. \quad \varphi \longmapsto f(\varphi) \overset{\text{Notation}}{=} \langle f, \varphi \rangle$$

s.t. f is continuous on $\mathcal{D}(\Omega)$ in the following sense

For any $\{\varphi_k\} \subset \mathcal{D}(\Omega)$, s.t

(i) $\operatorname{supp} \varphi_k \subset$ fixed compact $K \in \Omega$

(ii) $\forall \alpha \in (\alpha_1, \cdots, \alpha_n) \in \mathbb{Z}^n$, $\alpha_i \geq 0$

$$\lim_{k\to\infty} \|\partial^\alpha \varphi_k\|_{L^\infty(\Omega)} = 0 \quad \text{where } \partial^\alpha = \partial_{x_1}^{\alpha_1} \partial_{x_2}^{\alpha_2} \cdots \partial_{x_n}^{\alpha_n}$$

we have $\langle f, \varphi_k \rangle \to 0$ as $k \to \infty$.

f is continuous at $\varphi \equiv 0$. $\forall \varphi_k \to 0$ in $\mathcal{D}(\Omega)$

WTS $\underbrace{\langle f, \varphi_k \rangle}_{= f(\varphi_k)} \to \underbrace{\langle f, 0 \rangle}_{= f(0) = 0}$ $\quad \mathcal{D}(\Omega) = C_0^\infty(\Omega)$

$\mathcal{D}(\Omega) = X$. $\mathcal{D}'(\Omega) =$ set of all distributions on $\mathcal{D}(\Omega)$.

$= X'$ (dual space)

Ex. 2. Let $f \in L^1_{loc}(\Omega)$ (i.e. $\forall$ compact $K \subset \Omega$, $f \in L^1(K)$, i.e. $\int_K |f| dx < \infty$)

Define $F \in \mathcal{D}(\Omega)$ as $\langle F, \varphi \rangle \triangleq \int_\Omega f \varphi \, dx \in \mathbb{R}$ $\triangleq \langle f, \varphi \rangle$

Claim. $F \in \mathcal{D}'(\Omega)$. Suppose $\{\varphi_k\}_{k=1}^\infty$ satisfies conditions (i) and (ii) in the above definition

$$|\langle F, \varphi_k \rangle| = \left| \int_\Omega f \varphi_k \, dx \right| \leq \int_K |f(x)| \, \|\varphi_k\|_{L^\infty(\Omega)} dx \xrightarrow{k\to\infty} 0.$$

(by (ii) with $\alpha = (0, \cdots, 0)$ $\Rightarrow \|\varphi_k\|_{L^\infty(\Omega)} \to 0$)

F is called the distribution induced by $f(x)$ (often write $F = f$)

Ex. 3. Suppose $f \in \mathcal{D}'(\Omega)$, $g \in C^\infty(\Omega)$

Define: $\langle fg, \varphi \rangle = \langle f, \underbrace{g\varphi}_{\in C_0^\infty(\Omega) = \mathcal{D}(\Omega)} \rangle$ $\quad \forall \varphi \in \mathcal{D}(\Omega)$

$\Rightarrow fg := gf \in \mathcal{D}'(\Omega)$ (distr. $\cdot\, C^\infty$ = distr.)

Ex. 4. Fix an $a \in \Omega$, define

$$\delta_a \quad \mathcal{D}(\Omega) \to \mathbb{R}, \quad \langle \delta_a, \varphi \rangle = \varphi(a) \Rightarrow \delta_a \in \mathcal{D}'(\Omega)$$

δ-"function" $\delta_a(x) = \begin{cases} \infty & x = a \\ 0 & x \neq a \end{cases}$ and $\int_{\mathbb{R}^n} \delta_a(x) dx = 1$

$\Rightarrow \int_{\mathbb{R}^n} \delta_a(x) \varphi(x) dx = \varphi(a)$ if φ is continuous at $x = a$

- δ_a is the Dirac measure giving the unit mass to the point a
- Discrete case Krunecker delta $\delta_{ij} = \begin{cases} 1 & i = j \\ 0 & i \neq j \end{cases}$

<u>Definition</u> Let $\{f_k\}_{k=1}^{\infty} \subset \mathcal{D}'(\Omega)$, $f \in \mathcal{D}'(\Omega)$ we say $f_k \to f$ as $k \to \infty$ if $\forall \varphi \in \mathcal{D}(\Omega)$, $\langle f_k, \varphi \rangle \to \langle f, \varphi \rangle$ as $k \to \infty$.

<u>Theorem</u> (Spiky) Let $\{f_k\}_{k=1}^{\infty} \subset L^1(\Omega)$ s.t

(i) $\{f_k\}$ concentrates at $a \in \Omega$: $\forall$ small $\delta > 0$

$$\int_{\Omega \setminus B_\delta(a)} |f_k| \, dx \xrightarrow{k \to \infty} 0$$

(ii) $\|f_k\|_{L^1(\Omega)} \le$ const. M. $\forall k \ge 1$.

(iii) $\int_\Omega f_k(x) \, dx \to A$ as $k \to \infty$

Then f_k (understood as distributions) $\longrightarrow A\delta_a$ as $k \to \infty$

Proof. Just need to show $\forall \varphi \in \mathcal{D}(\Omega)$

$$\begin{array}{ccc} \langle f_k(x), \varphi \rangle & \xrightarrow{k \to \infty} & \langle A\delta_a, \varphi \rangle \\ \| & & \| \\ \int_\Omega f_k \varphi \, dx & & A\varphi(a) \end{array}$$

$$\text{LHS} = \int_\Omega f_k(x) (\varphi_k(x) - \varphi(a)) \, dx + \int_\Omega f_k(x) \varphi(a) \, dx$$

Since φ is continuous at $x = a$, $\forall \varepsilon > 0 \ \exists \delta > 0$ s.t

$$|\varphi(a) - \varphi(x)| < \varepsilon \quad \text{if } |x - a| < \delta, \ x \in \Omega.$$

Now
$$|I_k| \le \int_{\Omega \setminus B_\delta(a)} | \quad | \, dx + \int_{\Omega \cap B_\delta(a)} | \quad | \, dx$$
$$\le 2 \|\varphi\|_{L^\infty(\Omega)} \cdot \int_{\Omega \setminus B_\delta(a)} |f_k(x)| \, dx$$
$$+ \underbrace{\varepsilon \int_{\Omega \cap B_\delta(a)} |f_k| \, dx}_{\le \varepsilon M.}$$

Take $\limsup_{k\to\infty}$ of both sides

$\overset{(1)}{\Longrightarrow} \limsup_{k\to\infty} |I_k| \le 0 + M\varepsilon$. ε arbitrary

$\Longrightarrow \lim_{k\to\infty} |I_k| = 0$ $\square$

Ex. 5$_\triangle$ Fix an $a\in\Omega$, then $j_{\varepsilon,a} \to \delta_a$ as $\varepsilon\to 0$ in the sense of distributions. $(k=\frac{1}{\varepsilon})$

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Review $\Omega\subset\mathbb{R}^n$ bounded or not

$D(\Omega) = C_0^\infty(\Omega)$. $D'(\Omega) = \{ f: D(\Omega)\to\mathbb{R} \mid f$ is linear and $\underset{\Updownarrow}{\text{continuous}}\}$

f is continuous at $\varphi = 0$.

$(*)$
$$\left\{\begin{array}{l} \forall \{\varphi_k\}\subset D(\Omega) \quad \operatorname{supp}\varphi_k \subset K \text{ compact}, \forall k \\ \|D^\alpha \varphi_k\|_{L^\infty(\Omega)} \xrightarrow{k\to\infty} 0 \quad \forall \alpha = (\alpha_1,\dots,\alpha_n),\ \alpha_i\in\mathbb{Z}, \alpha_i\ge 0 \\ \Rightarrow f(\varphi_k) (= \langle f, \varphi_k\rangle) \to 0. \end{array}\right.$$

Derivatives of distributions

Motivation $f\in C^1(\Omega) \Rightarrow f, f_{x_i} \in L^1_{loc}(\Omega)$

$\left(\begin{array}{l}\forall K \text{ compact in } \Omega \\ \int_K |f|\,dx < \infty,\ \int_K |f_{x_i}|\,dx < \infty \end{array}\right)$

$$\langle f_{x_i}, \varphi\rangle = \int_\Omega f_{x_i}\varphi\,dx = \int_\Omega \left[(f\varphi)_{x_i} - f\varphi_{x_i}\right] dx$$

$$= \cancelto{0}{\int_{\partial\Omega} f\varphi \nu_i \,d\sigma} - \int_\Omega f\varphi_{x_i}\,dx$$

since $\varphi\in C_0^\infty(\Omega)$

$$= -\int_\Omega f\varphi_{x_i}\,dx$$

Definition $\forall f \in \mathcal{D}'(\Omega)$, define its distributional derivatives

$$\frac{\partial f}{\partial x_i} (= f_{x_i} = D_{x_i} f = D_i f) \text{ by}$$

$$\langle D_i f, \varphi \rangle \triangleq -\langle f, \varphi_{x_i} \rangle, \ \forall \varphi \in \mathcal{D}(\Omega)$$

Fact $D_i f \in \mathcal{D}'(\Omega)$

Linearity: $\langle D_i f, \alpha_1\varphi_1 + \alpha_2\varphi_2 \rangle = -\langle f, (\alpha_1\varphi_1 + \alpha_2\varphi_2)_{x_i} \rangle$

$$= -\langle f, \alpha_1(\varphi_1)_{x_i} + \alpha_2(\varphi_2)_{x_i} \rangle$$

$$= -[\alpha_1 \langle f, (\varphi_1)_{x_i} \rangle + \alpha_2 \langle f, (\varphi_2)_{x_i} \rangle]$$

$$= \alpha_1 \langle D_i f, \varphi_1 \rangle + \alpha_2 \langle D_i f, \varphi_2 \rangle$$

$D_i f$ is continuous at $\varphi = 0$: Take $\{\varphi_k\} \subset \mathcal{D}(\Omega)$ as in $(*)$,

$$\langle D_i f, \varphi_k \rangle = -\langle f, (\varphi_k)_{x_i} \rangle = -\langle f, \tilde{\varphi}_k \rangle \xrightarrow{k \to \infty} 0$$

(since $\{\tilde{\varphi}_k\}$ satisfies $*$, and f is continuous at $\varphi = 0$)

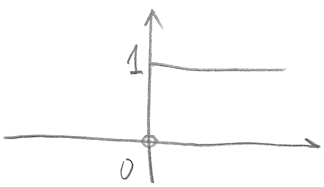
In general, $\forall \alpha \in \mathbb{Z}^n$, $\alpha = (\alpha_1, \dots, \alpha_n)$, $\alpha_i \geq 0$ define the distributional derivative $D^\alpha f$ by

$$\langle D^\alpha f, \varphi \rangle = (-1)^{|\alpha|} \langle f, D^\alpha \varphi \rangle, \ \forall \varphi \in \mathcal{D}(\Omega) \ (|\alpha| = \alpha_1 + \dots + \alpha_n)$$

$\Rightarrow D^\alpha f \in \mathcal{D}'(\Omega)$

Ex. Heaviside function $H(x) = \begin{cases} 0, & x < 0 \\ 1 & x \geq 0 \end{cases}$

- $H(x) \in C^1(\mathbb{R})$
- $H \in L^1_{loc}(\mathbb{R})$



Question:

$DH(x) = H'(x)$ (in the sense of distribution) $= ?$

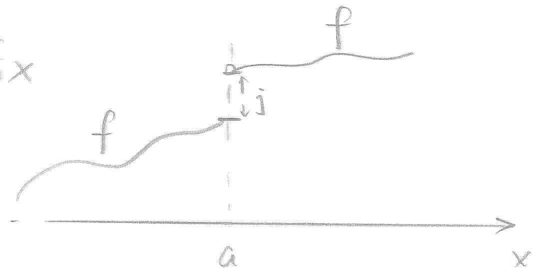
$\forall \varphi \in \mathcal{D}(\mathbb{R})$

$$\langle DH, \varphi \rangle = -\langle H, \varphi'(x) \rangle = -\int_{\mathbb{R}} H(x)\varphi'(x)\,dx$$

$$= \int_0^\infty \varphi'(x)\,dx = -\varphi(x)\Big]_0^\infty = \varphi(0)$$

$\Rightarrow DH = \delta_0$

Ex



$f \in L^1_{loc}(\mathbb{R})$

$f \in C^1((-\infty, a]) \cup C^1((a, +\infty))$

Question:

$Df = ?$ (in the sense of distribution)

$$Df = \underset{\in L^1_{loc}(\mathbb{R})}{f'(x)} + j\,\delta_a(x) \qquad f'(x) = \begin{cases} f'(x) \text{ classical derivatives } & (x \neq a) \\ \text{any value} & (x = a) \end{cases}$$

$\forall \varphi \in \mathcal{D}(\mathbb{R})$

$$\langle Df, \varphi \rangle \overset{def}{=} -\langle f, \varphi'(x) \rangle = -\int_{\mathbb{R}} f\varphi'(x)\,dx$$

$$= -\int_{-\infty}^a f\varphi'\,dx - \int_a^\infty f\varphi'\,dx$$

$$= (f\varphi)(-\infty) - (f\varphi)(a^-) + \int_{-\infty}^a f'\varphi\,dx$$

$$+ (f\varphi)(a^+) - (f\varphi)(\infty) + \int_a^\infty f'\varphi\,dx$$

$$= \int_{\mathbb{R}} f'\varphi\,dx + [f(a^+) - f(a^-)]\,\varphi(a)$$

$$= \langle f', \varphi \rangle + \langle j\,\delta_a, \varphi \rangle$$

Distributional solutions of PDEs

Consider $L: \mathcal{D}'(\Omega) \to \mathcal{D}'(\Omega)$

$$u \mapsto Lu \triangleq \sum_{|\alpha|=0}^{m} A_\alpha(x) D^\alpha u \ . \ A_\alpha(x) \in C^\infty(\Omega)$$

(in the sense of distribution)

$(f \in \mathcal{D}'(\Omega) \ . \ g \in C^\infty(\Omega)$

e.g.) $fg = gf \in \mathcal{D}'(\Omega)$

$\langle fg, \varphi \rangle \triangleq \langle f, g\varphi \rangle \ . \ \forall \varphi \in C_c^\infty(\Omega)$

$\forall \varphi \in \mathcal{D}(\Omega)$

$$\langle Lu, \varphi \rangle = \langle \sum_{|\alpha|=0}^{m} A_\alpha(x) D^\alpha u, \varphi \rangle$$

$$= \sum_{|\alpha|=0}^{m} \langle A_\alpha(x) D^\alpha u, \varphi \rangle$$

$$= \sum_{|\alpha|=0}^{m} \langle D^\alpha u, A_\alpha(x) \varphi \rangle$$

$$= \sum_{|\alpha|=0}^{m} (-1)^\alpha \langle u, D^\alpha [A_\alpha(x) \varphi] \rangle$$

$$= \langle u, \underbrace{\sum_{|\alpha|=0}^{m} (-1)^\alpha D^\alpha [A_\alpha(x) \varphi]}_{\triangleq L^* \varphi} \rangle$$

$$\langle Lu, \varphi \rangle = \langle u, L^* \varphi \rangle$$

L^* adjoint operator of L.

$L^* \quad \mathcal{D}(\Omega) \longrightarrow \mathcal{D}(\Omega)$

Ex. $L = \Delta$ Question: $L^* = ?$

$$m = 2 \quad A_\alpha(x) = \begin{cases} 1 & \alpha_i = 2 \text{ for some } i \in \{1, 2, \dots, n\}, \ \alpha_j = 0 \text{ if } j \neq i \\ 0 & \text{otherwise} \end{cases}$$

$$L^* \varphi = \sum_{i=1}^{n} (-1)^2 D_{x_i x_i} \varphi = \Delta \varphi$$

$L = \Delta : \mathcal{D}'(\Omega) \to \mathcal{D}'(\Omega)$ (in the sense of distribution)

$L^* = \Delta : \mathcal{D}(\Omega) \to \mathcal{D}(\Omega)$ (classical derivative)

We say Δ is symmetric: $\langle \Delta u, \varphi \rangle = \langle u, \Delta \varphi \rangle$

<u>Definition</u> Consider PDE

$$Lu = f \quad f \in \mathcal{D}'(\Omega) \text{ given}$$

(e.g. $-\Delta u = f(x)$)

If $u \in \mathcal{D}'(\Omega)$ and $Lu = f$ in the sense of distribution.

i.e. $\forall \varphi \in \mathcal{D}(\Omega)$

$$\underset{\left(\langle u, L^* \varphi \rangle\right)}{\overset{\parallel}{\langle Lu, \varphi \rangle}} = \langle f, \varphi \rangle$$

then we say u is a distributional solution of the PDE.

Remark: Classical solution must be a distributional solution

Ex. $-\Delta u = \delta_y$ in $\mathbb{R}^n$, $y \in \mathbb{R}^n$ fixed $(*1)$

<u>Definition</u> A distributional solution of $(*1)$ is called a fundamental solution (F.S)

Question Uniqueness? (No. If u is a FS, then $u+C$ is also an FS)

Recall: $u(x) = \begin{cases} -\frac{1}{2\pi} \ln|x-y| & n \geq 2 \quad (C_1 \ln r) \\ \dfrac{1}{(n-2)\,\omega_n |x-y|^{n-2}} & n \geq 3 \quad (C_2 \frac{1}{r^{n-2}}) \end{cases}$

ω_n: surface area of unit sphere in $\mathbb{R}^n$

$n\alpha_n = \omega_n$ (α_n: volume of $B_1(0)$)

Satisfies $-\Delta u = 0$ in $\mathbb{R}^n \setminus \{y\}$

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Review u is called a fundamental solution of

$$-\Delta u = \delta_y \quad \text{in } \mathbb{R}^n, \quad y \in \mathbb{R}^n \text{ fixed} \quad (*1)$$

if $u \in \mathcal{D}'(\mathbb{R}^n)$ and $\forall \varphi \in \mathcal{D}(\mathbb{R}^n) = C_0^\infty(\mathbb{R}^n) = C_c^\infty(\mathbb{R}^n)$

$$\langle -\Delta u, \varphi \rangle = \langle \delta_y, \varphi \rangle \quad (*2)$$

$$\langle -u, \Delta\varphi \rangle \qquad \overset{\text{def}}{=} \varphi(y)$$

Recall $u(x) = \begin{cases} -\frac{1}{2\pi} \ln|x-y| & n=2 \\ \frac{1}{(n-2)\omega_n |x-y|^{n-2}} & n \geq 3 \end{cases} \quad (*3)$

$\Delta u = 0, \ \forall x \neq y$

Claim The $u(x)$ defined as in $(*3)$ is a fundamental solution of $(*1)$

Proof

$$\int_{\mathbb{R}^n} u \Delta\varphi \, dx = \int_{B_\varepsilon(y)} u \Delta\varphi \, dx + \int_{\mathbb{R}^n \setminus B_\varepsilon(y)} u \Delta\varphi$$

$$= \int_{B_\varepsilon(y)} u \Delta\varphi \, dx + \int_{\partial B_\varepsilon(y)} u \frac{\partial\varphi}{\partial\nu} \, d\sigma - \int_{\partial B_\varepsilon(y)} \varphi \frac{\partial u}{\partial\nu} \, d\sigma + \int_{\mathbb{R}^n \setminus B_\varepsilon(y)} \underbrace{(\Delta u)}_{=0} \varphi \, dx^{\to 0}$$

$$\int_\Omega u\Delta\varphi\,dx = \int_{\partial\Omega} u \nabla\varphi\cdot\nu\,d\sigma - \int_{\partial\Omega}\varphi\nabla u\cdot\nu\,d\sigma$$

$$\Delta u \cdot \varphi\, dx$$

① $\int_{B_\varepsilon(y)} u \Delta\varphi \, dx \xrightarrow{\varepsilon \to 0} 0$

$$\left| \int_{B_\varepsilon(y)} u \Delta\varphi \, dx \right| \leq \int_{B_\varepsilon(y)} |u| |\Delta\varphi| \, dx \leq \|\Delta\varphi\|_{L^\infty} \int_{B_\varepsilon(y)} |u| \, dx$$

$n=2$

$$\int_{B_\varepsilon(y)} |u| \, dx = \int_{B_\varepsilon(y)} -\frac{1}{2\pi} |x-y| \, dx = \int_0^{2\pi} \int_0^\varepsilon -\frac{1}{2\pi} \ln r \, d\theta \quad \text{(translation)}$$

$$= -\int_0^\varepsilon (\ln r) \, r \, dr$$

$$= -\left[\frac{r^2}{2} \ln r - \frac{r^2}{4} \right]_0^\varepsilon = \frac{\varepsilon}{4} - \frac{\varepsilon^2}{2} \ln \varepsilon \longrightarrow 0$$

as $\varepsilon \to 0$

(Byproduct : $u \in L^1_{loc}(\mathbb{R}^n)$ but $u \notin L^1(\mathbb{R}^n)$ for $n=2$)

$n \geq 3$
$$\int_{B_\varepsilon(y)} |u|\, dx = \int_{B_\varepsilon(y)} \frac{1}{(n-2)\omega_n} |x-y|^{-(n-2)}\, dx$$

$$= \int_{\partial B_1(y)} \int_0^\varepsilon \frac{1}{(n-2)\omega_n} r^{-(n-2)} \cdot r^{n-1}\, dr\, d\sigma$$

(ω_n : area of unit sphere in $\mathbb{R}^n$)

$$= \frac{1}{n-2}\int_0^\varepsilon r\, dr = \frac{1}{n-2}\,\frac{1}{2} r^2 \Big|_0^\varepsilon = \frac{1}{2(n-2)}\varepsilon^2 \xrightarrow{\varepsilon \to 0} 0$$

($u \in L^1_{loc}(\mathbb{R}^n)$ also for $n \geq 3$)

② $\displaystyle\int_{\partial B_\varepsilon(y)} u \frac{\partial \varphi}{\partial \nu}\, d\sigma \xrightarrow{\varepsilon \to 0} 0$

$$\int_{\partial B_\varepsilon(y)} \left| u \frac{\partial \varphi}{\partial \nu}\right| d\sigma \leq \int_{\partial B_\varepsilon(y)} |u|\,|\nabla \varphi|\, d\sigma \quad (\because |\nabla\varphi \cdot \nu| \leq |\nabla \varphi|\cdot|\nu|)$$

$$\leq \|\nabla \varphi\|_{L^\infty} \int_{\partial B_\varepsilon(y)} |u|\, d\sigma \quad (*4)$$

$$(*4) \leq \|\nabla\varphi\|_{L^\infty}\left[-\frac{1}{2\pi}\cdot \ln \varepsilon\right]\cdot 2\pi\varepsilon \longrightarrow 0 \text{ as } \varepsilon \to 0$$

$n \geq 3$. $\displaystyle (*4) \leq \|\nabla\varphi\|_{L^\infty} \frac{1}{(n-2)\omega_n}\varepsilon^{-(n-2)}\omega_n \varepsilon^{n-1} = \|\nabla\varphi\|_{L^\infty}\, \varepsilon/(n-2) \longrightarrow 0$ as $\varepsilon \to 0$

③ Calculate $\displaystyle\int_{\partial B_\varepsilon(y)} \varphi \frac{\partial u}{\partial \nu}\, d\sigma$: $\displaystyle\frac{\partial u}{\partial \nu} = \nabla u \cdot \nu = \begin{cases} -\frac{1}{2\pi}\frac{1}{|x-y|}\frac{x-y}{|x-y|}\cdot\frac{y-x}{|y-x|} = \frac{1}{2\pi}\frac{1}{|x-y|} = \frac{1}{2\pi\varepsilon} & n=2 \\ \frac{1}{(n-2)\omega_n}(2-n)|x-y|^{1-n}\frac{x-y}{|x-y|}\cdot\frac{y-x}{|x-y|} = \frac{1}{\omega_n}|x-y|^{1-n} = \frac{1}{\omega_n}\varepsilon^{1-n} & n \geq 3 \end{cases}$

$\nabla|x| = \frac{x}{|x|}$. Moreover If f is radial, $f = f(|x|)$ then $\nabla f = f'(r)\frac{x}{|x|}$ where $r = |x|$

$n=2$
$$\int_{\partial B_\varepsilon(y)} \varphi(x)\frac{1}{2\pi\varepsilon}\, d\sigma = \frac{1}{2\pi\varepsilon}\int_{\partial B_\varepsilon(y)}\varphi(x)\, d\sigma$$

$$= \varphi(y_\varepsilon) \xrightarrow{\varepsilon \to 0} \varphi(y) \text{ (by continuity of } \varphi\text{)}.$$

$n \geq 3$:
$$\int_{\partial B_\varepsilon(y)} \varphi(x)\frac{1}{\omega_n \varepsilon^{n-1}}\, d\sigma = \frac{1}{\omega_n\varepsilon^{n-1}} \int_{\partial B_\varepsilon(y)} \varphi(x)\, dx = \varphi(y_\varepsilon) \xrightarrow{\varepsilon \to 0} \varphi(y)$$

Finally
$$\int_{\mathbb{R}^n} u \Delta \varphi\, dx = -\varphi(y)$$

$\square$

Method II Regularize u as $u_\delta(x) = \begin{cases} \frac{c}{2}\ln(r^2+\delta^2) & n=2,\ r=|x-y| \\ c/(r^2+\delta^2)^{\frac{n-2}{2}} & n\geq 3 \end{cases}$

$\forall \delta>0,\ u_\delta \in C^\infty(\mathbb{R}^n)$

$$-\int_{\mathbb{R}^n} u_\delta \Delta\varphi\, dx = -\int_{\mathbb{R}^n} (\Delta u_\delta)\varphi\, dx$$

WTS (i) $-\int_{-\mathbb{R}^n} (\Delta u_\delta)\varphi\, dx \xrightarrow{\delta\to 0} \left(-\int_{\mathbb{R}^n}(\Delta u)\varphi\, dx\right) = \varphi(y)$

(ii) $-\int_{\mathbb{R}^n} u_\delta \Delta\varphi \xrightarrow{\delta\to 0} -\int_{\mathbb{R}^n} u\Delta\varphi\, dx$ by DCT

Define

$$\Gamma(x,y) = \Gamma(x-y) = \begin{cases} -\frac{1}{2\pi}\ln|x-y| & n=2 \\ \frac{1}{(n-2)\omega_n}|x-y|^{-(n-2)} & n\geq 3 \end{cases}$$

Poisson Equation $-\Delta u = f(x)$ in $\mathbb{R}^n$

Define $u(x) = \int_{\mathbb{R}^n} \Gamma(x-y) f(y)\, dy$. $(*5)$

<u>Theorem</u> $\forall f \in C_0^2(\mathbb{R}^n)$, the $u(x)$ in $(*5)$ is $C^2(\mathbb{R}^n)$ & satisfies the Poisson equation in the classical sense.

"Formally" $-\Delta_x \int_{\mathbb{R}^n} \Gamma(x-y) f(y)\, dy = \int_{\mathbb{R}^n} -\Delta_x \Gamma(x-y) f(y)\, dy$

$$= \int_{\mathbb{R}^n} \underset{\delta(x-y)}{\underset{\parallel}{\delta_y(x)}} f(y)\, dy$$

$$= f(x)$$

Proof: 1. $u(x)$ in $(*5)$ is well defined ($\because \Gamma \in L^1_{loc}(\mathbb{R}^n)$, $f \in C_0^2(\mathbb{R})$)

2. $\forall$ fixed $x \in \mathbb{R}^n$, $u(x) = \int_{\mathbb{R}^n} \Gamma(z) f(x-z)\, dz$ $(z = x-y)$

$= \int_{\mathbb{R}^n} \Gamma(y) f(x-y)\, dy$

$e_i: (0, \dots, 0, 1, 0, \dots, 0)$
$\qquad\qquad\quad \uparrow$
$\qquad\qquad\quad i$-th

$$[u(x+he_i)-u(x)]\frac{1}{h} = \int_{\mathbb{R}^n} \frac{f(x+he_i-y)-f(x-y)}{h}\Gamma(y)\,dy$$

$$\text{(Mean value theorem)} = \int_{\mathbb{R}^n} f_{x_i}(x+se_i-y)\,\Gamma(y)\,dy \quad s\in(0,h)$$

$$|f_{x_i}(x+se_i-y)\Gamma(y)| \le \underbrace{\overset{\in L^1_{loc}}{|\Gamma(y)|}\cdot\|f_{x_i}\|_{L^\infty}\chi_{B_R(x)}(y)}_{\in L^1(\mathbb{R}^n)} \quad \forall y\in\mathbb{R}^n \quad R>0$$

s.t. $\operatorname{supp} f_{x_i}(x+se_i-y) \subseteq B_R(x)$

By the dominated convergence theorem,

$$\lim_{h\to 0}[u(x+he_i)-u(x)]\frac{1}{h} = \int_{\mathbb{R}^n}\lim_{h\to 0}\frac{f(x+he_i-y)-f(x-y)}{h}\Gamma(y)\,dy$$

$$= \int_{\mathbb{R}^n} f_{x_i}(x-y)\Gamma(y)\,dy \Rightarrow \frac{\partial u}{\partial x_i} \text{ exists}$$

Continuity of $\frac{\partial u}{\partial x_i}$ at x: Suppose $x^k \xrightarrow{k\to\infty} x$, we want to show that $\frac{\partial u}{\partial x_i}(x^k) \xrightarrow{k\to\infty} \frac{\partial u}{\partial x_i}(x)$

$$\frac{\partial u}{\partial x_i}(x^k) - \frac{\partial u}{\partial x_i}(x) = \int_{\mathbb{R}^n}[f_{x_i}(x^k-y)-f_{x_i}(x-y)]\,\Gamma(y)\,dy$$

$$|f_{x_i}(x-y)-f_{x_i}(x-y)|\,|\Gamma(y)| \overset{k \text{ is large}}{\le} \underbrace{2\|f_{x_i}\|_{L^\infty}\chi_{B_{R+1}}(x)\,|\Gamma(y)|}_{\in L^1_{loc}} \quad \forall y\in\mathbb{R}^n$$

By the dominated convergence theorem $L^1(\mathbb{R}^n)$

$$\lim_{k\to\infty}\frac{\partial u}{\partial x_i}(x^k) = \frac{\partial u}{\partial x_i}(x)$$

Similarly all $\frac{\partial^2 u}{\partial x_i \partial x_j}$ exist $= \int_{\mathbb{R}^n} f_{x_i x_j}(x-y)\,\Gamma(y)\,dy$

and are continuous in $\mathbb{R}^n$

$\Rightarrow u \in C^2(\mathbb{R})$

$$-\Delta u(x) = -\int_{\mathbb{R}^n} -\Delta_x f(x-y)\underset{\Gamma(x-y)}{\Gamma(y)}\,dy = \int_{\mathbb{R}^n} -\Delta_y f(y)\Gamma(x-y) = f(x) \quad \forall x\in\mathbb{R}^n$$

($\because$ we proved $\langle \Delta\varphi, u(x)\rangle = \varphi(y)$, take $\varphi = f \quad \forall \varphi\in C_0^\infty(\mathbb{R}^n)$)

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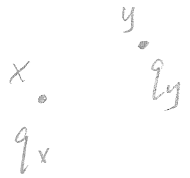
Review Poisson Equation $-\Delta u = f(x) \quad x \in \mathbb{R}^n \quad (P)$

- Define $u(x) = \int_{\mathbb{R}^n} \Gamma(x-y) f(y)\,dy \quad (*5)$
- If $f \in C_0^2(\mathbb{R})$, then the $u(x)$ in $(*5)$ is in $C^2(\mathbb{R}^n)$ and solves (P) in the classical sense.
- If $f \in L^1(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n)$ then $u \in C^1(\mathbb{R}^n)$ and solves (P) in the sense of distribution (HW4)

Physical Interpretation for $n=3$



$F = k_e \frac{x-y}{|x-y|^3} q_x q_y$ electric force on a charge q_x at x produced by a charge q_y at y.

Recall: $\Gamma(x-y) = \frac{1}{\omega_3 |x-y|}$, $-\nabla_x \Gamma(x-y) = \frac{1}{4\pi} \frac{x-y}{|x-y|^3}$

$\Gamma(x-y)$: electric potential at x induced by a positive unit point charge in y

Now if $f(x)$ represents density of charges at $x \in \mathbb{R}^3$,

$\Gamma(x-y) f(y)\,dy$: electric potential at x induced by charges in the region dy.

Sum up dy $\quad u(x) = \int_{\mathbb{R}^n} \Gamma(x-y) f(y)\,dy$ — total electric potential at x.

如果只有一点电荷，电荷密度为 δ_y.