4.1 Weak Derivatives

$\Omega \subseteq \mathbb{R}^n$ domain $\mathcal{D}(\Omega) = C_0^\infty(\Omega)$ $\mathcal{D}'(\Omega)$: { continuous linear functional defined on $\mathcal{D}(\Omega)$ }

$L^1_{loc}(\Omega) \subset \mathcal{D}'(\Omega)$ ($\forall f \in L^1_{loc}(\Omega)$ $\langle f, \varphi \rangle = \int_\Omega f\varphi\, dx$, $\varphi \in \mathcal{D}(\Omega)$)

$\forall f \in \mathcal{D}'(\Omega)$. $D^\alpha f$ always exists as a distribution. $\forall \alpha$.

$$\langle D^\alpha f, \varphi \rangle = (-1)^{|\alpha|} \langle f, D^\alpha \varphi \rangle$$

<u>Definition</u> Suppose $u \in L^1_{loc}(\Omega)$ and there the distributional derivative $D^\alpha u$ can be realized/regarded as an $L^1_{loc}(\Omega)$ function v. i.e. $D^\alpha u = v$ in the distributional sense:

$$\forall \varphi \in \mathcal{D}(\Omega). \quad \langle D^\alpha u, \varphi \rangle = (-1)^{|\alpha|} \int_\Omega u D^\alpha \varphi\, dx$$

$$= \int_\Omega v\varphi\, dx = \langle v, \varphi \rangle$$

Then we say v is the α-th weak derivative of u.

Write $v = D^\alpha u \; (= \partial^\alpha u)$

Remark: If $u = C^{|\alpha|}(\Omega)$ (hence $L^1_{loc}(\Omega)$), then classical $\partial^\alpha u$ = week $\partial^\alpha u$.

<u>Definition</u> We say that $u \in L^1_{loc}(\Omega)$ is k-times weakly differentiable if all weak $\partial^\alpha u$ $|\alpha| \le k$ exists.

Notation: $W^k(\Omega)$ = set of all such u's, linear space

Ex. 1 $u(x) = |x|$. $x \in \mathbb{R} = \Omega$ $u'(x)$ (distr) $= \begin{cases} -1 & x \le 0 \\ 1 & x > 0 \end{cases}$

Weak $u''(x)$ does not exist. (Argue by contradiction)

Suppose $u''(x)$ exists $= v(x) \in L^1_{loc}(\mathbb{R})$. Then

$$\forall \varphi \in C_0^\infty(\mathbb{R}). \quad \langle u''(x), \varphi \rangle = \langle v, \varphi \rangle$$

$$\overset{\text{def}}{=} (-1)^2 \langle u(x), \varphi'' \rangle = \int_\Omega u\varphi''\, dx \qquad = \int_\Omega v\varphi\, dx$$

$\Omega = \mathbb{R}$.

$$\int_\Omega u\varphi'' dx = \int_0^\infty x\varphi'' dx + \int_{-\infty}^0 (-x)\varphi'' dx \quad (x\varphi'' = (x\varphi')' - \varphi')$$

$$= (x\varphi')\big|_0^\infty - \int_0^\infty \varphi' dx - (x\varphi')\big|_{-\infty}^0 + \int_{-\infty}^0 \varphi' dx$$

$$= \varphi(0) + \varphi(0) = 2\varphi(0)$$

Now, take

$$\varphi_\varepsilon(x) = j\left(\frac{x}{\varepsilon}\right) \quad j(x) = \begin{cases} Ce^{\frac{1}{|x|^2-1}} & |x| < 1 \\ 0 & |x| \geq 1 \end{cases}$$

$\varphi_\varepsilon \in C_0^\infty(\mathbb{R}^n)$, $\operatorname{supp}\varphi_\varepsilon = B_\varepsilon(0)$

$\Omega = \mathbb{R}$ $$\int_\Omega v\varphi_\varepsilon dx = \underbrace{\int_{\mathbb{R}} v j\left(\frac{x}{\varepsilon}\right) dx}_{\to 0 \text{ as } \varepsilon \to 0.}$$

$$\left| v j\left(\frac{x}{\varepsilon}\right) \right| \leq |v(x)|\, j(0)\, \chi_{B_1}(x) \quad \text{if } \varepsilon < 1$$

By the Lebesgue dominated convergence theorem

$$\lim_{\varepsilon \to 0} \int_\Omega v\varphi_\varepsilon dx = \lim_{\varepsilon \to 0} \int_{\mathbb{R}} v j\left(\frac{x}{\varepsilon}\right) dx$$

$$\overset{\text{LDCT}}{=} \int_\Omega \lim_{\varepsilon \to 0} v j\left(\frac{x}{\varepsilon}\right) dx$$

$$= 0$$

任何发生断裂的函数不再有一阶弱导数

4.2. Approximating Bad functions by Good Ones

<u>Definition</u> $\forall\, u \in L^1_{loc}(\Omega)$ the regularization of u is $u_\varepsilon(x)$

$= \int_\Omega j_\varepsilon(x-y)\, u(y)\, dy$, $j_\varepsilon(x) = \frac{1}{\varepsilon^n} j\left(\frac{x}{\varepsilon}\right)$ $\int_{\mathbb{R}^n} j_\varepsilon(x) dx = 1$

$\operatorname{supp} j_\varepsilon(x) = B_\varepsilon(0)$.

- $\forall x \in \Omega$ $u_\varepsilon(x)$ is well defined for $0 < \varepsilon < \mathrm{dist}(x, \partial\Omega)$
- $\forall \varepsilon > 0$ small, $u_\varepsilon(x)$ is well defined on $\Omega_\varepsilon = \{x \in \Omega \mid \mathrm{dist}(x, \partial\Omega) > \varepsilon\}$

- If $u \in L^1(\Omega)$, then $u_\varepsilon(x)$ is well-defined on $\mathbb{R}^n$
  (same for $u \in L^p(\Omega)$, $p \geq 1$).
  (If Ω bounded, then $L^p(\Omega) \subset L^1(\Omega)$ by Hölder's inequality)
  (If Ω may not be bounded, $L^p_{loc}(\Omega) \subset L^1_{loc}(\Omega)$)

- If $u \in L^1(\Omega)$ & extend u by zero outside Ω:

$$\tilde{u}(x) = \begin{cases} u(x) & \text{if } x \in \Omega \\ 0 & \text{if } x \in \Omega^c \end{cases}$$

then $u_\varepsilon(x) = \int_\Omega j_\varepsilon(x-y)\, u(y)\, dy = \int_{\mathbb{R}^n} j_\varepsilon(x-y)\, \tilde{u}(y)\, dy$

$$= (j_\varepsilon * \tilde{u})(x)$$

convolution.

(often write $\tilde{u}$ as u) $\quad j_\varepsilon(x) = \frac{1}{\varepsilon^n} j\left(\frac{x}{\varepsilon}\right)$

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Lemma 1 (i) If $u \in L^1_{loc}(\Omega)$ then for fixed small $\varepsilon > 0$, $u_\varepsilon(x) \in C^\infty(\Omega_\varepsilon)$

(ii) If $u \in L^1(\Omega)$, then for fixed small $\varepsilon > 0$, $u_\varepsilon(x) \in C^\infty(\mathbb{R}^n)$, and $u_\varepsilon(x) \in C_0^\infty(\mathbb{R}^n)$ when Ω is bounded.

(iii) If $u \in L^p(\Omega)$ $\infty > p > 1$ then the same conclusions hold as in (ii).

Proof. (i) For all fixed $x \in \Omega_\varepsilon$, there is a $\delta > \varepsilon$, s.t. $\overline{B_\delta(x)} \subset \Omega$.

If z is close to x, then $B_\varepsilon(z) \subset B_\delta(x)$

Let $e_i = (0, \dots, 0, 1, 0, \dots, 0)$ Consider $\frac{u_\varepsilon(x+he_i) - u_\varepsilon(x)}{h}$

$$= \int_\Omega \frac{j_\varepsilon(x+he_i-y) - j_\varepsilon(x-y)}{h}\, u(y)\, dy.$$

$$\overset{MVT}{=} \int_\Omega \frac{\partial}{\partial x_i} j_\varepsilon(x+se_i-y)\, u(y)\, dy, \quad s \text{ between } 0 \text{ and } h.$$

$$\underbrace{\frac{\partial}{\partial x_i} j_\varepsilon(x+se_i-y)}_{\text{supp (w.r.t. } y) = B_\varepsilon(x+se_i) \subset B_\delta(x)} u(y) \le \overbrace{\left\|\frac{\partial}{\partial x_i} j_\varepsilon\right\|_{L^\infty(\mathbb{R}^n)} |u(y)|\, \chi_{B_{\delta}(x)}(y)}^{\text{dominating function}} \in L^1(\Omega)$$

Now by LDCT

$$\lim_{h\to 0} \frac{u_\varepsilon(x+he_i)-u_\varepsilon(x)}{h} = \int_\Omega \frac{\partial}{\partial x_i} j_\varepsilon(x-y)\, u(y)\, dy$$

Next WTS $\frac{\partial u_\varepsilon}{\partial x_i}$ is continuous on Ω_ε.

Suppose $\{x_k\} \subset \Omega_\varepsilon$ s.t. $x_k \xrightarrow{k\to\infty} x$.

$$\left|\frac{\partial u_\varepsilon}{\partial x_i}(x_k) - \frac{\partial u}{\partial x_i}(x)\right| = \left|\int_\Omega \left[\frac{\partial}{\partial x_i} j_\varepsilon(x_k-y) - \frac{\partial}{\partial x_i} j_\varepsilon(x-y)\right] u(y)\, dy\right|$$

$$\le \int_\Omega \left|\frac{\partial}{\partial x_i} j_\varepsilon(x_k-y) - \frac{\partial}{\partial x_i} j_\varepsilon(x-y)\right| |u(y)|\, dy$$

$$\le \underbrace{2\left\|\frac{\partial}{\partial x_i} j_\varepsilon\right\|_{L^\infty(\mathbb{R}^n)} u(y)\, \chi_{B_\delta(x)}(y)}_{L^1(\Omega) \text{ (the dominating function)}}$$

If k is large, then $x_k \approx x$ so that $B_\varepsilon(x) \subset B_\delta(x)$

$$\overset{LDCT}{\Longrightarrow} \frac{\partial u_\varepsilon}{\partial x_i}(x^k) \to \frac{\partial u_\varepsilon}{\partial x_i}(x) \text{ as } k\to\infty$$

$\Longrightarrow u_\varepsilon \in C^1(\Omega_\varepsilon)$. Similarly $u_\varepsilon \in C^k(\Omega_\varepsilon)$ $\forall k \ge 1$

(ii) $\forall$ fixed $x \in \Omega$. just take $\delta < \varepsilon$ s.t. $B_\varepsilon(x+se_i) \subset B_\delta(x)$

Do not need $B_\delta(x) \subset\subset \Omega$.

$\Longrightarrow u_\varepsilon(x) \in C^\infty(\mathbb{R}^n)$ by similar arguments.

when Ω is bounded, we have $u_\varepsilon \in C_0^\infty(\mathbb{R}^n)$

(iii) If $u \in L^p(\Omega)$ ($p > 1$) just use Hölder to see that the previous dominating function $\in L^1(\Omega)$

Lemma 2. If $u \in C^0(\Omega)$ then $\forall \Omega' \subset\subset \Omega$ $u_\varepsilon \xrightarrow{\varepsilon \to 0^+} u$ in $C_0(\overline{\Omega'})$

(i.e. the convergence is uniform in $\overline{\Omega'}$)

Proof. $\forall\, 0 < \varepsilon < \operatorname{dist}(\overline{\Omega'}; \partial\Omega)$

$$u_\varepsilon(x) = \int_\Omega j_\varepsilon(x-y)\, u(y)\, dy \quad \text{well-defined on } \overline{\Omega'}$$

$$= \int_{B_\varepsilon(x)} j_\varepsilon(x-y)\, u(y)\, dy \overset{z = \frac{x-y}{\varepsilon}}{=\!=\!=} \int_{B_1(0)} j(z)\, u(x-\varepsilon z)\, dz$$

$$u(x) = \int_{B_1(0)} j(z)\, dz \cdot u(x) = \int_{B_1(0)} j(z)\, u(x)\, dz$$

$$|u_\varepsilon(x) - u(x)| = \left| \int_{B_1(0)} j(z)\big(u(x-\varepsilon z) - u(x)\big)\, dz \right|$$

$$\leq \int_{B_1(0)} j(z)\, |u(x-\varepsilon z) - u(x)|\, dz$$

Take Ω'' such that $\Omega' \subset\subset \Omega'' \subset\subset \Omega$, then $u \in C^0(\overline{\Omega''})$

(u is uniformly continuous on $\overline{\Omega''}$)

Then $\forall\, \varepsilon > 0$ small, $\exists\, \sigma > 0$ s.t. $|u(x') - u(x)| < \delta$, if $|x - x'| < \sigma$

Now if $\varepsilon < \sigma$, then $|u(x-\varepsilon z) - u(x)| < \delta$

Thus $\forall\, x \in \overline{\Omega'}$ $|u_\varepsilon(x) - u(x)| \leq \int_{B_1(0)} j(z)\, \delta\, dz = \delta$ if $\varepsilon < \sigma$ $\square$

Lemma 3 $u \in L^1_{loc}(\Omega) \Rightarrow u_\varepsilon \to u$ a.e. as $\varepsilon \to 0$.

Proof: Recall Lebesgue's Differentiation Theorem $f : \mathbb{R}^n \to \mathbb{R}$

$f \in L^1_{loc}(\mathbb{R}) \Rightarrow$ (i) For a.e. $x_0 \in \mathbb{R}^n$ $\displaystyle\frac{1}{|B_r(x_0)|}\int_{B_r(x_0)} f\, dx \to f(x_0)$ as $r \to 0$

(ii) For a.e. $x_0 \in \mathbb{R}^n$ $\displaystyle\frac{1}{B_r(x_0)}\int_{B_r(x_0)} |f - f(x_0)| \to 0$ as $r \to 0$.

More generally $f \in L^p_{loc}(\mathbb{R}^n)$ $(1 \leq p < \infty)$

$$\Rightarrow \frac{1}{|B_r(x_0)|}\int_{B_r(x_0)} |f(x) - f(x_0)|^p\, dx \to 0 \text{ as } r \to 0.$$

Now, for a Lebesgue point x of u

$$|u_\varepsilon(x) - u(x)| = \int_{B_\varepsilon(x)} j_\varepsilon(x-y)\,[u(y) - u(x)]\,dy$$

ω_n volume of unit ball

$$\leq \frac{1}{\varepsilon^n} \int_{B_\varepsilon(x)} j\left(\frac{x-y}{\varepsilon}\right) |u(y) - u(x)|\,dy$$

$$\leq j(0) \frac{\omega_n}{\omega_n \varepsilon^n} \underbrace{\frac{1}{|B_\varepsilon(x)|} \int_{B_\varepsilon(x)} |u(y) - u(x)|\,dy}_{} \to 0 \text{ as } \varepsilon \to 0$$

$$\leq C \ldots$$

$C = j(0)\,\omega_n$

Lemma 4. Let $1 \leq p < \infty$. $u \in L^p_{loc}(\Omega)$ $(L^p(\Omega))$

Then $u_\varepsilon \to u$ in $L^p_{loc}(\Omega)$ $(L^p(\Omega))$

Applications:

(a) $L^p(\Omega) \cap C^\infty(\Omega)$ is dense in $L^p(\Omega)$ ($\Leftarrow$ Lm 1 + Lm 4)

($\forall\, u \in L^p(\Omega)$, $u_\varepsilon \in C^\infty(\mathbb{R}^n) \subseteq C^\infty(\Omega)$ by Lm 1

Also, $u_\varepsilon \in L^p(\Omega)$ and $u_\varepsilon \xrightarrow{\varepsilon \to 0} u$ in $L^p(\Omega)$ by Lm 4)

(b) If Ω is bounded, then $C^\infty(\overline{\Omega})$ is dense in $L^p(\Omega)$

($\Leftarrow$ Lm 1 + Lm 4)

(Note that we do not require any regularity on $\partial\Omega$)

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Lm 1 $u \in L^1_{loc}(\Omega) \Rightarrow u_\varepsilon(x) = \int_\Omega j_\varepsilon(x-y)\, u(y)\, dy \in C^\infty(\Omega_\varepsilon)$

$u \in L^p(\Omega)$ $(1 \le p \le \infty) \Longrightarrow$ $\cdot\ u_\varepsilon \in C^\infty(\mathbb{R}^n)$ $\quad \Omega_\varepsilon = \{x \in \Omega \mid dist(x, \partial\Omega) > \varepsilon\}$

$\cdot$ if Ω bounded $u_\varepsilon \in C_0^\infty(\mathbb{R}^n)$

$\cdot$ $supp\, u_\varepsilon \subseteq$ 

Lm 2 $u \in C^0(\Omega)$ $\forall\, \Omega' \subset\subset \Omega$

$$u_\varepsilon \xrightarrow{\varepsilon \to 0^+} u \quad \text{in } C^0(\overline{\Omega'})$$

Lm 3. $u \in L^1_{loc}(\Omega) \Rightarrow u_\varepsilon \xrightarrow{\varepsilon \to 0^+} u$ a.e. in Ω.

Lm. 4 $u \in L^p_{loc}(\Omega)$ $1 \le p < \infty$ $\Rightarrow u_\varepsilon \xrightarrow{\varepsilon \to 0^+} u$ in $L^p_{loc}(\Omega)$ $(L^p(\Omega))$

$(L^p(\Omega))$

Proof of Lemma 4.

(i) Suppose $u \in L^p_{loc}(\Omega)$ $(1 \le p < \infty)$ want to show

$$\forall\, \Omega' \subset\subset \Omega \quad \int_{\Omega'} |u_\varepsilon(x) - u(x)|^p\, dx \xrightarrow{\varepsilon \to 0} 0$$

Recall: $\forall x \in \Omega$, $\varepsilon > 0$ small

$$u_\varepsilon(x) := \int_{B_1(0)} j(z)\, u(x - \varepsilon z)\, dz \qquad u(x) = \int_{B_1(0)} j(z)\, u(x)\, dz$$

$$\| u_\varepsilon(x) - u(x) \|_{L^p(\Omega')} = \left\| \int_{B_1(0)} j(z) \big(u(x - \varepsilon z) - u(x) \big)\, dz \right\|_{L^p(\Omega')}$$

$$\overset{\text{Minkowski}}{\le} \int_{B_1(0)} j(z)\, \| u(\cdot - \varepsilon z) - u(\cdot) \|_{L^p(\Omega')}\, dz$$

Minkowski's inequalities $\forall\, 1 \le p \le \infty$

(a) $\|f+g\|_{L^p} \le \|f\|_{L^p} + \|g\|_{L^p}$ (b) $\left\| \sum_{i=1}^m f_i \right\|_{L^p} \le \sum_{i=1}^m \|f_i\|_{L^p}$

(c) $\left\| \int_Y f(\cdot, y)\, d\mu(y) \right\|_{L^p(X)} \le \int_Y \|f(\cdot, y)\|_{L^p(X)}\, d\mu(y)$

Now take a subdomain Ω'' s.t. $\Omega' \subset\subset \Omega'' \subset\subset \Omega$ $\quad \hat{u} \triangleq \begin{cases} u & x \in \Omega'' \\ 0 & x \in \mathbb{R}^n \setminus \Omega'' \end{cases}$

$$u \in L^p(\Omega'') \Rightarrow \hat{u} \in L^p(\mathbb{R}^n).$$

$$\| u(\cdot - \varepsilon z) - u(\cdot) \|_{L^p(\Omega')} \le \| \hat{u}(\cdot - \varepsilon z) - \hat{u}(\cdot) \|_{L^p(\mathbb{R}^n)}$$

$$\xrightarrow{\varepsilon \to 0^+} 0 \quad \text{uniformly for } z \in B_1(0)$$

by the continuity of L^p norm

$(1 \le p < \infty)$

$$\le 2 \| \hat{u}(\cdot) \|_{L^p(\mathbb{R}^n)}$$

(dominating function)

$$\overset{\text{LDCT}}{\Longrightarrow} \quad \| u_\varepsilon - u \|_{L^p(\Omega')} \xrightarrow{\varepsilon \to 0} 0 \qquad \square$$

(ii) If $u \in L^p(\Omega)$, then $\Omega' = \Omega$ and extend $u \equiv 0$ on Ω^c at the beginning. $\square$

Applications (cont'd)

(c) $C_0^\infty(\Omega)$ is dense in $L^p(\Omega)$

Proof: $\forall\, u \in L^p(\Omega)$ Define $\Omega_k = \{ x \in \Omega \mid |x| < k,\ \mathrm{dist}(x, \partial\Omega) > \frac{1}{k} \}$

$k \in \mathbb{N}$ $\quad \Omega_k \subset\subset \Omega$, $\quad \Omega_k \xrightarrow{k \to \infty} \Omega$ $\quad \left(\Omega = \bigcup_{k=1}^{\infty} \Omega_k \right)$

Let $\underbrace{u_k(x) = u(x) \chi_{\Omega_k}(x)}$ $\in L^p(\Omega)$

$\longrightarrow u(x)$ in $L^p(\Omega)$ as $k \to \infty$ $\quad (*1)$

For fixed $k \ge 1$,

$(u_k)_\varepsilon \xrightarrow{\varepsilon \to 0^+} u_k$ in $L^p(\Omega)$ (Lemma 4) $\quad \Big\} (*2)$

$\operatorname{supp}(u_k)_\varepsilon$ is fatter than Ω_k by the amount of ε]

$\Longrightarrow$ If $\varepsilon > 0$ small, $\operatorname{supp}(u_k)_\varepsilon \subset\subset \Omega$.

Now $\forall \delta > 0$, by $(*1)$, $\exists\, u_{k_0}$ s.t. $\|u_{k_0} - u\|_{L^p(\Omega)} \leq \frac{\delta}{2}$

by $(*2)$ $\exists\, \varepsilon_0 > 0$ s.t. $\|(u_{k_0})_{\varepsilon_0} - u_{k_0}\|_{L^p(\Omega)} \leq \frac{\delta}{2}$ & $\operatorname{supp}(u_{k_0})_{\varepsilon_0} \subset\subset \Omega$

Then $\|(u_{k_0})_{\varepsilon_0} - u\|_{L^p(\Omega)} \leq \delta$. $(u_{k_0})_{\varepsilon_0} \in C_0^\infty(\Omega)$ $\square$

(d) $u \in L^1_{loc}(\Omega)$, $v_1, v_2 \in L^1_{loc}(\Omega)$ are α-th weak derivatives of u. $\Longrightarrow$ $v_1 = v_2$ a.e. on Ω (uniqueness of weak derivatives)

Proof. $\forall \varphi \in C_0^\infty(\Omega)$ ($= \mathcal{D}(\Omega)$)

$$\int_\Omega v_1 \varphi\, dx = \int_\Omega v_2 \varphi\, dx = (-1)^\alpha \int_\Omega u\, \partial^\alpha \varphi\, dx$$

$$\Longrightarrow \int_\Omega (v_1 - v_2)\varphi\, dx = 0.$$

$\forall$ fixed $y \in \Omega$, take $\varphi(x) = j_\varepsilon(y - x)$ $\operatorname{supp}\varphi = B_\varepsilon(y) \subset\subset \Omega$.
if $\varepsilon > 0$ small

$$\Longrightarrow \underbrace{\int_\Omega (v_1 - v_2)(x)\, j_\varepsilon(y - x)\, dx}_{(v_1 - v_2)_\varepsilon(y)} = 0 \quad (*1)$$

$$\Longrightarrow \underset{\equiv 0}{(v_1 - v_2)_\varepsilon(y)} \xrightarrow{\varepsilon \to 0} v_1 - v_2 \quad \text{in } L^1_{loc}(\Omega)$$

$v_1 = v_2$ a.e.

Lemma 5. If $u \in L^1_{loc}(\Omega)$ weak $\partial^\alpha u$ exists & $\in L^1_{loc}(\Omega)$

then $\forall \varepsilon > 0$, $\forall x \in \Omega_\varepsilon$, $\partial^\alpha u_\varepsilon(x)$ (classical) $= (\partial^\alpha u)_\varepsilon(x)$

Proof: By Lemma 1. $\forall \varepsilon > 0$, $\forall x \in \Omega_\varepsilon$, $u_\varepsilon(x) \in C^\infty(\Omega_\varepsilon)$

(From the pf of Lemma 1)

$$\partial^\alpha u_\varepsilon(x) \text{ (classical)} = \int_\Omega \partial_x^\alpha j(x-y)\, u(y)\, dy$$

$$= \int_\Omega (-1)^{|\alpha|} \partial_y^\alpha j_\varepsilon(x-y)\, u(y)\, dy$$

$$\overset{\text{def of weak } \partial^\alpha u}{=} \int_\Omega \underbrace{j_\varepsilon(x-y)}_{\in C_0^\infty(\Omega)} \partial^\alpha u(y)\, dy$$

$$= \int_\Omega j_\varepsilon(x-y)\, \partial^\alpha u(y)\, dy$$

$$= (\partial^\alpha u)_\varepsilon(x)$$

Application (e) Suppose $u \in L^1_{loc}(\Omega)$, weak $\frac{\partial u}{\partial x_i}$ $(i = 1, 2, \cdots, n)$ exist and $L^1_{loc}(\Omega)$ Weak ∇u a.e. on Ω. Then

$$u \equiv \text{const. } C \in \mathbb{R} \quad \text{a.e. on } \Omega.$$

Proof. By Lemma 5 $\nabla(u_\varepsilon) = (\nabla u)_\varepsilon \equiv 0$ in Ω_ε.

$\Rightarrow u_\varepsilon \equiv$ const C_ε in Ω_ε (by calculus)

By Lemma 3. $u_\varepsilon \to u$ a.e. in Ω as $\varepsilon \to 0$.

$\Rightarrow \lim_{\varepsilon \to 0} C_\varepsilon = C$ $\quad$ ($C \in \mathbb{R}$ or $C = \pm\infty$)

and $u \equiv C$ a.e. on Ω.

Since $C = \pm\infty$ would imply $u \notin L^1_{loc}(\Omega)$, we conclude that $u = \text{const } C \in \mathbb{R}$ a.e. on Ω. $\square$

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<u>Theorem 1</u> Let $u, v \in L^1_{loc}(\Omega)$. Then

$$v = \text{weak } \partial^\alpha u \iff \exists \{u_k\}_{k\ge1}^\infty \subset C^\infty(\Omega) \text{ such that}$$

$$u_k \to u \ \underline{\text{in } L^1_{loc}(\Omega)} \ \&$$

$$\text{classical } \partial^\alpha u_k \to v \ \underline{\text{in } L^1_{loc}(\Omega)}$$

$\langle \partial^\alpha u_k, \varphi \rangle \to \langle v, \varphi \rangle \quad \forall \varphi \in \mathcal{D}(\Omega)$

Proof: ($\Leftarrow$) $\forall \varphi \in \mathcal{D}(\Omega)$ $(= C_0^\infty(\Omega))$

$$\int_\Omega (\partial^\alpha u_k)\varphi(x)\,dx \xlongequal{\text{Integration by parts}} (-1)^{|\alpha|}\int_\Omega u_k(x)\,\partial^\alpha \varphi(x)\,dx$$

Let $k \to \infty$ $\downarrow$ $\downarrow$

$$\int_\Omega v \cdot \varphi\,dx \overset{\checkmark}{=\!=} (-1)^{|\alpha|}\int_\Omega u\,\partial^\alpha \varphi(x)\,dx$$

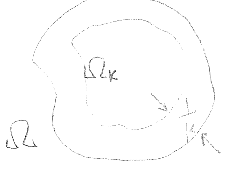
$\Rightarrow v = \text{weak } \partial^\alpha u$.

$$|-| \le \int_\Omega |\partial^\alpha u_k - v|\,|\varphi(x)|\,dx \le \|\varphi\|_{L^\infty} \int_{\text{supp}\varphi \subset \Omega} |\partial^\alpha u_k - v|\,dx \xrightarrow{k\to\infty} 0$$

Remark: In "$\Leftarrow$", we only need $\{u_k\}_{k=1}^\infty \subset C^{|\alpha|}(\Omega)$.

($\Rightarrow$) $\forall k \ge 1$. Let $\Omega_k = \{x \in \Omega \mid |x| < k, \text{dist}(x, \partial\Omega) > \frac{1}{k}\}$ (技巧)

(Ω_k有界并且远离边界，从而紧)



$\Rightarrow \Omega_k \subset\subset \Omega$.

Define $u_k(x) = \int_{\Omega_k} j_{\frac{1}{k}}(x-y)\,u(y)\,dy = (u|_{\Omega_k})_{\varepsilon=\frac{1}{k}}$

Recall $u \in L^1_{loc}(\Omega)$, $u_\varepsilon(x) = \int j_\varepsilon(x-y)\,u(y)\,dy$, $x \in \Omega_\varepsilon$

$$\left(j_\varepsilon(x) = \frac{1}{\varepsilon^n} j\left(\frac{x}{\varepsilon}\right) \right)$$

$$u \in L^1(\Omega_k) \Longrightarrow \underline{u_k \in C_0^\infty(\mathbb{R}^n)}$$

$\forall\, \Omega' \subset\subset \Omega$, $\exists\, k_0 \geq 1$ s.t. $\Omega' \subset\subset \Omega_k$ if $k \geq k_0$

$\forall\, x \in \Omega'$, $u_k(x) = \int_{\Omega_k} j_{\frac{1}{k}}(x-y)\,u(y)\,dy$ $\quad (k \geq k_0)$

(支撑集包含在 $B_{\frac{1}{k}}$ 上) $\longrightarrow$ $= \int_{\underline{\Omega_k \cap B_{\frac{1}{k}}(x)}} j_{\frac{1}{k}}(x-y)\,u(y)\,dy$

(固定区域) $\longrightarrow$ $= \int_{\underline{\Omega_{k_0}}} j_{\frac{1}{k}}(x-y)\,u(y)\,dy$ if k is large

(可用 Lemma 4) $\longrightarrow$ $= (u|_{\Omega_k})_{\frac{1}{k}}(x) \xrightarrow{\text{Lemma 4}} u|_{\Omega_{k_0}}$ in $L^1(\Omega_{k_0})$

hence in $L^1(\Omega')$

By Lemma 5 (with "Ω" $= \Omega_{k_0}$)

$$\partial^\alpha \left[(u|_{\Omega_0})_{\frac{1}{k}}(x) \right] = \left[(\partial^\alpha u)|_{\Omega_{k_0}} \right]_{\frac{1}{k}}(x) = (v|_{\Omega_{k_0}})_{\frac{1}{k}}(x)$$

$\qquad \|$

$\partial^\alpha u_k(x)$

$\forall\, x \in (\Omega_{k_0})_{\frac{1}{k}} \supset \Omega'$

(if k is large)

$\downarrow$ Lm 4

v in $L^1(\Omega')$

$\Longrightarrow \partial^\alpha u_k \xrightarrow{k\to\infty} v$ in $L^1(\Omega')$

$\square$

4.3 Chain Rule

<u>Theorem 2</u> (chain rule). If $f \in C^1(\mathbb{R})$, $f' \in L^\infty(\mathbb{R})$, $u(x) \in W^1(\Omega)$

$\Longrightarrow f(u(x)) \in W^1(\Omega)$ & <u>weak</u> $\partial_{x_i}(f(u)) = f'(u(x))\partial_{x_i} u$

$i = 1, \cdots, n$.

(i.e. $\nabla(f(u)) = f'(u)\nabla u$)

Proof: $u \in W^1(\Omega) \xrightarrow{Th1^{(\Rightarrow)}} \exists \{u_k\}_{k=1}^\infty \subset C^\infty(\Omega)$ s.t.

$u_k \to u$, $\nabla u_k \to$ weak ∇u.
(*1) (*2)

Consider $\{f(u_k)\}_{k=1}^\infty \subset C^1(\Omega)$.

$\forall\ \Omega' \subset\subset \Omega$ • WTS: (a) • $f(u_k) \to f(u)$ in $L^1(\Omega)$

(b) • $\partial x_i f(u_k) \to f'(u)\partial_{x_i} u$ in $L^1(\Omega')$

$\xrightarrow[(\Leftarrow)]{Th1}$ $f(u) \in W^1(\Omega)$ $\partial x_i f(u) = f'(u)\partial_{x_i} u$ (weak).

(a) $\int_{\Omega'} |f(u_k) - f(u)|\,dx \overset{MVT}{=} \int_{\Omega'} |f'(\xi)||u_k - u|\,dx$

$\leq \|f\|_{L^\infty} \int_{\Omega'} |u_k - u|\,dx \xrightarrow{k\to\infty} 0$

by (*1)

(b) $\int_{\Omega'} |f'(u_k)\nabla u_k - f'(u)\nabla u|\,dx \leq \int_{\Omega'} |f'(u_k)\nabla u_k - f'(u_k)\nabla u|\,dx$

$+ \int_{\Omega'} |f'(u_k)\nabla u - f'(u)\nabla u|\,dx$.

$I \leq \int_{\Omega'} |f'(u_k)||\nabla u_k - \nabla u|\,dx \leq \|f\|_{L^\infty} \int_{\Omega'} |\nabla u_k - \nabla u|\,dx$

$\xrightarrow{k\to\infty} 0$ by (*2)

$$\mathrm{II} \quad \left| f'(u_k)\nabla u - f'(u)\nabla u \right| \le 2\|f'\|_{L^\infty} |\nabla u| \in L^1(\Omega')$$

(dominating function)

$(*1) \Longrightarrow u_k \to u$ in $L^1(\Omega')$

Riesz-Fischer

$\Longrightarrow \exists\, u_{k_j}$ (subsequence) $\longrightarrow u$ a.e. in Ω'

$\Longrightarrow \quad f'(u_{k_j}) \longrightarrow f'(u)$ a.e. in Ω'

LDCT $\Longrightarrow$ $\mathrm{II} \to 0$ as $k_j \to \infty$ 將原函數列換成其為

是 R-F 的子列 $\square$

Theorem 3. If $u \in W^1(\Omega)$ then $u^+ = \max\{0, u\} \in W^1(\Omega)$

$u^- = \min\{0, u\} \in W^1(\Omega)$

$|u| \in W^1(\Omega)$

& $\nabla u^+(x) = \begin{cases} \nabla u(x) & \text{if } u(x) > 0 \\ 0 & \text{if } u(x) \le 0 \end{cases}$

$\nabla u^-(x) = \begin{cases} \nabla u(x) & \text{if } u(x) < 0 \\ 0 & u(x) \ge 0 \end{cases}$

$$\nabla |u|(x) = \begin{cases} \nabla u(x) & \text{if } u(x) > 0 \\ -\nabla u(x) & \text{if } u(x) < 0 \\ 0 & \text{if } u(x) = 0 \end{cases}$$

Proof. For $\varepsilon > 0$, define $f_\varepsilon(t) = \begin{cases} (t^2+\varepsilon^2)^{\frac{1}{2}} - \varepsilon & \text{if } t > 0 \\ 0 & \text{if } t \le 0 \end{cases}$

$$f_\varepsilon'(t) = \frac{1}{2}(t^2+\varepsilon^2)^{-\frac{1}{2}} \cdot 2t = \frac{(t)^+}{(t^2+\varepsilon^2)^{\frac{1}{2}}}$$

- $f_\varepsilon \in C^1(\mathbb{R})$, $f_\varepsilon' \in [0, 1)$, $0 \le f_\varepsilon(t) \le (t)^+ \quad \forall t \in \mathbb{R}$.

• $f_\varepsilon(u(x)) \in W^1(\Omega)$ weak $\nabla f_\varepsilon(u) = f_\varepsilon'(u)\, \nabla u$ (by chain rule)

$$= \frac{u^+ \nabla u}{\sqrt{u^2+\varepsilon^2}}$$

By definition of weak $\nabla f_\varepsilon(u)$, $\forall \varphi \in C_0^\infty(\Omega)$,

$$\int_\Omega f_\varepsilon(u) \nabla \varphi \, dx = -\int_\Omega \frac{u^+ \nabla u}{\sqrt{u^2+\varepsilon^2}} \varphi \, dx$$

$$= -\int_{u>0} \frac{u \nabla u}{\sqrt{u^2+\varepsilon^2}} \varphi \, dx$$

i.e. $\{x \in \Omega \; u(x) > 0\}$

$\varepsilon \to 0$ (LDCT) H.W. ; $\varepsilon \to 0$ (LDCT)

$$\int u^+ \nabla \varphi \, dx \qquad -\int_{u>0} \varphi \nabla u \, dx$$

$$\| $$

$$\int_\Omega [\varphi \nabla u] \chi_{\{u>0\}} \, dx$$

$\Longrightarrow$ weak $\nabla u^+ = \nabla u \, \chi_{\{u>0\}}$

$u^-(x) = \min\{0, u(x)\} = -\max\{0, -u(x)\} = -(-u(x))^+$

$|u| = u^+ - u^-$ $\square$

<u>Corollary</u> $u \in W^1(\Omega) \Longrightarrow \nabla u \equiv 0$ a.e. on any set Γ where $u \equiv$ const C.

Proof: $\nabla u = \nabla(u-c)$ in Ω

$= \nabla((u-c)^+ + (u-c)^-)$ in Ω

$\overset{\text{Thm 3}}{=} 0 + 0$ in Γ ($u \equiv C$ in Γ)

$= 0$ in Γ $\square$

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4.4 Sobolev Spaces

$1 \le p \le \infty$, $k \ge 0$ integer $\Omega \subseteq \mathbb{R}^n$ domain

$W^{k,p}(\Omega) = \{ u \in W^k(\Omega) \mid \partial^\alpha u \in L^p(\Omega),\ \forall \alpha \text{ such that } |\alpha| \le k \}$

$1 \le p < \infty$

$L^p(\Omega) = \{ u : \Omega \to \mathbb{R} \cup \{\pm\infty\} \mid u \text{ measurable},\ \int_\Omega |u|^p dx < \infty \}$

$\|u\|_{L^p(\Omega)} = \left(\int_\Omega |u|^p dx \right)^{\frac{1}{p}}$

$p = \infty$

$L^\infty(\Omega) = \{ u : \Omega \to \mathbb{R} \cup \{\pm\infty\} \mid u \text{ measurable, } \operatorname{ess\,sup} |u| < \infty \}$

$\|u\|_{L^\infty(\Omega)} = \operatorname{ess\,sup} |u|$

Norm on $W^{k,p}(\Omega)$

$$\|u\|_{W^{k,p}(\Omega)} = \begin{cases} \left[\int_\Omega \sum_{|\alpha|=0}^{k} |\partial^\alpha u(x)|^p dx \right]^{\frac{1}{p}} & 1 \le p < \infty \\ \sum_{|\alpha|=0}^{k} \|\partial^\alpha u\|_{L^\infty(\Omega)} & p = \infty \end{cases}$$

equivalent

$$|||u|||_{W^{k,p}(\Omega)} := \sum_{|\alpha|=1}^{k} \|\partial^\alpha u\|_{L^p(\Omega)} = \sum_{|\alpha|=0}^{k} \left[\int_\Omega |\partial^\alpha u|^p dx \right]^{\frac{1}{p}}$$

why $\sim$? $\|u\|_{W^{k,p}(\Omega)} \le |||u|||_{W^{k,p}(\Omega)}$ $\left(\because \sum a_i^p \le \left(\sum a_i\right)^p \ a_i \ge 0,\ p \ge 1 \right)$

$$|||u|||_{W^{k,p}(\Omega)} \le \left(\sum_{|\alpha|=0}^{k} \|\partial^\alpha u\|_{L^p}^p \right)^{\frac{1}{p}} \left(\sum_{|\alpha|=0}^{k} 1 \right)^{\frac{1}{q}}$$

(Hölder applied to $\sum_{|\alpha|=0}^{k} |\|\partial^\alpha u\| \cdot 1|$)

$$\le C \|u\|_{W^{k,p}(\Omega)}$$

Suppose $1 \le p < \infty$. check

- $\|u\|_{W^{k,p}}$ is a norm
- $|||u|||_{W^{k,p}}$ is a norm

i.e. $\|cu\|_{W^{k,p}} = |c| \cdot \|u\|_{W^{k,p}}$ ✓

$\|u\|_{W^{k,p}} = 0 \Rightarrow \|u\|_{L^p(\Omega)} \Rightarrow u = 0$ a.e

$$\|u+v\|_{W^{k,p}} \left(\overset{?}{\le} \|u\|_{W^{k,p}} + \|v\|_{W^{k,p}} \right)$$

$$= \left(\int_\Omega \sum_{|\alpha|=0}^{k} |\partial^\alpha u + \partial^\alpha v|^p dx \right)^{\frac{1}{p}}$$

$$\| \quad \left\{ \left(\sum_{|\alpha|=0}^{k} |\partial^\alpha u + \partial^\alpha v|^p \right)^{\frac{1}{p}} \right\}^p$$

Discrete Minkowski

Def $\|a\|_{\ell^p} = \left(\sum_{i=1}^{m} |a_i|^p \right)^{\frac{1}{p}}$

$\|a+b\|_{\ell^p} \le \|a\|_{\ell^p} + \|b\|_{\ell^p}$

$$\overset{\text{discrete}}{\underset{\text{Minkowski}}{\le}} \left\{ \int_\Omega \Bigg[\underbrace{\left(\sum_{|\alpha|=0}^{k} |\partial^\alpha u|^p \right)^{\frac{1}{p}}}_{f} + \underbrace{\left(\sum_{|\alpha|=0}^{k} |\partial^\alpha v|^p \right)^{\frac{1}{p}}}_{g} \Bigg]^p dx \right\}^{\frac{1}{p}}$$

$$= \left(\int_\Omega |f+g|^p dx \right)^{\frac{1}{p}} \le \|f\|_{L^p(\Omega)} + \|g\|_{L^p(\Omega)}$$

$$= \|u\|_{W^{k,p}} + \|v\|_{W^{k,p}} \qquad \square$$

<u>Theorem 3</u> $\forall\, 1 \le p \le \infty$ $k \ge 0$ integer. $W^{k,p}(\Omega)$ is a Banach space

Proof. Suppose $\{u_m\}$ is Cauchy in $W^{k,p}(\Omega)$

$$\Rightarrow \begin{cases} \{u_m\} \text{ is Cauchy in } L^p(\Omega) \\ \{\partial^\alpha u_m\} \text{ is Cauchy in } L^p(\Omega) \end{cases} \xrightarrow[\text{complete}]{L^p(\Omega)} \begin{cases} u_m \to u_\infty \ \underline{\text{in } L^p(\Omega)} \\ \partial^\alpha u_m \to V_\alpha \ \underline{\text{in } L^p(\Omega)} \end{cases}$$

Aim. $V_\alpha = \text{weak } \partial^\alpha u_\infty,\ \forall\, \alpha \le k$

Why: $\forall\, \varphi \in C_0^\infty(\Omega) (= \mathcal{D}(\Omega))$

$$\int_\Omega \partial^\alpha u_m(x)\, \varphi(x)\, dx = (-1)^\alpha \int_\Omega u_m(x)\, \partial^\alpha \varphi(x)\, dx$$

作差后用Hölder $\downarrow$ (by def of weak $\partial^\alpha u_m$) $\downarrow$

$$\int_\Omega V_\alpha\, \varphi(x)\, dx = (-1)^\alpha \int_\Omega u_\infty(x)\, \partial^\alpha \varphi(x)\, dx$$

$\Longrightarrow$ $|||\cdot|||_{W^{k,p}(\Omega)}$ is complete

So is $\|\cdot\|_{W^{k,p}(\Omega)}$ by equivalency $\square$

<u>Theorem 4</u> $W^{k,p}(\Omega)$ is separable if $1 \le p < \infty$

$W^{k,p}(\Omega)$ is reflexive if $1 < p < \infty$

Recall: <u>Definition</u> ① X normed linear space: X is called separable if $\exists\, A = \{a_i\}_{i=1}^\infty \subset X$ such that $\bar{A} = X$.

<u>Prop 1</u>
- Any subset of separable space is separable.
- The product of two separable spaces is separable

  $X \times Y = \{(x, y) \mid x \in X,\ y \in Y\}$ $\|(x,y)\|_{X\times Y} = \sqrt{\|x\|_X^2 + \|y\|_Y^2}$
- The image of a separable space under a continuous map is separable.

② X Banach space $X^* = \{ u^* : X \to \mathbb{R} \mid \text{linear, bounded} \}$

— dual space of X^*

$$J : X \longrightarrow (X^*)^*$$

$$x \longmapsto [\, u^* \mapsto u^*(x) \,]$$

X is called <u>reflexive</u> if $J(X) = (X^*)^*$

<u>Thm</u> (Weak compactness) X reflexive Banach

$\{x_k\} \subset X$ bounded

$\Rightarrow \exists$ subsequence $\{x_{k_j}\}_{j=1}^{\infty} \subset \{x_k\}_{k=1}^{\infty}$ & $x_\infty \in X$

s.t. $x_{k_j} \xrightarrow{\text{weak convergence}} x_\infty$ as $j \to \infty$.

(i.e. $\forall u^* \in X^*$, $u^*(x_{k_j}) \longrightarrow u^*(x_\infty)$)

<u>Prop 2.</u> A <u>closed subspace</u> of a reflexive Banach space is reflexive.

Prop 3. $\Omega \subseteq \mathbb{R}^n$ domain

- $L^p(\Omega)$ = Banach space for $1 \le p \le \infty$ (Riesz-Fischer)
- separable for $1 \le p < \infty$
- Reflexive for $1 < p < \infty$.

Proof of Thm 4. Define the mapping

$$T : W^{k,p}(\Omega) \longrightarrow \prod_{|\alpha|=0}^{k} L^p(\Omega) =: Y.$$

$$u \longmapsto (\partial^\alpha u)_{|\alpha|=0}^{k}$$

$\Rightarrow \quad \|Tu\|_Y = |||u|||_{W^{k,p}(\Omega)}$, so T is linear and isometric.

Let $M = T(W^{k,p}(\Omega))$

$W^{k,p}(\Omega)$ Banach. T is linear and isometric

$\Longrightarrow M$ is closed linear subspace of Y (*1)

Why (*1). $\forall \{u_n\} \subset W^{k,p}(\Omega)$, $T(u_n) \longrightarrow v_\infty$ in Y

want to show $v_\infty \in M$

$\Rightarrow T(u_n)$ is Cauchy in Y. $|||u_n - u_{n+q}|||_{W^{k,p}}$

$$= \|T(u_n) - T(u_{n+q})\|_Y$$

$\{u_n\}$ is Cauchy in $W^{k,p}$

$\Longrightarrow u_n \to u_\infty \xrightarrow{T \text{ is continuous}} T(u_\infty) = v_\infty \in M$

Fact: $L^p(\Omega)$ $(1<p<\infty)$ is reflexive

$\Rightarrow Y$ is reflexive

$\Rightarrow M \underset{\text{subspace}}{\overset{\text{closed}}{\subseteq}} Y$ is reflexive.

$\Rightarrow W^{k,p}$ $(1<p<\infty)$ is reflexive ($\because T: W^{k,p}(\Omega) \to M$ linear isometric)

Separability: $T: W^{k,p}(\Omega) \longrightarrow Y$ linear. isometric

$$T^{-1}: M = T(W^{k,p}(\Omega)) \longrightarrow X. \text{ linear, continuous}$$

$$X = W^{k,p}(\Omega)$$

$L^p(\Omega)$ $(1 \le p < \infty)$

$\Rightarrow W^{k,p}(\Omega)$ is separable by Prop 1.

$\varphi \in C(\Omega)$, $\operatorname{supp}\varphi := \overline{\{x \in \Omega : \varphi(x) \neq 0\}}$

Partition of unity:

Covering of Ω: Let $\{U_i\}_{i=1}^{\infty}$ be bounded and open sets in Ω. s.t.

bounded or not.

(i) $\overline{U_i} \subset \Omega$

(ii) Every compact $K \subset \Omega$ intersects only finitely many U_i's.

(iii) $\bigcup_{i=1}^{\infty} U_i = \Omega$

A partition of unity subordinate to the covering $\{U_i\}$ of Ω is $\{\varphi_i\}_{i=1}^{\infty} \subset C_0^{\infty}(\Omega)$ s.t.

(a) $\varphi_i \geq 0$

(b) $\operatorname{supp}\varphi_i \subset U_i$

(c) $\sum_{i=1}^{\infty} \varphi_i(x) = 1$, $\forall x \in \Omega$.

(实际上是有限和)

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Theorem 5. The partition of unity exists

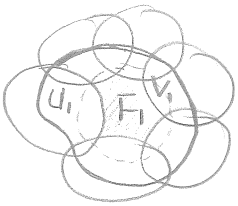
Proof: Step 1 Construct a new open covering such that

$\overline{V_i} \subset U_i$ $\bigcup_{i=1}^{\infty} V_i = \Omega$ Let $F_1 = \overline{U_1} \setminus \bigcup_{i=2}^{\infty} U_i$

$= \overline{U_1} \cap \left(\bigcup_{i=2}^{\infty} U_i\right)^c$

$\Rightarrow$ F_1 is closed

Since $\overline{U_1} \subset \Omega$, $\partial U_1 \subset \Omega = \bigcup_{i=1}^{\infty} U_i$



$$\Rightarrow \partial U_1 \subset \bigcup_{i=2}^{\infty} U_i \quad \text{(一个开集的边界不包含于自身)}$$

$$\Rightarrow F_1 \subset U_1 \Rightarrow \operatorname{dist}(F_1, \partial\Omega) > 0$$

Define $V_1 = \{x \in U_1 \mid \operatorname{dist}(x, F_1) < \frac{1}{2}\operatorname{dist}(F_1, \partial U_1)\}$

(取法不唯一)

$$\Rightarrow \begin{cases} V_1 \text{ open}, \ F_1 \subset V_1 \subset \overline{V_1} \subset U_1 \\ V_1 \cup \left(\bigcup_{i\geq 2} U_i\right) = \Omega \end{cases}$$

Let $F_2 = \overline{U}_2 \setminus \left(\bigcup_{i\geq 3} U_i \cup V_1\right)$

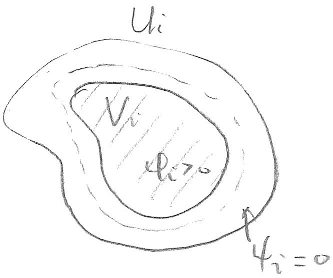
- F_2 is closed.
- $F_2 \cup \left(\bigcup_{i\geq 3} U_i \cup V_1\right) = \Omega$
- $F_2 \subset U_2$. $\exists\, V_2$ open such that $F_2 \subset V_2 \subset \overline{V_2} \subset U_2$

$$\Rightarrow V_1 \cup V_2 \cup \left(\bigcup_{i\geq 3} U_i\right) = \Omega.$$

Similarly, we obtain the V_i's that are desired.

Step 2. Construct $\varphi_i \in C_0^\infty(U_i)$ such that $\varphi_i \geq 0$, $\varphi_i > 0$ on V_i

$\forall i \geq 0$.



Let $\varphi_i(x) = (\chi_{V_i})_\varepsilon$

$$= \int_{V_i} j^{\varepsilon}(x-y)\chi_{V_i}(y)\,dy$$

$$= \int_{V_i} j_\varepsilon(x-y)\,dy$$

Since V_i is bounded, $\psi_i(x) \in C_0^\infty(\mathbb{R}^n)$

$\operatorname{supp} \psi_i \subseteq \{x \in U_i \mid \operatorname{dist}(x, V_i) < \varepsilon\} \subset\subset U_i$ if $\varepsilon < \frac{1}{2}\operatorname{dist}(\overline{U}_i, \partial U_i)$

$\psi_i(x) > 0$, $\forall x \in \overline{V}_i$ ($\because |B_\varepsilon(x) \cap V_i| > 0$ (positive measure))

ψ_i为一个积分，正函数在正测度集上积分大于0.

Step 3. Let $\psi(x) = \sum_{i=1}^{\infty} \psi_i(x)$

For all fixed $x_0 \in \Omega := \bigcup_{i=1}^{\infty} V_i \Rightarrow$ there exists some $i \geq 1$

such that $x_0 \in V_i$, $\psi_i(x) > 0$.

Take small $\delta > 0$ s.t. $\overline{B_\delta(x_0)} \subset \Omega$

$\Longrightarrow$ Only finitely many U_i's intersect $\overline{B_\delta(x_0)}$

recall $\operatorname{supp} \psi_i \subset\subset U_i$

$\Longrightarrow$ On $B_\delta(x_0)$, $\psi(x)$ is well-defined & $\psi \in C^\infty(B_\delta(x_0))$

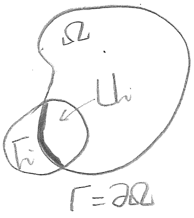
$$\Longrightarrow \psi \in C^\infty(\Omega) \quad 1 = \frac{\psi(x)}{\psi(x)} = \sum_{i=1}^{\infty} \frac{\psi_i(x)}{\psi(x)} \triangleq \sum_{i=1}^{\infty} \varphi_i(x) \qquad \square$$

Remark: Suppose Ω is bounded and $\overline{\Omega} \subset \bigcup_{i=1}^{k} U_i$

U_i's are open and bounded. Then there exist $\varphi_i \in C_0^\infty(U_i)$

such that $0 \leq \varphi_i \leq 1$, $\sum_{i=1}^{k} \varphi_i(x) = 1$, $\forall x \in \overline{\Omega}$ (HW)

Ex.



$\int_\Gamma f(x)\, d\sigma \quad \Gamma_i := \{x \in U_i \mid x_n = g(x_1, \dots, x_{n-1})\}$

$$= \int_\Gamma f(x) \Big(\sum_{i=1}^{k} \varphi_i\Big) d\sigma = \sum_{i=1}^{k} \int_{\mathbb{R}^{n-1}} (f \varphi_i)(x_1, \dots, x_{n-1}, g(x_1, \dots, x_{n-1})) \sqrt{1 + |\nabla g|^2}\, dx_1 \cdots dx_{n-1}$$

Density Theorem If $1 \le p < \infty$, $k \ge 0$ integer, then $C^\infty(\Omega) \cap W^{k,p}(\Omega)$ is dense in $W^{k,p}(\Omega)$. (Rk: Lemma 4 $\Rightarrow k=0$)

Proof WTS. $\forall\, u \in W^{k,p}(\Omega)$, $\forall\, \varepsilon > 0$, $\exists\, v \in C^\infty(\Omega) \cap W^{k,p}(\Omega)$

s.t. $\|u - v\|_{W^{k,p}(\Omega)} < \varepsilon$

$\Omega_j \triangleq \{ x \in \Omega \mid |x| \le R + j,\ \mathrm{dist}(x, \partial\Omega) > \frac{1}{j} \}$ (R large enough)

$\Omega_j \subset\subset \Omega_{j+1} \subset\subset \Omega$. $\Omega = \bigcup_{j=1}^{\infty} \Omega_j$

$\{\Omega_j\}_{j=1}^{\infty}$ does not satisfy ~~the finite~~ intersection property

Let $U_j = \Omega_{j+1} \setminus \bar{\Omega}_{j-1}$, $j \ge 0$ $(\Omega_{-1} = \Omega_0 = \emptyset)$

$U_0 = \Omega_1 \setminus \bar{\Omega}_{-1}$, $U_1 = \Omega_2 \setminus \bar{\Omega}_0 = \Omega_2$

$U_2 = \Omega_3 \setminus \bar{\Omega}_1$, $U_3 = \Omega_4 \setminus \Omega_2$ --- continue

$\Rightarrow \{U_j\}_{j=0}^{\infty}$ satisfies the conditions in Theorem 5

$\Rightarrow \exists$ partition of unity subordinate to $\{U_j\}_{j=0}^{\infty}$: $\sum_{j \ge 0} \varphi_j(x) = 1$

$0 \le \varphi_j \le 1$

$\mathrm{supp}\, \varphi_j \subset\subset U_j$, $j \ge 0$

Since $u \in W^{k,p}(\Omega)$, $\varphi_j u \in W^{k,p}(\Omega)$

$\mathrm{supp}(\varphi_j u) \subset\subset U_j$, $j \ge 0$

Consider $(\varphi_j u)_h(x) = \int_{\Omega \leftrightarrow U_j} j_h(x-y)(\varphi_j u)(y)\, dy$ $h > 0$ small.

Lemma 5 $\Rightarrow \partial^\alpha[(\varphi_j u)_h] = [\partial^\alpha(\varphi_j u)]_h$ on $\Omega_h := \{ x \in \Omega : \mathrm{dist}(x, \partial\Omega) > h \}$

(*)

Since for h small, $\forall x \in \mathbb{R}^n \setminus \Omega_h$, $B_h(x) \cap \operatorname{supp}(\varphi_j u) = \emptyset$.

$\Rightarrow$ each side of (*) is 0

$\Rightarrow$ (*) holds on $\mathbb{R}^n$ $\forall \alpha$ such that $|\alpha| \le k$

By Lemma 4 $\quad [\partial^\alpha(\varphi_j u)]_h \xrightarrow{h \to 0} \partial^\alpha(\varphi_j u)$ in $L^p(\Omega)$

$$\overset{(*)}{\Rightarrow} (\varphi_j u)_h \xrightarrow{h \to 0} \varphi_j u \text{ in } W^{k,p}(\Omega)$$

Now for $\forall \varepsilon > 0$, for each $j = 1, 2, \dots$, take $h_j > 0$ small such that $\|(\varphi_j u)_{h_j} - \varphi_j u\|_{W^{k,p}(\Omega)} < \frac{\varepsilon}{2^i}$

Let $v(x) = \sum_{j=0}^{\infty} (\varphi_j u)_{h_j}(x) \quad \operatorname{supp}(\varphi_j u)_{h_j} \subset\subset U_j$

$\forall$ subdomain $\Omega' \subset\subset \Omega$, Ω' intersects only finitely many $\operatorname{supp}(\varphi_j u)_{h_j} \Rightarrow v(x)$ is well-defined & $v \in C^\infty(\Omega')$

Ω' is arbitrary $\Rightarrow v \in C^\infty(\Omega)$

Moreover, $\|v - u\|_{W^{k,p}(\Omega)} = \left\| \sum_{j \ge 0} (\varphi_j u)_{h_j} - \sum_{j \ge 0} \varphi_j u \right\|_{W^{k,p}}$

$$\le \sum_{j \ge 0} \|(\varphi_j u)_{h_j} - \varphi_j u\|_{W^{k,p}}$$

$$\le \sum_{j \ge 0} \frac{\varepsilon}{2^{j+1}} = \varepsilon.$$

& $v = (v - u) + u \in W^{k,p}(\Omega)$ $\square$

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If Ω is bounded and $\partial\Omega \in C^1$, then $C^\infty(\bar{\Omega})$ is dense in $W^{k,p}(\Omega)$

(Recall, $C^\infty(\bar{\Omega})$ is dense in $L^p(\Omega)$ provided Ω being bounded, no need $\partial\Omega \in C^1$)

4.5 Sobolev Inequalities

$W^{1,p}(\Omega) = \{ u \in L^p(\Omega) \mid \text{weak } \nabla u \in L^p(\Omega) \}$

Recall $C_0^\infty(\Omega)$ is dense in $L^p(\Omega)$

Q: is $C_0^\infty(\Omega)$ dense in $W^{k,p}(\Omega)$?

A: No. In fact, $\overline{C_0^\infty(\Omega)}^{W^{k,p}(\Omega)} \triangleq W_0^{k,p}(\Omega) \subsetneq W^{k,p}(\Omega) \quad (k \geq 1)$

边介为0?

Sobolev Inequalities $(n \geq 2)$

(i) If $\Omega \subseteq \mathbb{R}^n$ a domain (may or may not be bounded) and $1 \leq p < n$.

$p^* = \dfrac{np}{n-p}$, then $\|u\|_{L^{p^*}(\Omega)} \leq C(n,p) \|\nabla u\|_{L^p(\Omega)} \quad \forall u \in W_0^{1,p}(\Omega)$

$C(n,p) = \dfrac{p(n-1)}{n-p} \dfrac{1}{\sqrt{n}} \quad (*1)$

(ii) If $\Omega \subseteq \mathbb{R}^n$ bounded domain and $p > n$, then $\forall u \in W_0^{1,p}(\Omega)$, we have

$\underbrace{u \in C^0(\bar{\Omega})}$, $\quad \|u\|_{L^\infty(\Omega)} \leq \tilde{C}(n,p) |\Omega|^{\frac{1}{n} - \frac{1}{p}} \|\nabla u\|_{L^p(\Omega)} \quad (*2)$

$\uparrow$

($\exists v \in C^0(\bar{\Omega})$, s.t. $u = v$ a.e. Ω

i.e. $\exists$ a continuous representative)

(iii) If $\Omega \subseteq \mathbb{R}^n$ a bounded domain, then $\forall u \in W_0^{1,n}(\Omega)$ and $1 \leq q < \infty$

we have $\|u\|_{L^q(\Omega)} \leq C(q, n, |\Omega|) \|\nabla u\|_{L^n(\Omega)} \quad (*3)$

Proof (i) Step 1 Special case $p = 1$ $u \in C_0^1(\Omega)$. Extend $u \equiv 0$ on Ω^c

$\Rightarrow u \in C_0^1(\mathbb{R}^n)$

For any $i = 1, \cdots, n$

$$u(x_1,\dots,x_i,\dots,x_n) \overset{FTC}{=} \int_{-\infty}^{x_i} \frac{\partial u}{\partial x_i}(x_1,\dots,x_i,\dots,x_n)\,dx_i$$

$$|u(x_1,\dots,x_i,\dots,x_n)| \le \int_{-\infty}^{\infty} \left|\frac{\partial u}{\partial x_i}(x_1,\dots,x_i,\dots,x_n)\right| dx_i$$

$$|u(x_1,\dots,x_n)|^{\frac{n}{n-1}} \le \left[\prod_{i=1}^{n} \int_{-\infty}^{\infty} \left|\frac{\partial u}{\partial x_i}\right| dx_i\right]^{\frac{1}{n-1}}$$

$$\int_{-\infty}^{\infty} |u|^{\frac{n}{n-1}} dx_1 \le \underbrace{\left[\int_{-\infty}^{\infty} \left|\frac{\partial u}{\partial x_1}\right| dx_1\right]^{\frac{1}{n-1}}}_{与\ x_1\ 无关} \underbrace{\int_{-\infty}^{\infty} \left[\prod_{i=2}^{n} \int_{-\infty}^{\infty} \left|\frac{\partial u}{\partial x_i}\right| dx_i\right]^{\frac{1}{n-1}} dx_1}$$

(Hölder $\int |f_1 \cdots f_m|\,dx \le \|f_1\|_{L^{p_1}} \cdots \|f_m\|_{L^{p_m}}$, $\frac{1}{p_1} + \dots + \frac{1}{p_m} = 1$)

$$\le \prod_{i=2}^{n} \left[\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \left|\frac{\partial u}{\partial x_i}\right| dx_i\,dx_1\right]^{\frac{1}{n-1}}$$

($f_i = \left[\int_{-\infty}^{\infty} \left|\frac{\partial u}{\partial x_i}\right| dx_i\right]^{\frac{1}{n-1}}$, $p_i = n-1$

$n-1$ terms $i = 2,\dots,n$)

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} |u|^{\frac{n}{n-1}} dx_1\,dx_2 \le \left(\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \left|\frac{\partial u}{\partial x_2}\right| dx_2\,dx_1\right)^{\frac{1}{n-1}} \cdot$$

$$\int_{-\infty}^{\infty} \left(\int_{-\infty}^{\infty} \left|\frac{\partial u}{\partial x_1}\right| dx_1\right)^{\frac{1}{n-1}} \prod_{i=3}^{n} \left(\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \left|\frac{\partial u}{\partial x_i}\right| dx_i\,dx_1\right)^{\frac{1}{n-1}} dx_2$$

$$\le \left[\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \left|\frac{\partial u}{\partial x_2}\right| dx_2\,dx_1\right] \left[\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \left|\frac{\partial u}{\partial x_1}\right| dx_1\,dx_2\right]^{\frac{1}{n-1}} \cdot \prod_{i=3}^{n} \left[\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \left|\frac{\partial u}{\partial x_1}\right| dx_i\,dx_1\,dx_2\right]^{\frac{1}{n-1}}$$

$$\int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} |u|^{\frac{n}{n-1}} dx_1 \cdots dx_n \le \prod_{i=1}^{n} \left[\int_{\mathbb{R}^n} \left|\frac{\partial u}{\partial x_i}\right| dx\right]^{\frac{1}{n-1}}$$

$$\Rightarrow \left[\int_{\mathbb{R}^n} |u|^{\frac{n}{n-1}} dx\right]^{\frac{n-1}{n}} \le \prod_{i=1}^{n} \left[\int_{\mathbb{R}^n} \left|\frac{\partial u}{\partial x_i}\right| dx\right]^{\frac{1}{n}}$$

(Inequality $\sqrt[n]{a_1 \cdots a_n} \le \frac{a_1 + \dots + a_n}{n}$)

$$\le \frac{1}{n} \int_{\mathbb{R}^n} \sum_{i=1}^{n} \left|\frac{\partial u}{\partial x_i}\right| dx$$

$$\leq \frac{\sqrt{n}}{n} \int_{\mathbb{R}^n} \left(\sum_{i=1}^{n} \left| \frac{\partial u}{\partial x_i} \right|^2 \right)^{\frac{1}{2}} dx \qquad \left(\left(\sum_{i=1}^{n} a_i^2 \right)^{\frac{1}{2}} \geq \frac{1}{\sqrt{n}} \sum_{i=1}^{n} |a_i| \right)$$

$$\leq \frac{\sqrt{n}}{n} \| \nabla u \|_{L^1(\mathbb{R}^n)}$$

$$\| u \|_{L^{\frac{n}{n-1}}(\mathbb{R}^n)} \leq \frac{1}{\sqrt{n}} \| \nabla u \|_{L^1(\mathbb{R}^n)}$$

$$\Longrightarrow \quad \| u \|_{L^{\frac{n}{n-1}}(\Omega)} \leq \frac{1}{\sqrt{n}} \| \nabla u \|_{L^1(\Omega)} \quad (*4)$$

Step 2. $1 < p < n$. $u \in C_0^1(\Omega)$

Consider $|u|^\alpha$ $\alpha > 1$ to be chosen

$\because \alpha > 1$ $|u|^\alpha \in C_0^1(\Omega)$ $\left(\frac{\partial}{\partial x_i} |u|^\alpha = \alpha |u|^{\alpha-1} \cdot \frac{\partial}{\partial x_i} u = \alpha |u|^{\alpha-1} \frac{\partial u}{\partial x_i} \cdot \operatorname{sign}(u) \right)$

By $(*4)$
$$\| |u|^\alpha \|_{L^{\frac{n}{n-1}}(\Omega)} \leq \frac{1}{\sqrt{n}} \| \nabla(|u|^\alpha) \|_{L^1(\Omega)}$$

$$= \frac{\alpha}{\sqrt{n}} \| |u|^{\alpha-1} \nabla u \|_{L^1(\Omega)}$$

$$\leq \frac{\alpha}{\sqrt{n}} \| \nabla u \|_{L^p(\Omega)} \| |u|^{\alpha-1} \|_{L^q(\Omega)}$$

$$\left(\int_\Omega |u|^{\frac{\alpha n}{n-1}} dx \right)^{\frac{n-1}{n}} \cdot \left(\int_\Omega |u|^{(\alpha-1)q} dx \right)^{-\frac{1}{q}} \leq \frac{\alpha}{\sqrt{n}} \| \nabla u \|_{L^p(\Omega)}$$

▷ $\frac{\alpha n}{n-1} = (\alpha-1) q = p^* = \frac{np}{n-p}$ 同时满足

▷ $\frac{n-1}{n} - \frac{1}{q} = \frac{n-1}{n} - \left(1 - \frac{1}{p}\right) = \frac{p(n-1) - np + n}{np}$

Step 3 Lastly, for general $u \in W_0^{1,p}(\Omega)$ by the definition of $W_0^{1,p}(\Omega)$, $\exists\, u_k \in C_0^\infty(\Omega)$ s.t. $u_k \to u$ in $W^{1,p}(\Omega)$

$$\begin{cases} u_k \to u \text{ in } L^p(\Omega) \ \& \text{ a.e. in } \Omega \text{ by passing to a subsequence } (*4a) \\ \nabla u_k \to \nabla u \text{ in } L^p(\Omega) \end{cases}$$

$(*4b)$

Apply $(*1)$ to u_k

$$\|u_k\|_{L^{\frac{np}{n-p}}(\Omega)} \leq C(n,p)\,\|\nabla u_k\|_{L^p(\Omega)} \quad (*5)$$

$$\overset{\text{Fatou}}{\Longrightarrow} \|u_k\|_{L^{\frac{np}{n-p}}(\Omega)} \overset{(*4)}{\leq} \liminf_{k\to\infty}\|u_k\|_{L^{\frac{np}{n-p}}(\Omega)}$$

$$\overset{(*5)}{\leq} C(n,p)\liminf_{k\to\infty}\|\nabla u_k\|_{L^p(\Omega)}$$

$$\overset{(*4b)}{=} C(n,p)\,\|\nabla u\|_{L^p(\Omega)} \qquad \square$$

(ii) Let $\tilde{u} = \dfrac{\sqrt{n}\,u}{\|\nabla u\|_{L^p(\Omega)}}$. Assume first $|\Omega| = 1$

Recall we obtaied in the proof of (i) that $\forall\, u \in C_0^1(\Omega)$

$$\forall \alpha > 1 \quad \||u|^\alpha\|_{L^{\frac{n}{n-1}}(\Omega)} \leq \frac{\alpha}{\sqrt{n}}\,\||u|^{\alpha-1}\|_{L^q(\Omega)} \cdot \|\nabla u\|_{L^p(\Omega)} \quad \left(\frac{1}{p}+\frac{1}{q}=1\right)$$

$$/\left(\frac{\sqrt{n}}{\|\nabla u\|_{L^p}}\right)^\alpha$$

$$\Longrightarrow \||\tilde{u}|^\alpha\|_{L^{\frac{n}{n-1}}(\Omega)} \leq \alpha\,\||\tilde{u}|^{\alpha-1}\|_{L^q(\Omega)}$$

$$\Longrightarrow \|\tilde{u}\|_{L^{\frac{\alpha n}{n-1}}(\Omega)} \leq \alpha^{\frac{1}{\alpha}}\,\|\tilde{u}\|_{L^{(\alpha-1)q}(\Omega)}^{(1-\frac{1}{\alpha})} \leq \alpha^{\frac{1}{\alpha}}\,\|\tilde{u}\|_{L^{\alpha q}(\Omega)}^{(1-\frac{1}{\alpha})}$$

$$\left(\because \|\tilde{u}\|_{L^{(\alpha-1)q}(\Omega)} \overset{\text{Hölder}}{\leq} \|\tilde{u}\|_{L^{\alpha q}(\Omega)}\right)$$

Take $\alpha = \delta^m$, $m = 1, 2, \dots$ $\delta = \dfrac{\frac{n}{n-1}}{\frac{p}{p-1}} > 1$ ($\because p > n$)

$$\Rightarrow \|\tilde{u}\|_{L^{\delta^m \frac{n}{n-1}}(\Omega)} \le (\delta^m)^{\delta^{-m}} \|\tilde{u}\|^{1-\delta^{-m}}_{L^{\delta^{m-1}\frac{n}{n-1}}(\Omega)}$$

$$\alpha q = \alpha \frac{1}{1-\frac{1}{p}} = \frac{\alpha p}{p-1}$$

$$\delta^m q = \delta^{m-1}\delta q = \delta^{m-1} \frac{\frac{n}{n-1}}{\frac{p}{p-1}} \frac{\alpha p}{p-1} = \delta^{m-1}\frac{\alpha n}{n-1}$$

$m \to \infty \downarrow$

$\|\tilde{u}\|_{L^\infty(\Omega)}$

$\Downarrow$

$\|\tilde{u}\|_{L^\infty(\Omega)} < X$

$\Rightarrow$

$\|u\|_{L^\infty(\Omega)} \le \dfrac{X}{\sqrt{\Omega}} \|\nabla u\|_{L^p(\Omega)}$

$\dfrac{X}{\sqrt{\Omega}} = \tilde{C}(n,p)$

$$\le (\delta^m)^{\delta^{-m}} \left[(\delta^{m-1})^{\delta^{-(m-1)}} \|\tilde{u}\|^{1-\delta^{-(m-1)}}_{L^{\delta^{m-2}\frac{n}{n-1}}(\Omega)} \right]^{1-\delta^{-m}}$$

$$\le \delta^{[m\delta^{-m} + (m-1)\delta^{-(m-1)}]} \|\tilde{u}\|^{(1-\delta^{-m})(1-\delta^{-(m-1)})}_{\delta^{m-2}\frac{n}{n-1}}$$

$\dots$

$$\le \delta^{[m\delta^{-m} + (m-1)\delta^{-(m-1)} + \dots + 1\cdot\delta^{-1}]} \cdot \|\tilde{u}\|^{(1-\delta^{-m})(1-\delta^{-(m-1)})\cdots(1-\delta^{-1})}_{\delta^0 \frac{n}{n-1} = \frac{n}{n-1}}$$

$$\le \delta^{\sum_{k=1}^{m} k\delta^{-k}}$$

$$< \delta^{\sum_{k=1}^{\infty} k\delta^{-k}} := X < \infty.$$

(Hw: $X \to \infty$ as $p \to n^+$ i.e. $\delta \to 1^+$)

$$\left(\|\tilde{u}\|_{\frac{n}{n-1}} = \frac{\sqrt{n}}{\|\nabla u\|_p} \|u\|_{\frac{n}{n-1}} \right.$$

If $|\Omega| \neq 1$, let $\Omega' = \{ y = \frac{x}{|\Omega|^{\frac{1}{n}}} \mid x \in \Omega \}$. Then $|\Omega'| = 1$

If $u \in C_0^1(\Omega)$ then $v(y) \triangleq u(x) \in C_0^1(\Omega')$

$$\|v\|_{L^\infty(\Omega')} \le \tilde{C}(n,p) \|\nabla_y v\|_{L^r(\Omega')}$$

$$\|u\|_{L^\infty(\Omega)} = \left[\int_{\Omega'} |\nabla_y v|^r dy\right]^{\frac{1}{r}} \xrightarrow{y = \frac{x}{|\Omega|^{\frac{1}{n}}}}$$

$$\left[\int_{\Omega} |\nabla_x u|^r |\Omega|^{\frac{r}{n}} \frac{1}{|\Omega|} dx\right]^{\frac{1}{r}}$$

$$\left(\nabla_y v = \nabla_x u \cdot |\Omega|^{\frac{1}{n}} \right)$$

$$= |\Omega|^{\frac{1}{n} - \frac{1}{r}} \|\nabla u\|_{L^r(\Omega)}$$

$u \in C_0^1(\Omega)$, general $u \in W_0^{1,r}(\Omega)$

take $u_k \in C_0^\infty(\Omega)$, $u_k \to u$ in $W^{1,r}(\Omega)$

$\square$