Notes for *Introduction to Representation Theory* 2022/23

March 30, 2023

Contents

[1] We've skipped this section in class for now, but we'll come back to it later

These are notes for the course *Introduction to Representation Theory* held in 2022/23. Most of this is based on the books [Vinb] and [Stein]. The text will be expanded throughout the term. Note that the text is *not* proofread, so be careful when using it! I'm of course happy to here about typos and mistakes you find, and I'd appreciate if you could send any errors you notice (or other comments you might have) to matz@math.ku.dk.

1 Basic Definitions

Our general setup will be as follows:

- G is a group; very soon we will impose some additional conditions on G
- K is a field; we will usually assume that $K = \mathbb{R}$ or $K = \mathbb{C}$
- V is a *finite dimensional* vector space over K.

We will write $\mathrm{GL}(V)$ for the group of invertible linear endomorphisms $V \longrightarrow V$.

Remark 1.1.
- *The restriction to $K \in \{\mathbb{R}, \mathbb{C}\}$ is usually not essential, but can be convenient to avoid having to deal with some exceptional cases. As long as the characteristic of K is 0 (and K is algebraically closed if necessary) most results will carry over.*

- *The assumption that V is finite dimensional is more critical as infinite dimensional representation theory is significantly more involved.*

Initially, we define two notions of a representation:

Definition 1.2. A *linear representation* of G on V (over K) is a group homomorphism

$$\pi : G \longrightarrow \mathrm{GL}(V).$$

We will usually denote this representation by (π, V) or just by π if V is understood from the context.

- V is called the *representation space* of π.
- $\dim_K V =: \dim \pi =: \deg \pi$ is called the *degree* (or *dimension*) of π.

Definition 1.3. Let $n \in \mathbb{N}$. A *matrix representation* of G over K of dimension (or degree) n is a group homomorphism

$$\pi : G \longrightarrow \mathrm{GL}_n(K).$$

Remark 1.4. *Given a matrix representation $\pi : G \longrightarrow \mathrm{GL}_n(K)$ we can canonically attach a linear representation by using the canonical isomorphism of groups $\mathrm{GL}_n(K) \longrightarrow \mathrm{GL}(K^n)$ given via the standard basis $\{e_1, \dots, e_n\}$ of K^n.*

Conversely, given a linear representation $\pi : G \longrightarrow \mathrm{GL}(V)$ and a basis $\mathcal{B} = \{v_1, \dots, v_n\}$ of V over K, we can attach a matrix representation $\pi_{\mathcal{B}} : G \longrightarrow \mathrm{GL}_n(K) \simeq \mathrm{GL}(K^n)$ (the last isomorphism is the canonical one from before) which satisfies

$$\pi_{\mathcal{B}}(g)e_j = \varphi_{\mathcal{B}}(\pi(g)v_j)$$

where $\varphi_{\mathcal{B}} : V \longrightarrow K^n$ is the isomorphism characterized by $\varphi_{\mathcal{B}}(v_i) = e_i$, $i = 1, \dots, n$. In other words, $\pi_{\mathcal{B}}(g)$ is the matrix $(a_{ij})_{i,j=1,\dots,n}$ such that

$$\pi(g)v_j = \sum_{i=1}^{n} a_{ij} v_j.$$

This matrix representation depends on the choice of basis. To make this construction more 'canonical' we introduce the notion of equivalence *next.*

Definition 1.5.
- Let (π_1, V_1), (π_2, V_2) be two linear representations of a group over the same field K. We call π_1 and π_2 *equivalent*, written as $(\pi_1, V_1) \simeq (\pi_2, V_2)$, or simply $\pi_1 \simeq \pi_2$, if there exists a vector space isomorphism $\Phi : V_1 \longrightarrow V_2$ such that

$$\pi_1(g) = \Phi^{-1} \circ \pi_2(g) \circ \Phi$$

for all $g \in G$.

- If $\pi_1 : G \longrightarrow \mathrm{GL}_{n_1}(K)$, $\pi_2 : G \longrightarrow \mathrm{GL}_{n_2}(K)$ are two matrix representations of G, we call π_1 and π_2 *equivalent*, denoted by $\pi_1 \simeq \pi_2$, if $n_1 = n_2 =: n$ and there exists $A \in \mathrm{GL}_n(K)$ such that
$$\pi_1(g) = A^{-1}\pi_2(g)A$$
for all $g \in G$.

Note that equivalence of (linear or matrix) representations is an equivalence relation on the set of all (linear or matrix) representations.

Lemma 1.6. *Let (π, V) be a linear representation of G, and let $\mathcal{B}_1$, $\mathcal{B}_2$ be two bases for V. Then $\pi_{\mathcal{B}_1} \simeq \pi_{\mathcal{B}_2}$.*

Conversely, if π_1, π_2 are two equivalent matrix representations of G, the corresponding linear representations on K^n are then equivalent as well.

Proof. The second assertion is immediate from the above definitions. For the first assertion note that the equivalence of $\pi_{\mathcal{B}_1}$ and $\pi_{\mathcal{B}_2}$ will be realized by the base change matrix that changes $\mathcal{B}_1$ to $\mathcal{B}_2$. □

Remark 1.7. *This lemma therefore implies that we have a bijection between the set of equivalence classes of matrix representations of G over K and the set of equivalence classes of linear representations of G over K given by the maps above.*

Convention: We'll usually only be interested in linear or matrix representations up to equivalence as all important properties are preserved under equivalency. We will therefore in the following often not distinguish between linear and matrix representations, and just speak of *representations* of G.

1.1 Examples

Example 1.8 (One-dimensional representations of $\mathbb{R}$)**.** Let $K = \mathbb{R}$ or $K = \mathbb{C}$. Suppose $\alpha \in K$. Then
$$\pi_\alpha : \mathbb{R} \longrightarrow \mathrm{GL}_1(K) = K^\times,\ \pi_\alpha(t) := e^{\alpha t}$$
defines a one-dimensional representation of the additive group $\mathbb{R}$.

In fact, if we assume that the representation $\pi : \mathbb{R} \longrightarrow K^\times$ is differentiable, these are all possible one-dimensional representation. (Here by a 'differentiable representation' we just understand a group homomorphism $\pi : \mathbb{R} \longrightarrow K^\times$ that happens to be a differentiable function from $\mathbb{R}$ to K.) To see this let $\pi : \mathbb{R} \longrightarrow K^\times$ be a differentiable representation. Then for all $t \in \mathbb{R}$,
$$\frac{d}{ds}\pi(t+s)_{|s=0} = \frac{d}{ds}\left(\pi(t)\pi(s)\right)_{|s=0} = \pi'(0)\pi(t).$$
We know from analysis that π therefore must be of the form $\pi(t) = \beta e^{\alpha t}$ for suitable $\alpha, \beta \in K$. Since $\pi(0) = 1$, we get $\beta = 1$, hence $\pi = \pi_\alpha$.

Example 1.9 (n-dimensional representations of $\mathbb{R}$)**.** Let $K = \mathbb{R}$ or $K = \mathbb{C}$. For $A \in M_n(K)$ we define the *matrix exponential* e^A by

$$e^A = \sum_{k\geq 0} \frac{A^k}{k!} = \mathbf{1}_n + A + \frac{1}{2}A^2 + \frac{1}{6}A^3 + \dots$$

where $\mathbf{1}_n$ denotes the $n \times n$ identity matrix.

We list several properties of this matrix exponential (see [Hall, Proposition 2.3]; or you can check these properties yourself)

- The series defining e^A converges absolutely for any $A \in M_n(K)$ with respect to the usual matrix norm on $M_n(K)$.
- $\forall A \in M_n(K)$: $e^A \in \mathrm{GL}_n(K)$
- $e^{\mathbf{0}_n} = \mathbf{1}_n$ where $\mathbf{0}_n$ denotes the $n \times n$ zero matrix
- $\forall B \in \mathrm{GL}_n(K), A \in M_n(K)$: $B^{-1}e^A B = e^{B^{-1}AB}$
- If $A, B \in M_n(K)$ with $AB = BA$, then $e^{A+B} = e^A e^B$. In particular, $e^{-A}e^A = \mathbf{1}_n$, that is, e^{-A} is the inverse matrix of e^A.

Using these properties, we see that if $A \in M_n(K)$, then

$$\pi_A : \mathbb{R} \longrightarrow \mathrm{GL}_n(K),\ \pi_A(t) = e^{tA}$$

defines an n-dimensional representation of the additive group $\mathbb{R}$. In fact, one can show similarly as above that if $\pi : \mathbb{R} \longrightarrow \mathrm{GL}_n(K)$ is a differentiable representation, we can find $A \in M_n(K)$ such that $\pi = \pi_A$. (Again, we understand 'differentiable representation' in the naive way as being a representation $\pi : \mathbb{R} \longrightarrow \mathrm{GL}_n(K)$ which at the same time happens to be a differentiable map $\mathbb{R} \longrightarrow M_n(K)$.)

Example 1.10 (Rotations in $\mathbb{R}^2$)**.** We define

$$\pi : \mathbb{R} \longrightarrow \mathrm{GL}_2(\mathbb{R}),\ \pi(t) := \begin{pmatrix} \cos(t) & \sin(t) \\ -\sin(t) & \cos(t) \end{pmatrix}$$

Geometrically, the matrix t represents the counterclockwise rotation by angle t around the origin in $\mathbb{R}$. To see that this is in fact a representation (which can also be checked by hand using trigonometric identities), we'll find a matrix $A \in M_2(\mathbb{R})$ such that $\pi = \pi_A$. In fact, let

$$A = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

To see that this is the correct choice, note that

$$A^2 = -\mathbf{1}_2,\ A^3 = -A,\ A^4 = \mathbf{1}_2.$$

We can therefore compute

$$e^{tA} = \sum_{k\geq 0} \frac{A^k}{k!} = \sum_{k=2m, m\geq 0} \frac{(-1)^m}{(2m)!}\mathbf{1}_2 + \sum_{k=2m+1, m\geq 0} \frac{(-1)^m}{(2m+1)!} A = \cos(t)\mathbf{1}_2 + \sin(t)A = \pi(t).$$

Example 1.11. Consider the symmetric group S_n on n elements. For $i, j = 1, \dots, n$ write $E_{ij} \in M_n(\mathbb{C})$ for the matrix given by

$$(E_{ij})_{kl} = \begin{cases} 1 & \text{if } (i,j) = (k,l) \\ 0 & \text{else.} \end{cases}$$

Define

$$\pi : S_n \longrightarrow \mathrm{GL}_n(\mathbb{C}),\ \pi(\sigma) = E_{1\sigma(1)} + \ldots + E_{n\sigma(n)}$$

It can easily be seen that each of the matrices $\pi(\sigma)$ is indeed invertible and that $\pi(\mathrm{id}) = \mathbf{1}_n$. To see that it defines a group homomorphism note that

$$E_{ij}E_{kl} = \begin{cases} E_{il} & \text{if } j = k \\ \mathbf{0}_n & \text{else.} \end{cases}$$

Hence for $\sigma, \tau \in S_n$ we get

$$\pi(\sigma)\pi(\tau) = \sum_{i=1}^{n} E_{i\sigma(i)} \sum_{j=1}^{n} E_{j\tau(j)} = \sum_{i,j:\sigma(i)=j} E_{i\tau(j)} = \sum_{i=1}^{n} E_{i\tau\circ\sigma(i)} = \pi(\tau \circ \sigma)$$

so that π is indeed a group homomorphism.

Note that this representation can also be defined by $\pi : S_n \longrightarrow \mathrm{GL}(\mathbb{C}^n)$, $\pi(\sigma)e_j = e_{\sigma(j)}$ where $e_1, \dots, e_n$ denotes the standard basis of $\mathbb{C}^n$.

Example 1.12. Suppose X is a finite set, G a group.

Recall: A group action of G on X is a map

$$G \times X \longrightarrow X\ (g, x) \mapsto g \cdot x$$

such that $e \cdot x = x$ and $g \cdot (h \cdot x) = (gh) \cdot x$ for all $x \in X$, $g, h \in G$ (here e denotes the neutral element of G). Equivalently, a group action of G on X is a group homomorphism $G \longrightarrow S(G) :=$ group of bijections $X \to X$.

Suppose we are given a group action of G on X. Let V_X denote the K-vector space of all (formal) linear combinations of elements in X. Then V_X is a K-vector space of dimension $|X|$. In generalization of the previous example, we define

$$\pi : G \longrightarrow \mathrm{GL}(V_X),\ \pi(g) \sum_{x \in X} a_x x := \sum_{x \in X} a_x g \cdot x.$$

This is indeed a representation of G which can be checked similarly as in the previous example.

Note that this representation is sometimes defined slightly differently: Let $K[X]$ denote the K-vector space of all functions $X \longrightarrow K$. Then given a group action of G on X we define

$$\tilde{\pi} : G \longrightarrow \mathrm{GL}(K[X])$$

by $(\tilde{\pi}(g)f)(x) := f(g^{-1} \cdot x)$ for $f \in \mathrm{GL}(K[X])$, $x \in X$. $\tilde{\pi}$ is in fact equivalent to π as can be seen from the vector space isomorphism

$$K[X] \longrightarrow V_X,\ f \mapsto \sum_{x \in X} f(x)x.$$

(In fact, $\tilde{\pi}$ is the dual of π as will become clear later.)

Example 1.13 (Left-/right-regular representation)**.** Let G be a group, and let $\mathbb{C}[G]$ denote the $\mathbb{C}$-vector space of all functions $G \longrightarrow \mathbb{C}$. G acts on itself from the left and right, that is, we have a left action

$$G \times G \longrightarrow G,\ (g, x) \mapsto gx$$

and a right action

$$G \times G \longrightarrow G,\ (g, x) \mapsto xg^{-1}.$$

These induce corresponding representations on $\mathbb{C}[G]$ as in the previous example, namely the *left regular representation of* G

$$L : G \longrightarrow \mathrm{GL}(\mathbb{C}[G]),\ (L(g)f)(x) = f(g^{-1}x),$$

and the *right regular representation of* G

$$R : G \longrightarrow \mathrm{GL}(\mathbb{C}[G]),\ (R(g)f)(x) = f(xg).$$

Note that when G is not finite one often considers the left-/right-regular representations on slightly smaller subspaces of functions (e.g. continuous functions).

1.2 Invariant subspaces

Definition 1.14. Suppose (π, V) is a representation of a group G, and W is a subvectorspace of V. W is called *invariant* (with respect to π) if

$$\pi(g)W \subseteq W$$

for all $g \in G$.

If $W \subseteq V$ is an invariant subspace, then (π_W, W) with $\pi_W(g) := \pi(g)_{|W}$, $g \in G$, is again a representation of G, and called a *subrepresentation* of π.

Remark 1.15.

- *We will sometimes say that a representation (σ, U) is a subrepresentation of (π, V) if σ is equivalent to an actual subrepresentation of π.*
- *If W_1, W_2 are two invariant subspaces of V, then $W_1 \cap W_2$ and $W_1 + W_2$ are both invariant again.*

Example 1.16. Recall the representation $\pi : S_n \longrightarrow \mathrm{GL}(\mathbb{C}^n)$, $\pi(\sigma)e_j = e_{\sigma(j)}$ from Example 1.11. Let $W = \mathbb{C}(e_1 + \ldots + e_n) \subseteq \mathbb{C}^n$. Then W is an invariant subspace, and (π_W, W) is in fact equivalent to the *trivial representation* of S_n, that is, $\pi_W(\sigma) = \mathrm{id}_W$ for all $\sigma \in S_n$.

Definition/Lemma 1.17. *Suppose (π, V) is a representation of G and $W \subseteq V$ is an invariant subspace. We define the* quotient representation $\pi_{V/W} : G \longrightarrow \mathrm{GL}(V/W)$ *by*

$$\pi_{V/W}(g)(v + W) = \pi(g)v + W.$$

Then $(\pi_{V/W}, V/W)$ is a well-defined representation of G on V/W. Moreover, if there exists an invariant subspace $W' \subseteq V$ such that $V = W \oplus W'$, then $\pi_{V/W} \simeq \pi_{W'}$.

Proof. To check that $\pi_{V/W}$ is well-defined, let $v_1, v_2 \in V$ with $v_1 + W = v_2 + W$. Then for all $g \in G$,

$$\pi(g)v_1 - \pi(g)v_2 = \pi(g)(v_1 - v_2) \in W$$

since W is invariant and $v_1 - v_2 \in W$. Hence $\pi_{V/W}$ is well-defined.

For the last assertion note that $W' \longrightarrow V/W$, $w' \mapsto w' + W$ defines an isomorphism of vector spaces, and carries $\pi_{W'}$ to $\pi_{V/W}$. $\square$

Definition 1.18. A representation (π, V) of G is called *irreducible* if whenever $W \subseteq V$ is an invariant subspace, it follows that $W = \{0\}$ or $W = V$.

Example 1.19. Every 1-dimensional representation is irreducible.

Example 1.20. Recall the representation $\pi : \mathbb{R} \longrightarrow \mathrm{GL}_2(\mathbb{R})$, $\pi(t) = \left(\begin{smallmatrix} \cos(t) & \sin(t) \\ -\sin(t) & \cos(t) \end{smallmatrix}\right)$ from Example 1.10. π is irreducible. To see this, suppose that this was not true, that is, there exists a (necessarily) 1-dimensional invariant subspace $W \subseteq \mathbb{R}^2$. Let $v = (v_1, v_2) \in \mathbb{R}^2$ be a vector spanning W. Then

$$\pi(\frac{\pi}{2})v = \begin{pmatrix} -v_2 \\ v_1 \end{pmatrix}$$

which is not contained in $\mathbb{R}v$, and therefore a contradiction to our assumption.

Example 1.21. Recall the representation $\pi : S_n \longrightarrow \mathrm{GL}_n(\mathbb{C})$ from Example 1.11. We saw that $W = \mathbb{C}(e_1 + \ldots + e_n)$ is an invariant subspace so that π is not irreducible. Define

$$W' = \{x \in \mathbb{C}^n \mid x_1 + \ldots + x_n = 0\}.$$

Then clearly $\mathbb{C}^n = W \oplus W'$, and W' is invariant itself. Since W is 1-dimensional, the subrepresentation (π_W, W) is irreducible. We now also show that $(\pi_{W'}, W')$ is irreducible as well.

For this first note that $\{e_1 - e_2, e_1 - e_3, \ldots, e_1 - e_n\}$ is a basis of W'. Now let $U \subseteq W'$ be an invariant subspace, $U \neq \{0\}$. Let $u \in U$, $u \neq 0$, and write $u = a_1 e_1 + \ldots + a_n e_n$ for suitable $a_1, \ldots, a_n \in \mathbb{C}$. Since $u \notin W$ there must be i, j, $i \neq j$ such that $a_i \neq a_j$. Let $(ij) \in S_n$ denote the permutation transposing i and j. Then

$$W \ni \pi((ij))u - u = a_j e_i + a_i e_j - (a_i e_i + a_j e_j) = (a_j - a_i)(e_i - e_j) \neq 0$$

so that $e_i - e_j \in U$. If $i = 1$, it follows that then also $e_1 - e_k = \pi((jk))(e_1 - e_j) \in U$ and thus $U = W'$. If $i \neq 1$, we first apply $\pi((1i))$ to reduce to the case $i = 1$.

2 Complete Reducibility and Maschke's Theorem

2.1 Basic definitions and properties

Definition 2.1. A representation (π, V) is *completely reducible* if whenever $W \subseteq V$ is an invariant subspace, there exists an invariant subspace $W' \subseteq V$ such that $V = W \otimes W'$. In other words, every invariant subspace of V admits an invariant complement.

Example 2.2. Every irreducible representation is completely reducible. The representation $\pi : S_n \longrightarrow \mathrm{GL}_n(\mathbb{C})$ as above is also completely reducible.

Example 2.3. The representation

$$\pi : \mathbb{R} \longrightarrow \mathrm{GL}_2(\mathbb{C}),\ \pi(t) := \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix}$$

is *not* completely reducible. The subspace $\mathbb{C}e_1$ spanned by e_1 in $\mathbb{C}^2$ is clearly invariant, but it does not have an invariant complement: If W' is a complement of $\mathbb{C}e_1$ in $\mathbb{C}^2$ it must have dimension 1, hence $W' = \mathbb{C}v$ for some suitable $v = (v_1, v_2)^t \in \mathbb{C}^2$, $v_2 \neq 0$. But then

$$\pi(1)v - v = \begin{pmatrix} v_2 \\ 0 \end{pmatrix} \in W \cap W' = \{0\}$$

which is a contradiction since $v_2 \neq 0$.

In the next section we will prove that real and complex representation of finite or compact groups are always completely reducible. As the last example shows, this is not necessarily the case for other groups.

Proposition 2.4. *Suppose (π, V) is a completely reducible representation of G and (π_W, W) a subrepresentation of (π, V). Then (π_W, W) is also completely reducible*

Proof. Let (π, V) and (π_W, W) be as in the statement of the proposition. Let $U \subseteq W$ be an invariant subspace. Since π is completely reducible, there exists an invariant subspace $U' \subseteq V$ such that $V = U \oplus U'$. Then $W = U \oplus (U' \cap W)$ and by Remark 1.15 $U' \cap W$ is also invariant. Hence we've found an invariant complement to U in W so that π_W is completely reducible. $\square$

Corollary 2.5. *Suppose (π, V) is a completely reducible representation of G. Then there exists minimal invariant subspaces $W_1, \ldots, W_r$ of V such that*

$$V = W_1 \oplus \ldots \oplus W_r.$$

Note that an invariant subspace $W \subseteq V$, $W \neq 0$, is minimal if and only if the subrepresentation (π_W, W) is irreducible.

Proof. Recall that we assume throughout that all our representations are finite dimensional. If π is irreducible, V is a minimal invariant subspace of itself, hence there is nothing to show. We therefore assume that π is not irreducible. Let $W \subseteq V$ be an invariant subspace, $W \neq \{0\}$, $W \neq V$. By assumption there exists an invariant complement $W' \subseteq V$ so that $V = W \oplus W'$. Note that $\dim W, \dim W' < \dim V$, and by Proposition 2.4 both (π_W, W) and $(\pi_{W'}, W')$ are completely reducible again. We continue this process until the resulting subrepresentations are irreducible. Note that this requires only finitely many iterations since $\dim V < \infty$ and $\dim W, \dim W' < \dim V$. □

Theorem 2.6. *Suppose (π, V) is a finite dimensional representation of G. Suppose $W_1, \ldots, W_r \subseteq V$ are minimal invariant subspaces (all non-zero) such that*

$$V = W_1 + \ldots + W_r \tag{1}$$

(we don't assume that this sum is direct). Then π is completely reducible. Moreover, for every invariant subspace $W \subseteq V$ there exists $0 \leq l \leq r$ and $i_1, \ldots, i_l \in \{1, \ldots, r\}$ (if $l = 0$, the set of i_j's is understood to be empty) such that

$$V = W \oplus W_{i_1} \oplus \ldots \oplus W_{i_l}. \tag{2}$$

Proof. It will suffice to prove (2). Let $W \subseteq V$ be an invariant subspace. Let $I = \{i_1, \ldots, i_l\} \subseteq \{1, \ldots, r\}$ be a maximal (potentially empty) subset such that the collection of subspaces $W, W_{i_1}, \ldots, W_{i_l}$ is linearly independent, that is

$$W + W_{i_1} + \ldots + W_{i_l} = W \oplus W_{i_1} \oplus \ldots \oplus W_{i_l}.$$

We call this sum U. If $U = V$, we are done. Hence we assume that $U \subsetneq V$. Then because of (1) there exists $j \in \{1, \ldots, r\}$ such that $W_j \not\subseteq U$, so in particular, $W_j \neq W_{i_k}$, $k = 1, \ldots, l$. Since the W_j is minimal invariant, the intersection $W_j \cap W_{i_k}$ is therefore empty for each $k = 1, \ldots, l$, and for the same reason, $W_j \cap W = \emptyset$. Hence the collection of subspaces $W, W_{i_1}, \ldots, W_{i_l}, W_j$ is linearly independent in contradiction to the maximality assumption of I. Thus $U = V$. □

Example 2.7. Complete reducibility of π_A equivalent to A being diagonalizable.

2.2 Orthogonal and Unitary Representations

From now on V will denote a finite dimensional real or complex vector space, and K accordingly denotes $\mathbb{R}$ or $\mathbb{C}$.

Recall: An *inner product* on V is a map

$$\beta : V \times V \longrightarrow K$$

such that

- $\forall v, w \in V$: $\beta(v, w) = \overline{\beta(w, v)}$ (where the bar denotes complex conjugation; if $K = \mathbb{R}$ this acts trivially and β is symmetric),

- $\forall v_1, v_2, w \in V$, $\alpha_1, \alpha_2 \in K$: $\beta(\alpha_1 v_1 + \alpha_2 v_2, w) = \alpha_1 \beta(v_1, w) + \alpha_2 \beta(v_2, w)$,
- $\forall v \in V \setminus \{0\}$: $\beta(v, v) > 0$.

Definition 2.8. A representation (π, V) of a group G is called *orthogonal* (if $K = \mathbb{R}$) or *unitary* (if $K = \mathbb{C}$) if there exists an inner product β on V which is invariant under π, that is

$$\beta(\pi(g)v, \pi(g)w) = \beta(v, w)$$

for all $v, w \in V$ and all $g \in G$.

Proposition 2.9. *Suppose (π, V) is an orthogonal or unitary representation of G. Then π is completely reducible.*

Proof. Suppose (π, V) is orthogonal or unitary, and let β be a π-invariant inner product on V. Let $W \subseteq V$ be an invariant subspace. Define

$$W' = \{v \in V \mid \forall w \in W : \ \beta(v, w) = 0\}.$$

From linear algebra we know that $V = W \oplus W'$. Hence it will suffice to show that W' is invariant. Let $g \in G$, $w' \in W'$ and $w \in W$. Then

$$\beta(\pi(g)w', w) = \beta(\pi(g)w', \pi(g)\pi(g^{-1})w) = \beta(w', \pi(g^{-1})w)$$

where we used the π-invariance of β for the last equality. Since W is invariant, $\pi(g^{-1})w \in W$, hence $\beta(w', \pi(g^{-1})w) = 0$ by definition of W'. It follows that $\pi(g)w' \in W'$ so that W' is indeed invariant. ☐

2.3 Complete reducibility for finite and compact groups

Theorem 2.10 (Maschke's Theorem)**.** *Every finite dimensional real or complex representation of a* finite *group G is completely reducible.*

Proof. Let (π, V) be a finite dimensional real or complex representation of G. Let β_0 be an (arbitrary) inner product on V (such an inner product can, for example, be found by fixing an isomorphism $V \simeq K^n$ and taking the standard inner product on K^n). Define for $v, w \in W$

$$\beta(v, w) = \frac{1}{|G|} \sum_{g \in G} \beta_0(\pi(g)v, \pi(g)w).$$

It is easily seen that β satisfies the first two properties of an inner product. For the positive definiteness note that if $v \in V$, $v \neq 0$, then $\beta_0(\pi(g)v, \pi(g)v) > 0$ for each $g \in G$ so that β is positive definite as well. β is in fact also π-invariant: We have

$$\begin{aligned} \beta(\pi(h)v, \pi(h)w) = \frac{1}{|G|} \sum_{g \in G} \beta_0(\pi(g)\pi(h)v, \pi(g)\pi(h)w) &= \frac{1}{|G|} \sum_{g \in G} \beta_0(\pi(gh)v, \pi(gh)w) \\ &= \frac{1}{|G|} \sum_{x \in G} \beta_0(\pi(x)v, \pi(x)w) = \beta(v, w) \end{aligned}$$

for all $h \in G$ and all $v, w \in V$. Hence (π, V) is orthogonal (if $K = \mathbb{R}$) or unitary (if $K = \mathbb{C}$) so that by Proposition 2.9 π is completely reducible. □

Suppose now that G is a compact topological group (see Appendix 12.1 for the definitions and examples). All representations considered here are finite dimensional real or complex continuous representations.

Theorem 2.11. *Every finite dimensional continuous real or complex representation (π, V) of a compact topological group G is completely reducible.*

By Proposition 2.9 it will suffice to show the following:

Lemma 2.12. *Let (π, V) and G be as in Theorem 2.11. Then π is orthogonal or unitary (depending on whether it is real or complex).*

Proof. Recall that we fix a right invariant Haar measure on G. We will assume that we normalized it so that G has measure 1. Let β_0 be an (arbitrary) inner product on V. Define β for $v, w \in V$ by

$$\beta(v, w) = \int_G \beta_0(\pi(g)v, \pi(g)w)\, dg$$

(note that $G \ni g \mapsto \beta_0(\pi(g)v, \pi(g)w) \in K$ defines a continuous function $G \to K$). Then β clearly satisfies the first two properties of an inner product. For the positive definiteness use that β_0 is positive definite together with property (13) of the Haar measure. We finally check that β is π-invariant: For $h \in G$, $v, w \in V$ we have

$$\beta(\pi(h)v, \pi(h)w) = \int_G \beta_0(\pi(gh)v, \pi(gh)w)\, dg = \int_G \beta_0(\pi(g)v, \pi(g)w)\, dg = \beta(v, w)$$

where we used that the Haar measure is right-invariant. □

3 Constructing new representations from old ones

We continue to assume that all representations are finite dimensional and that $K = \mathbb{R}$ or $K = \mathbb{C}$.

3.1 The contragredient/dual representation

Suppose (π, V) is a representation of G. Recall that the dual V' of V is the K-vector space of all linear maps $f : V \longrightarrow K$.

Definition 3.1. The *dual* (or *contragredient*) of (π, V) is the representation (π', V') of G defined by

$$\pi'(g)f(v) = f(\pi(g^{-1})v)$$

for $g \in G$, $f \in V'$, $v \in V$.

Note that this indeed defines a representation of G: If $g, h \in G$, then

$$\pi'(gh)f(v) = f(\pi((gh)^{-1})v) = f(\pi(h^{-1})\pi(g^{-1})v) = (\pi'(h)f)(\pi(g^{-1})v) = \pi'(g) \circ \pi'(h)f(v).$$

Example 3.2. We want to work out what the dual representation looks like in terms of matrices. Suppose $V = K^n$ so that we canonically identify π with the matrix representation $\pi : G \longrightarrow \mathrm{GL}_n(K)$. Let $e_1, \ldots, e_n$ denote the standard basis of K^n, and let $\langle \cdot, \cdot \rangle$ denote the standard inner product on K^n. Then a basis for the dual of K^n is given by the linear forms $\langle \cdot, e_j \rangle$, $j = 1, \ldots, n$, and we identify $(K^n)'$ with K^n with respect to this basis. We can then compute

$$\pi'(g)\langle \cdot, e_j \rangle(e_i) = \langle \pi(g^{-1})e_i, e_j \rangle = (\pi(g^{-1})e_i)^t e_j = e_i^t \pi(g^{-1})^t e_j$$

hence as a matrix representation we obtain

$$\pi'(g) = \left(\pi(g)^{-1}\right)^t.$$

Proposition 3.3. *Let (π, V) be as above. Then*

1. $(\pi'', V'') \simeq (\pi, V)$
2. *π is irreducible if and only if π' is irreducible.*

Proof. (i) is clear. For (ii) suppose π is irreducible and let $U \subseteq V'$ be invariant. Define

$$W := \{v \in V \mid \forall f \in U : f(v) = 0\}.$$

Then for $v \in W$ and $g \in G$ we get that $f(\pi(g)v) = \pi'(g^{-1})f(v) = 0$ for every $f \in U$ since U is invariant. Hence W is an invariant subspace of V, and thus $W = \{0\}$ or $W = V$ since π is irreducible. But then $U = V'$ or $U = \{0\}$ and therefore π' is irreducible as well. The other direction follows from this together with (i). □

3.2 Direct Sums

Definition 3.4. Suppose (π_i, V_i), $i = 1, 2$ are two representations of a group G over the same field K. We define the *direct sum* $(\pi_1 \oplus \pi_2, V_1 \oplus V_2)$ as the representation of G given by

$$\pi_1 \oplus \pi_2(g)(v_1, v_2) = (\pi_1(g)v_1, \pi_2(g)v_2).$$

If $(\pi_1, V_1), \ldots, (\pi_r, V_r)$ are representations of G, we inductively also define $(\pi_1 \oplus \ldots \oplus \pi_r, V_1 \oplus \ldots \oplus V_r)$.

Remark 3.5. *We clearly have $(\pi_1 \oplus \pi_2, V_1 \oplus V_2) \simeq (\pi_2 \oplus \pi_1, V_2 \oplus V_1)$, and more generally if σ is a permutation of $\{1, \ldots, r\}$, $(\pi_1 \oplus \ldots \oplus \pi_r, V_1 \oplus \ldots \oplus V_r) \simeq (\pi_{\sigma(1)} \oplus \ldots \oplus \pi_{\sigma(r)}, V_{\sigma(1)} \oplus \ldots \oplus V_{\sigma(r)})$. We therefore don't always need to fix a specific order when taking direct sums of representations, that is, if we are given representations (π_i, V_i) for i in some finite index set I, we might write $\bigoplus_{i \in I}(\pi_i, V_i)$ and this is well-defined up to equivalence.*

Example 3.6. We want to work out how direct sums look in terms of matrix representations. Suppose $(\pi_1, V_1), \ldots, (\pi_r, V_r)$ are representations, and $\mathcal{B}_1, \ldots, \mathcal{B}_r$ are bases for $V_1, \ldots, V_r$, respectively. Let $\pi_{i,\mathcal{B}_i} : G \longrightarrow \mathrm{GL}_{n_i}(K)$ denote the corresponding matrix representation, where $n_i = \dim \pi_i$. Then $\mathcal{B} = \mathcal{B}_1 \cup \ldots \cup \mathcal{B}_r$ is a basis of $V = V_1 \oplus \ldots \oplus V_r$ (we use the ordering of $\mathcal{B}$ obtained by concatenating the $\mathcal{B}_i$'s), and writing $\pi := \pi_1 \oplus \ldots \oplus \pi_r$ and $n = n_1 + \ldots + n_r$ we obtain

$$\pi_{\mathcal{B}} : G \longrightarrow \mathrm{GL}_n(K), \quad \pi_{\mathcal{B}}(g) = \begin{pmatrix} \pi_{1,\mathcal{B}_1}(g) & & & \\ & \pi_{2,\mathcal{B}_2}(g) & & \\ & & \ddots & \\ & & & \pi_{r,\mathcal{B}_r}(g) \end{pmatrix}$$

which is a block diagonal matrices with blocks of size $n_i \times n_i$ on the diagonal.

We collect some basic results on direct products:

Theorem 3.7. *A representation (π, V) of G is completely reducible if and only if there exists irreducible representations $(\pi_1, V_1), \ldots, (\pi_r, V_r)$ of G such that $\pi \simeq \pi_1 \oplus \ldots \oplus \pi_r$.*

Proof. This is just Corollary 2.5 and Theorem 2.6 in terms of direct sums of representations. □

Theorem 3.8. *Suppose $(\pi, V) \simeq (\pi_1, V_1) \oplus \ldots \oplus (\pi_r, V_r)$ with $\pi_1, \ldots, \pi_r$ irreducible. Then if $W \subseteq V$ is invariant, there exists $I, J \subseteq \{1, \ldots, r\}$ such that*

$$(\pi_W, W) \simeq \bigoplus_{i \in I} (\pi_i, V_i), \text{ and } (\pi_{V/W}, V/W) \simeq \bigoplus_{j \in J} (\pi_j, V_j).$$

Proof. Suppose $W \subseteq V$ is invariant. By Theorem 2.6 there exists $J \subseteq \{1, \ldots, r\}$ such that

$$(\pi, V) \simeq (\pi_W, W) \oplus \bigoplus_{j \in J} (\pi_j, V_j)$$

so that

$$(\pi_{V/W}, V/W) \simeq \bigoplus_{j \in J} (\pi_j, V_j).$$

For π_W note that $(\pi_W, W) \simeq (\pi_{V/\bigoplus_{j \in J} V_j}, V/\bigoplus_{j \in J} V_j)$ so that this case also follows from the case of the quotient representation. □

Corollary 3.9. *Suppose (π_i, V_i), $i = 1, \ldots, r$, are pairwise non-equivalent irreducible subrepresentations of (π, V). Then $V_1, \ldots, V_r$ are linearly independent subspaces of V.*

Proof. Suppose $V_1, \ldots, V_r$ are not linearly independent. Let $1 \leq m < r$ such that $V_1, \ldots, V_m$ are linearly independent, but $V_1, \ldots, V_m, V_{m+1}$ are not. Then $V_{m+1} \cap (V_1 \oplus \ldots \oplus V_m)$ is a non-zero invariant subspace of V, and hence also of V_{m+1}. Since π_{m+1} is irreducible, this implies that $V_{m+1} \cap (V_1 \oplus \ldots \oplus V_m) = V_{m+1}$, that is, $V_{m+1} \subseteq (V_1 \oplus \ldots \oplus V_m)$. By Theorem 3.8 π_{m+1} is therefore equivalent to $\bigoplus_{i \in I} \pi_i$ for some suitable $I \subseteq \{1, \ldots, m\}$. Because of irreducibility, $|I| = 1$, say $I = \{i_0\}$, but then $\pi_{m+1} \simeq \pi_{i_0}$ in contradiction to the pairwise¨ non-equivalence of the π_i's. □

Corollary 3.10. *Suppose*

$$(\pi, V) \simeq (\pi_1, V_1) \oplus \ldots \oplus (\pi_r, V_r) \simeq (\sigma_1, W_1) \oplus \ldots \oplus (\sigma_s, W_s)$$

for some irreducible representations $(\pi_1, V_1), \ldots, (\pi_r, V_r)$, $(\sigma_1, W_1) \ldots, (\sigma_s, W_s)$ (all non-zero). Then $r = s$ and there exists a bijection $\varphi : \{1, \ldots, r\} \longrightarrow \{1, \ldots, r\}$ such that $(\pi_i, V_i) \simeq (\sigma_{\varphi(i)}, W_{\varphi(i)})$ for $i = 1, \ldots, r$.

Proof. We argue by induction on r. When $r = 1$, $\pi \simeq \pi_1$ is irreducible, hence $s = 1$ and $\pi_1 \simeq \sigma_1$.

Now suppose that $r > 1$. By Theorem 3.8 there exists $I \subseteq \{1, \ldots, r\}$ such that

$$(\sigma_1, W_1) \simeq \bigoplus_{i \in I} (\pi_i, V_i).$$

Since σ_1 is irreducible, $|I| = 1$, and after reordering the π_j's we can without loss or generality assume that $I = \{1\}$, that is, $(\sigma_1, W_1) \simeq (\pi_1, V_1)$. But then

$$\bigoplus_{j=1}^{s} (\sigma_j, W_j) \simeq (\pi_{V/W_1}, V/W_1) \simeq (\pi_{V/V_1}, V/V_1) \simeq \bigoplus_{i=2}^{r} (\pi_i, V_i)$$

so that we can now apply the induction hypothesis and obtain the assertion of the corollary. □

3.3 Tensor products

Definition 3.11. Let (π, V), (σ, W) be two representations of G over K. We define the *tensor product* of π and σ as the representation $\pi \otimes \sigma$ of G on the tensor product $V \otimes W$ of V and W given by

$$\pi \otimes \sigma(g) v \otimes w = \pi(g) v \otimes \sigma(g) w$$

Recall that $\dim V \otimes W = \dim V \cdot \dim W$, so if π has dimension n and σ has dimension m, $\pi \otimes \sigma$ will have dimension nm.

Remark 3.12.

- *Clearly, $(\pi \otimes \sigma, V \otimes W) \simeq (\sigma \otimes \pi, W \otimes V)$.*
- *If π has dimension 1, then we can identify π with a homomorphism $G \longrightarrow K^\times$, and $\pi \otimes \sigma : G \longrightarrow \mathrm{GL}(V \otimes W)$ is equivalent to the representation $G \longrightarrow \mathrm{GL}(W)$, $g \mapsto \pi(g)\sigma(g)$.*

Example 3.13. We can write the tensor product in terms of matrices as follows: Suppose $\mathcal{B} = \{v_1, \ldots, v_n\}$ is a basis of V, and $\mathcal{C} = \{w_1, \ldots, w_m\}$ is a basis of W. Let $\varphi_{\mathcal{B}} : V \longrightarrow K^n$, and $\varphi_{\mathcal{C}} : W \longrightarrow K^m$ denote the corresponding isomorphisms which satisfy $\varphi_{\mathcal{B}}(v_i) = e_i$ and $\varphi_{\mathcal{C}}(w_j) = f_j$ (where the e_i and f_j denote the standard basis of K^n, K^m, respectively). Note that we have a canonical isomorphism $K^n \otimes K^m \longrightarrow M_{n\times m}(K)$ given by $e_i \otimes f_j \mapsto e_i f_j^t$ so that we get an isomorphism

$$\varphi_{\mathcal{B},\mathcal{C}} : V \otimes W \longrightarrow M_{n\times m}(K),\ \varphi_{\mathcal{B},\mathcal{C}}(v \otimes w) = \varphi_{\mathcal{B}}(v)\varphi_{\mathcal{C}}(w)^t.$$

Recall the matrix representations $\pi_{\mathcal{B}} : G \longrightarrow \mathrm{GL}_n(K)$, $\sigma_{\mathcal{C}} : G \longrightarrow \mathrm{GL}_m(K)$ which are characterized by

$$\varphi_{\mathcal{B}}(\pi(g)v_i) = \pi_{\mathcal{B}}(g)e_i, \text{ and } \varphi_{\mathcal{C}}(\sigma(g)w_j) = \sigma_{\mathcal{C}}(g)f_j.$$

Hence using the isomorphism $\varphi_{\mathcal{B},\mathcal{C}}$ we see that $\pi \otimes \sigma$ is equivalent to the representation

$$\Pi : G \longrightarrow \mathrm{GL}(M_{n\times m}(K))$$

given by

$$\Pi(g)A = \pi_{\mathcal{B}}(g)A\sigma_{\mathcal{C}}(g)^t.$$

Example 3.14. Suppose (π, V) is a representation of G of dimension n. Consider

$$\pi \otimes \pi' : G \longrightarrow \mathrm{GL}(V \otimes V').$$

Fix some basis for V and the corresponding dual basis for V'. Then we know from Example 3.2 that in terms of matrices $\pi'(g) = (\pi(g)^{-1})^t$ (we suppress the dependency on the fixed basis here). Hence combined with the preivous example, $\pi \otimes \pi'$ is equivalent in terms of matrices to

$$G \longrightarrow \mathrm{GL}(M_n(K)),\ g \mapsto \left(A \mapsto \pi(g)A\pi(g)^{-1}\right),$$

that is, $\pi \otimes \pi' \simeq \mathrm{Ad} \circ \pi$, where $\mathrm{Ad} : \mathrm{GL}_n(K) \longrightarrow \mathrm{GL}(M_n(K))$ is the adjoint representation $\mathrm{Ad}(x)A = xAx^{-1}$ from Exercise 9. We know from Exercise 9 that Ad is not irreducible, hence $\pi \otimes \pi' \simeq \mathrm{Ad} \circ \pi$ will not be irreducible either.

Definition 3.15. Suppose H is another group, (π, V) is a representation of G and (σ, W) is a representation of H. We define the *external tensor product* of π and σ as the representation $\pi \times \sigma$ of $G \times H$ on $V \otimes W$ given by

$$(\pi \times \sigma)(g, h)v \otimes w = \pi(g)v \otimes \sigma(h)w$$

for $(g, h) \in G \times H$ and $v \otimes w \in V \otimes W$.

We will later show that if π and σ are both irreducible, then so is $\pi \times \sigma$. This is not true for $\pi \otimes \sigma$ as can be seen from Example 3.14.

Remark 3.16. *If $H = G$, and $\Delta : G \hookrightarrow G \times G$, $\Delta(g) = (g, g)$, denotes the diagonal embedding, then*

$$(\pi \times \sigma) \circ \Delta = \pi \otimes \sigma.$$

Example 3.17. Consider the embedding $\iota_1 : G \hookrightarrow G \times H$, $\iota_1(g) = (g, 1)$. Then $(\pi \times \sigma) \circ \iota_1 \simeq \pi^{\dim \sigma}$ as G-representations (that is, $\pi^{\dim \sigma}$ is the direct sum of $\dim \sigma$-many copies of π). This can be seen as follows: Fix a basis $w_1, \ldots, w_m$ of W. Then $V \otimes W = \bigoplus_{i=1,\ldots,m} V \otimes w_i$, and each $V \otimes w_i$ is invariant under $\pi \times \sigma \circ \iota_1$, and the corresponding subrepresentation is isomorphic to (π, V).

Similarly, if we consider the embedding $\iota_2 : H \longrightarrow G \times H$, $\iota_2(h) = (1, h)$, then $(\pi \times \sigma) \circ \iota_2 \simeq \sigma^{\dim \pi}$ as H-representations.

4 Intertwining Operators and Schur's Lemma

4.1 Intertwining Operators

Definition 4.1. Suppose (π_i, V_i), $i = 1, 2$, are representations of G over the same field K. An *intertwining operator* from π_1 to π_2 is a linear map $\Phi : V_1 \longrightarrow V_2$ such that

$$\Phi \circ \pi_1(g) = \pi_2(g) \circ \Phi$$

for all $g \in G$. In this situation we say that Φ *intertwines* π_1 with π_2.

Remark 4.2.

- *$\pi_1 \simeq \pi_2$ is equivalent with the existence of a bijective intertwining operator from π_1 to π_2.*
- *In general, an intertwining operator does* not *need to be a bijection. For example, the zero map $\Phi : V_1 \longrightarrow V_2$, $v \mapsto 0$, intertwines π_1 with π_2.*

Example 4.3. Suppose (π_1, V_1), (π_2, V_2) are representations of G, and put $(\pi, V) = (\pi_1 \oplus \pi_2, V_1 \oplus V_2)$. Let $\Phi : V_1 \longrightarrow V$, $v_1 \mapsto (v_1, 0)$. Then Φ intertwines π_1 with π.

Theorem 4.4.

(i) Suppose $(\pi_1, V_1), (\pi_2, V_2)$ are irreducible representations of G and Φ is an intertwining operator from π_1 to π_2. Then either $\Phi \equiv 0$ or Φ is an isomorphism.

(ii) Suppose (π, V) is a representation of G, and $V = V_1 \oplus \ldots \oplus V_r$ with $V_1, \ldots, V_r$ minimal invariant subspaces of V such that $\pi_{V_i} \not\simeq \pi_{V_j}$ for all $i \neq j$. Then for every invariant subspace $U \subseteq V$, $U \neq 0$, there exists $\{i_1, \ldots, i_l\} \subseteq \{1, \ldots, r\}$ such that

$$U = V_{i_1} \oplus \ldots \oplus V_{i_l}.$$

The indices $i_1, \ldots, i_l$ are uniquely determined by U up to ordering.

(Note that we already know this to be true up to isomorphism, see Theorem 3.8.)

Proof. (i) Suppose Φ intertwines π_1 with π_2, and π_1 and π_2 are both irreducible. Let $v \in \ker \Phi$. Then for all $g \in G$,

$$\Phi(\pi_1(g)v) = \pi_2(g)\Phi(v) = 0$$

so that $\ker \Phi$ is an invariant subspace of V_1. Because of the irreducibility of π_1, Φ is therefore either injective or $\equiv 0$. Now let $w \in \operatorname{Im} \Phi$, say $w = \Phi(v)$ for a suitable $v \in V_1$. Then for all $g \in G$, $\pi_2(g)w = \pi_2(g)\Phi(v) = \Phi(\pi_1(g)v)$ so that the image of Φ is also an invariant subspace of V_2. Using the irreducibility of π_2 we therefore get that Φ is either $\equiv 0$ or surjective. Putting this together with the first half of the argument we obtain that either $\Phi \equiv 0$ or Φ is an isomorphism.

(ii) Since (π, V) is completely reducible, (π_U, U) will be so as well, hence there exist minimal invariant subspaces $U_1, \ldots, U_l \subseteq U$ such that $U = U_1 \oplus \ldots \oplus U_l$. For $i = 1, \ldots, l$ and $j = 1, \ldots, r$ consider the linear maps

$$\Phi_{ij} : U_i \hookrightarrow U \hookrightarrow V \longrightarrow V_j,$$

where the first two maps are the natural inclusions and the last map is the projection from V onto V_j along the complement $\bigoplus_{k \neq j} V_k$. It is easily checked that Φ_{ij} intertwines π_{U_i} with π_{V_j}, hence since U_i and V_j are both minimal invariant, Φ_{ij} is either an isomorphism or $\equiv 0$ by the first part of the theorem. If Φ_{ij} is an isomorphism, it must in fact be the identity map by definition. Moreover, since $V = V_1 + \ldots + V_r$, for each i there must exist j_i such that Φ_{ij_i} is the identity. Hence $U = V_{1j_1} \oplus \ldots \oplus V_{lj_l}$.

□

Remark 4.5. *If Φ is an intertwining operator, then* $\ker \Phi$ *and* $\operatorname{Im} \Phi$ *are always invariant subspaces independently of whether π_1 and π_2 are irreducible or not.*

4.2 Schur's Lemma

Theorem 4.6 (Schur's Lemma). *Suppose (π, V) is an irreducible finite dimensional complex representation of G. Let Φ be an intertwining operator from π to itself. Then there exists $\alpha \in \mathbb{C}$ such that $\Phi = \alpha \mathrm{id}_V$.*

Proof. Since V is finite dimensional and complex, Φ must have an eigenvalue, say $\alpha \in \mathbb{C}$, with eigenvector $v \in V$, $v \neq 0$, that is,

$$(\Phi - \alpha \mathrm{id}_V)v = 0.$$

Therefore the endomorphism $\Phi - \alpha \mathrm{id}_V : V \longrightarrow V$ is not an isomorphism. But $\Phi - \alpha \mathrm{id}_V$ still intertwines π with itself, hence by Theorem 4.4 it must vanish identically. □

Remark 4.7.

- *Schur's Lemma is* not *true for real representation as $\mathbb{R}$ is not closed. One can, however, replace $\mathbb{C}$ by another algebraically closed field.*
- *The finite dimensionality of the representations (which we always assume) is also essential for Schur's Lemma, though under certain circumstances a version of Schur's Lemma can also be proven for certain infinite dimensional representations using the spectral theorem for self-adjoint operators.*

4.3 Consequences of Schur's Lemma

Corollary 4.8. *Suppose (π, V_1), (π_2, V_2) are irreducible finite dimensional complex representations of G. Then*

$$\dim_{\mathbb{C}} \operatorname{Hom}(\pi_1, \pi_2) = \begin{cases} 1 & \text{if } \pi_1 \simeq \pi_2, \\ 0 & \text{else,} \end{cases}$$

where $\operatorname{Hom}(\pi_1, \pi_2)$ *denotes the $\mathbb{C}$-vector space of all intertwining operators $\Phi : V_1 \longrightarrow V_2$ from π_1 to π_2.*

Proof. If $\pi_1 \not\simeq \pi_2$, then by Theorem 4.4 we have $\mathrm{Hom}(\pi_1, \pi_2) = \{0\}$. Now suppose that $\pi_1 \simeq \pi_2$ and fix an isomorphism $\Phi_0 : V_1 \longrightarrow V_2$ intertwining π_1 with π_2. Let $\Phi \in \mathrm{Hom}(\pi_1, \pi_2)$. Then $\Phi_0^{-1} \circ \Phi : V_1 \longrightarrow V_1$ is an intertwining operator of π_1 with itself, hence by Schur's Lemma there exists $\alpha \in \mathbb{C}$ with $\Phi = \alpha \Phi_0$ so that $\dim_{\mathbb{C}} \mathrm{Hom}(\pi_1, \pi_2) = 1$. □

Corollary 4.9. *Suppose (π, V) is an irreducible finite dimensional complex representation of G. Let U be a finite dimensional complex vector space, and let σ denote the trivial representation of G on U (that is, $\sigma(g) = \mathrm{id}_U$ for all $g \in G$).*

Suppose $W \subseteq V \otimes U$, $W \neq 0$, is a minimal invariant subspace with respect to the representation $\pi \otimes \sigma$ of G. Then there exists $u \in U$ such that $W = V \otimes u$.

Proof. Let $m = \dim U$ and let $\{u_1, \ldots, u_m\}$ denote a basis of U. Recall that $\pi \otimes \sigma$ is equivalent to a direct sum of m-copies of π. Hence if $W \subseteq V \otimes U$ is minimal invariant, we have $((\pi \otimes \sigma)_W, W) \simeq (\pi, V)$. Fix an isomorphism $\Phi_0 : W \longrightarrow V$. We define m homomorphisms $\phi_i : V \otimes U \longrightarrow V$ by the property that for all $x \in V \otimes U$ we have

$$x = \phi_1(x) \otimes u_1 + \ldots + \phi_m(x) \otimes u_m.$$

Then for every $x \in V \otimes U$, $g \in G$, we get on the one hand that

$$\pi \otimes \sigma(g)x = (\pi(g)\phi_1(x)) \otimes u_1 + \ldots + (\pi(g)\phi_m(x)) \otimes u_m$$

and on the other,

$$\pi \otimes \sigma(g)x = \phi_1(\pi \otimes \sigma(g)x) \otimes u_1 + \ldots + \phi_m(\pi \otimes \sigma(g)x) \otimes u_m.$$

Hence $\phi_i(\pi \otimes \sigma(g)x) = \pi(g)\phi_i(x)$ for all i, $g \in G$, $x \in V \otimes U$ so that the restrictions $(\phi_i)_{|W} \in \mathrm{Hom}((\pi \otimes \sigma)_W, \pi)$. Hence by Corollary 4.8 there exist $\alpha_i \in \mathbb{C}$ such that $(\phi_i)_{|W} = \alpha_i \Phi_0$. Hence for all $x \in W$ we get

$$x = \phi_1(x) \otimes u_1 + \ldots + \phi_m(x) \otimes u_m = \Phi_0(x) \otimes (\alpha_1 u_1 + \ldots + \alpha_m u_m) =: \Phi_0(x) \otimes u.$$

□

Corollary 4.10. *Suppose (π, V) is an irreducible finite dimensional complex representation of G, and (σ, W) is an irreducible finite dimensional complex representation of another group H. Then $(\pi \times \sigma, V \otimes W)$ is an irreducible representation of $G \times H$.*

Proof. Let $U \subseteq V \otimes W$, $U \neq 0$, be an invariant subspace with respect to $\pi \times \sigma$. Let $\pi_0 : G \longrightarrow \mathrm{GL}(W)$ denote the trivial representation of G on W. Then U is also an invariant subspace of the representation $(\pi \otimes \pi_0, V \otimes W)$ of G. By Corollary 4.9 there exists $w \in W$, $w \neq 0$, such that $V \otimes \{w\} \subseteq U$.

For $v \in V$ we now define

$$W_v := \{w' \in W \mid v \otimes w' \in U\}.$$

W_v is clearly σ-invariant, hence $W_v = \{0\}$ or $W_v = W$ because of the irreducibility of σ. Since $w \in W_v$ for all v, it follows that $W_v = W$ for all $v \in V$. Hence $U = V \otimes W$ so that $\pi \times \sigma$ is irreducible. □

Corollary 4.11. *Suppose G is abelian, and (π, V) is an irreducible finite dimensional complex representation of G. Then π has dimension 1.*

Proof. Since G is abelian, we get for all $g, h \in G$,

$$\pi(g) \circ \pi(h) = \pi(gh) = \pi(hg) = \pi(h) \circ \pi(g)$$

so that $\pi(h) : V \longrightarrow V$ is an intertwining operator of π with itself for every $h \in G$. Hence by Schur's Lemma, there exists $\alpha_h \in \mathbb{C}$ such that $\pi(h) = \alpha_h \mathrm{id}_V$. Clearly, the map $G \longrightarrow \mathbb{C}^\times = \mathrm{GL}_1(\mathbb{C})$, $h \mapsto \alpha_h$ is a group homomorphism, hence π is equivalent to this 1-dimensional representation, and therefore must be 1-dimensional itself because of irreducibility. □

4.4 Commutator subgroups

The main result Corollary 4.17 of this section are a consequence of Corollary 4.11.

Recall:

Definition 4.12. Let G be a group. The *commutator subgroup* $[G, G]$ of G is the subgroup of G generated by all the commutators

$$[g, h] := ghg^{-1}h^{-1} \in G, \ g, h \in G.$$

Remark 4.13. *G is abelian if and only if $[G, G] = \{1\}$.*

Lemma 4.14. *The commutator subgroup $[G, G]$ is a normal subgroup of G, and the abelianization $A(G) := G/[G, G]$ of G is an abelian group.*

Remark 4.15. *The abelianization of G is the largest abelian quotient of G.*

Proof of Lemma 4.14. First let $g, h, x \in G$. Then

$$x[g, h]x^{-1} = [xgx^{-1}, xhx^{-1}] \in [G, G]$$

so that $[G, G]$ is a normal subgroup of G. Now if $\tilde{x} := x[G, G], \tilde{y} := y[G, G] \in A(G)$, then

$$\tilde{x}\tilde{y}\tilde{x}^{-1}\tilde{y}^{-1} = xyx^{-1}y^{-1}[G, G] = 1[G, G]$$

so that $A(G)$ is abelian. □

Lemma 4.16. *Suppose N is a normal subgroup of G, and let $p : G \longrightarrow G/N$ denote the projection. Then*

$$\{\textit{representations of } G/N\} \longrightarrow \{\textit{representations of } G \textit{ with kernel containing } N\}, \ \pi \mapsto \pi \otimes p$$

is a bijection. This bijection is also true, when we add properties such as 'finite-dimensional', 'complex', or 'irreducible', etc on both sides

Proof. This is clear from the definition of p. □

Corollary 4.17. *Let $p : G \longrightarrow G/[G,G] = A(G)$ denote the projection. Then the map*

$$\{1-\textit{dimensional complex representations of } G\} \\ \longrightarrow \{\textit{irreducible finite dimensional complex representations of } A(G)\}$$

given by $\pi \mapsto \pi \circ p$ is well-defined and a bijection.

Proof. By Lemma 4.16 and Corollary 4.11 it will suffice to show that the irreducible finite dimensional complex representations of G whose kernel contain $[G,G]$ are exactly the 1-dimensional ones. Every such representations whose kernel contains $[G,G]$ is 1-dimensional by Corollary 4.11. On the other hand, if π is a 1-dimensional representation of G, then the image of π is a commutative group, hence $\pi([g,h]) = 1$ for all $g, h \in G$ so that $[G,G] \leq \ker \pi$. $\square$

Example 4.18. Let $G = S_n$. Let $\mathrm{sgn} : G \longrightarrow \{\pm\}$ denote the signum character, that is $\mathrm{sgn}(\sigma) = 1$ if σ is even, and $\mathrm{sgn}(\sigma) = -1$ if σ is odd. One can show that $[S_n, S_n] = A_n = \ker \mathrm{sgn}$, the *alternating group* on n-elements. Since $S_n/A_n \simeq \{\pm 1\}$ it follows from Corollary 4.17 that S_n has exactly two 1-dimensional complex representations, namely the trivial one, and sgn.

5 Matrix Coefficients

Recall that

$$\mathbb{C}[G] = \mathbb{C} - \text{v.s. of all maps } f : G \longrightarrow \mathbb{C}.$$

Definition 5.1. Suppose (π, V) is a finite dimensional complex representation. Let $\mathcal{B} = \{v_1, \ldots, v_n\}$ be a basis for V and write

$$\pi_{\mathcal{B}}(g) = (a_{ij}(g))_{i,j} \in \mathrm{GL}_n(\mathbb{C}) \tag{3}$$

with respect to this basis. The space

$$M(\pi) := \mathrm{span}_{\mathbb{C}}\{a_{ij} \mid i, j = 1, \ldots, n\} \subseteq \mathbb{C}[G]$$

is called the *space of matrix coefficients of* π and the functions $a_{ij} : G \longrightarrow \mathbb{C}$ are called *matrix elements* of π.

Definition 5.2. We fix some notation:

- For a linear map $\phi : V \longrightarrow V$ we write

$$\mathrm{tr}_\phi : G \longrightarrow \mathbb{C},\ g \mapsto \mathrm{tr}(\pi(g) \circ \phi).$$

- For $v \in V$ and $f \in V'$ (V' the dual space of V) we write

$$\mu_{f,v} : G \longrightarrow \mathbb{C},\ g \mapsto f(\pi(g)v).$$

Proposition 5.3. *We have*

$$M(\pi) = \{\mathrm{tr}_\phi \mid \phi \in \mathrm{Hom}(V,V)\} = \mathrm{span}_{\mathbb{C}}\{\mu_{f,v} \mid v \in V, f \in V'\}.$$

In particular, $M(\pi)$ is independent of the choice of basis of V used to initially define $M(\pi)$.

Proof. Let $\{v_1,\dots,v_n\}$ be as above, and let $\{f_1,\dots,f_n\}$ be a dual basis of V'.

Let $f = \alpha_1 f_1 + \ldots + \alpha_n f_n \in V'$ and $v = \beta_1 v_1 + \ldots + \beta_n v_n \in V$. Then for $g \in G$

$$\mu_{f,v}(g) = f(\pi(g)) = \sum_{i,j} \alpha_i\beta_j f_i(\pi(g)v_j) = \sum_{i,j} \alpha_i\beta_j a_{ij}(g)$$

(with $a_{ij}(g)$ as in (3)). Hence $M(\pi)$ equals the span of the $\mu_{f,v}$.

Now if $A \in \mathrm{Hom}(V,V)$ write A as a matrix with respect to $\mathcal{B}$, that is $(\alpha_{ij})_{i,j} \in M_n(K)$. Then for $g \in G$,

$$\mathrm{tr}_A(g) = \mathrm{tr}\left((a_{ij}(g))_{i,j}(\alpha_{kl})_{k,l}\right) = \sum_{i,j=1,\dots,n} a_{ij}(g)\alpha_{ji}$$

so that $M(\pi)$ also equals the collection of the tr_A. □

Remark 5.4. *The map $V' \times V \longrightarrow M(\pi)$, $(f,v) \mapsto \mu_{f,v}$ is bilinear, hence we obtain a unique linear map $V' \otimes V \longrightarrow M(\pi)$.*

Example 5.5.

- If π is one-dimensional, π is just a homomorphism $G \longrightarrow \mathbb{C}^\times$ and $M(\pi) = \mathbb{C}\pi$.
- If $\pi \simeq \sigma$, then $M(\pi) = M(\sigma)$.
- If $\pi \simeq \pi_1 \oplus \ldots \oplus \pi_r$ (the π_j don't need to be irreducible), then $M(\pi) = M(\pi_1) + \ldots + M(\pi_r)$.

Lemma 5.6. *The vector space $M(\pi)$ is invariant under left and right translation by elements of g, that is, for all $F \in M(\pi)$ and all $g \in G$, we have $F(g\cdot), F(\cdot g) \in M(\pi)$.*

Definition 5.7. The representation $\mathrm{Reg} : G \times G \longrightarrow \mathrm{GL}(\mathbb{C}[G])$ defined by

$$\mathrm{Reg}(g,h)F(x) = F(h^{-1}xg)$$

is called the *two-sided regular representation* of G. For each finite dimensional complex representation π of G we write

$$\mathrm{Reg}_\pi : G \times G \longrightarrow \mathrm{GL}(M(\pi))$$

for the corresponding subrepresentation of Reg on $M(\pi)$.

Proof of Lemma 5.6. Let $f \in M(\pi)$. We can assume that $F = \mathrm{tr}_\varphi$ for some $\varphi \in \mathrm{Hom}(V,V)$. Then for all $g, x \in G$:

$$F(gx) = \mathrm{tr}(\pi(gx)\varphi) = \mathrm{tr}(\pi(g)\pi(x)\varphi\pi(g)\pi(g)^{-1}) = \mathrm{tr}(\pi(x)\varphi\pi(g)) = \mathrm{tr}_{\varphi\pi(g)}(x)$$

where we used that the trace is invariant under conjugation. Hence $F(g\cdot) = \mathrm{tr}_{\varphi\pi(g)} \in M(\pi)$. Also, $F(xg) = \mathrm{tr}(\pi(x)\pi(g)\varphi) = \mathrm{tr}_{\pi(g)\varphi}(x)$, hence $F(\cdot g) = \mathrm{tr}_{\pi(g)\varphi} \in M(\pi)$. □

Theorem 5.8. *Suppose (π, V) is an irreducible finite dimensional complex representation. Then $\pi \times \pi' \simeq \mathrm{Reg}_\pi$ via the isomorphism*

$$\Phi : V \otimes V' \longrightarrow M(\pi),\ v \otimes f \mapsto \mu_{f,v}.$$

Proof. We show first that Φ is an intertwining operator. Let $g, h, x \in G$. Then

$$\begin{aligned}\Phi(\pi \otimes \pi'(g,h)v \otimes f)(x) &= \Phi((\pi(g)v) \otimes (\pi'(h)f))(x) = \mu_{\pi'(h)f,\pi(g)v}(x)\\ &= f(\pi(h^{-1})\pi(x)\pi(g)v) = \mu_{f,v}(h^{-1}xg) = \left(\mathrm{Reg}_\pi(g,h)\mu_{f,v}\right)(x)\end{aligned}$$

so that indeed Φ intertwines $\pi \times \pi'$ with Reg_π. Since Φ is surjective by definition, it remains to show that Φ is injective. By Corollary 4.10 $(\pi \times \pi', V \otimes V')$ is irreducible. Since the kernel of an intertwining operator is invariant, it must therefore be either $\{0\}$ or the whole space. Because we already know that the map is surjective, the kernel must thus be $\{0\}$. □

Corollary 5.9. *Suppose (π, V) is an irreducible finite dimensional complex representation. Then $\dim M(\pi) = (\dim \pi)^2$.*

Proof. This is clear from Theorem 5.8. □

Corollary 5.10. *Suppose π_1, π_2 are irreducible finite dimensional complex representations of G. If $\pi_1 \not\simeq \pi_2$, then $\mathrm{Reg}_{\pi_1} \not\simeq \mathrm{Reg}_{\pi_2}$.*

More generally, if $\pi_1, \ldots, \pi_m$ are irreducible finite dimensional complex representations of G which are pairwise non-equivalent, then $M(\pi_1), \ldots, M(\pi_m)$ are linearly independent subspaces of $\mathbb{C}[G]$.

Proof. The second assertion follows immediately from the first one together with Theorem 2.6. It will therefore suffice to show that if $\mathrm{Reg}_{\pi_1} \simeq \mathrm{Reg}_{\pi_2}$, then $\pi_1 \simeq \pi_2$. Suppose that $\mathrm{Reg}_{\pi_1} \simeq \mathrm{Reg}_{\pi_2}$. Let σ_i denote the restriction of Reg_{π_i} to the subgroup $G \times \{1\}$ of G. Then clearly, $\sigma_i \simeq R_{M(\pi_i)}$, where $R_{M(\pi_i)}$ denotes the right regular representation of G on $M(\pi_i)$.

We claim that $R_{M(\pi)} \simeq \pi_i^{\dim \pi_i}$ (i.e., the direct sum of $\dim \pi_i$-many copies of π_i). Assuming this claim, we can conclude that if $\mathrm{Reg}_{\pi_1} \simeq \mathrm{Reg}_{\pi_2}$, then $\sigma_1 \simeq \sigma_2$ which then implies that $\pi_1 \simeq \pi_2$ because of the claim and irreducibility of π_1 and π_2.

To prove the claim recall that $M(\pi_i) \simeq V_i \otimes V_i'$. For $g, x \in G$, $\mu_{f,v} \in M(\pi_i)$ we get

$$\left(R_{M(\pi_i)}(g)\mu_{f,v}\right)(x) = \mu_{f,v}(xg) = f(\pi_i(xg)v) = \mu_{f,\pi_i(g)v}(x)$$

so that $(R_{M(\pi_i)}, M(\pi_i)) \simeq (1 \otimes \pi_i, V_i \otimes V_i') \simeq \pi_i^{\dim V_i'}$ where $(1, V_i)$ denotes the trivial representation of G on V_i. Hence the claim follows from $\dim V_i' = \dim V_i$. □

6 Matrix coefficients for finite groups

In this section we assume throughout that G is a finite group and all representations are complex even if not explicitly mentioned.

Theorem 6.1. *(i) Every complex irreducible representation of G is finite dimensional.*

(ii) Up to equivalence G has only finitely many (in fact at most $|G|$-many) irreducible complex representations.

(iii) Every irreducible complex representation of G is equivalent to a subrepresentation of the right regular representation $R : G \longrightarrow \mathrm{GL}(\mathbb{C}[G])$.

Proof. Let (π, V) be an irreducible complex representation of G.

(i) Let $v \in V$, $v \neq 0$, and let $U = \mathrm{span}_{\mathbb{C}}\{\pi(g)v \mid g \in G\}$. Then $U \neq 0$ is an invariant subspace of V, hence $U = V$ because of the irreducibility of π. Since G is finite, $\dim V = \dim U \leq |G| < \infty$.

(iii) By (i) and the proof of Corollary 5.10 we know that $R_{M(\pi)} \simeq \pi^{\dim \pi}$, hence π is equivalent to a subrepresentation of $(R_{M(\pi)}, M(\pi))$ which is in turn a subrepresentation of $(R, \mathbb{C}[G])$.

(ii) Suppose $\pi_1, \ldots, \pi_r$ are pairwise inequivalent irreducible complex representations. The collection $M(\pi_1), \ldots, M(\pi_r)$ are by Corollary 5.9 therefore linearly independent so that

$$r \leq \dim M(\pi_1) + \ldots + \dim M(\pi_r) = \dim M(\pi_1) \oplus \ldots \oplus M(\pi_r) \leq \dim \mathbb{C}[G] = |G|.$$

$\square$

Theorem 6.2. *Suppose $\pi_1, \ldots, \pi_r$ is a complete list of representatives for the equivalence classes of irreducible complex representations of G. Then*

$$\mathbb{C}[G] = M(\pi_1) \oplus \ldots \oplus M(\pi_r). \tag{4}$$

Proof. We already know that the right hand side of (4) is contained in $\mathbb{C}[G]$. Note that if we define for $x \in G$ the map

$$\delta_x : G \longrightarrow \mathbb{C},\ \delta_x(g) = \begin{cases} 1 & \text{if } x = g, \\ 0 & \text{else,} \end{cases}$$

then $\{\delta_x \mid x \in G\}$ is a basis of $\mathbb{C}[G]$. It will therefore suffice to show that $\delta_x \in \mathbb{C}[G]$ for each $x \in G$.

By Maschke's Theorem, $(R, \mathbb{C}[G])$ is completely reducible, that is $R \simeq \pi_1^{m_1} \oplus \ldots \oplus \pi_r^{m_r}$ for suitable integers $m_1, \ldots, m_r \in \mathbb{N}_0$. Define $\epsilon : \mathbb{C}[G] \longrightarrow \mathbb{C}$ by $\epsilon(f) = f(1)$ where $1 \in G$ denotes the unit element of G. Hence $\epsilon \in \mathbb{C}[G]'$ and we can consider

$$\mu_{\epsilon,\delta_x}(g) = \epsilon(R(g)\delta_x) = \delta_x(g) = \begin{cases} 1 & \text{if } x = g, \\ 0 & \text{else.} \end{cases}$$

Hence

$$\delta_x = \mu_{\epsilon,\delta_x} \in M(R) = M(\pi_1^{m_1} \oplus \ldots \oplus \pi_r^{m_r}) = M(\pi_1) + \ldots + M(\pi_r)$$

which finishes the proof. $\square$

Corollary 6.3. *Let $\pi_1, \ldots, \pi_r$ be as in Theorem 6.2. Then*

$$|G| = (\dim \pi_1)^2 + \ldots + (\dim \pi_r)^2 .$$

Proof. This is immediate from Theorem 6.2 and $\dim \mathbb{C}[G] = |G|$. $\square$

Corollary 6.4. *Let the notation be as in Theorem 6.2. Then*

$$R \simeq \pi_1^{\dim \pi_1} \oplus \ldots \oplus \pi_r^{\dim \pi_r}.$$

Proof. Recall that $R_{M(\pi)} \simeq \pi_i^{\dim \pi_i}$, hence the corollary follows from Theorem 6.2. $\square$

Example 6.5. Suppose G is finite and abelian. Then $\dim \pi_i = 1$ for all i. Hence up to equivalence G has exactly $|G|$-many inequivalent irreducible representations, each of dimension 1.

Example 6.6. Let $G = S_3$. Then $|G| = 6$, and the only way to write this as a sum of squares is as $1+1+1+1+1+1$ or $1+1+2^2$. In the exercises it was shown that if all irreducible representations of a group are 1-dimensional, then G is abelian, which is not the case here. (Alternatively, we know from Example 4.18 that S_3 has exactly two 1-dimensional representations.) Hence G must have two non-equivalent 1-dimensional representations (which we already found), and one irreducible 2-dimensional representation. In fact, we already know what this 2-dimensional representation is: It is the permutation representation of S_3 on the 2-dimensional vector space

$$V = \{(x_1, x_2, x_3) \in \mathbb{C}^3 \mid x_1 + x_2 + x_3 = 0\}.$$

Example 6.7. Let $G = S_4$. Then $|G| = 24$. We know that G has exactly two non-equivalent 1-dimensional representations, namely the trivial representation χ_0 and the signum representation $\chi = \mathrm{sgn}$, and we also found an irreducible 3-dimensional representation, namely the permutation representation π on the 3-dimensional space

$$V = \{(x_1, x_2, x_3, x_4) \in \mathbb{C}^4 \mid x_1 + x_2 + x_3 + x_4 = 0\}.$$

We obtain another irreducible 3-dimensional on V by multiplying with χ, that is, $(\chi\pi, V)$, and one can show that this is not equivalent to π. Now $24 - 1 - 1 - 3^2 - 3^2 = 2^2$ so that we are missing one irreducible 2-dimensional representation. To find this two-dimensional representation consider the Klein group

$$H = \{1, (12)(34), (13)(24), (14)(23)\}$$

which is a normal subgroup of S_4 with the property that $S_4/H \simeq S_3$ (we identify those two groups). Let (σ, W) denote the irreducible 2-dimensional representation of S_3 and define $\tilde{\sigma} : S_4 \longrightarrow \mathrm{GL}(W)$ by $\tilde{\sigma}(g) = \sigma(gH)$, which then gives the missing 2-dimensional representation of S_4.

6.1 Invariants[2]

Suppose G is a finite group and $H \leq G$ a subgroup. Then G acts on the coset space G/H by left multiplication, that is by $g \cdot xH = gxH$. We get a corresponding representation

$$\pi_H : G \longrightarrow \mathrm{GL}(\mathbb{C}[G/H]),\ \pi_H(g)f(xH) = f(g^{-1}xH).$$

Lemma 6.8. *The linear map*

$$\varphi : \mathbb{C}[G/H] \longrightarrow \mathbb{C}[G]^H := \{f : G \longrightarrow \mathbb{C} \mid \forall g \in G, h \in H :\ f(gh) = f(g)\},\ f \mapsto f \circ p,$$

where $p : G \longrightarrow G/H$ is the projection, is a bijective intertwining operator of π_H with $L_{\mathbb{C}[G]^H}$, where $L_{\mathbb{C}[G]^H}$ denotes the subrepresentation of the left regular representation of G on the subspace $\mathbb{C}[G]^H \subseteq \mathbb{C}[G]$.

Proof. Clear from the definitions. □

Theorem 6.9. *Suppose $(\pi_1, V_1), \ldots, (\pi_r, V_r)$ is a complete list of representatives for the equivalence classes of irreducible complex representations of G. Let $\nu_i : V_i \otimes V_i' \longrightarrow M(\pi_i)$ be the isomorphism from Theorem 5.8. Then*

$$\mathbb{C}[G/H] \simeq \bigoplus_{i=1}^{r} \nu_i(V_i^H \otimes V_i')$$

where V_i^H denotes the subspace of H-invariants, that is,

$$V_i^H = \{v \in V_i \mid \forall h \in H :\ \pi_i(h)v = v\}.$$

Proof. Recall that $\mathbb{C}[G] = M(\pi_1) \oplus \ldots \oplus M(\pi_r)$, and each $M(\pi_j)$ is invariant under right translation. Hence

$$\mathbb{C}[G/H] \simeq \mathbb{C}[G]^H = M(\pi_1)^H \oplus \ldots \oplus M(\pi_r)^H$$

where $M(\pi_j)^H$ denotes the subspace if H-fixed vectors (under right translation) in $M(\pi_j)$. We rewrite this space as follows:

$$\begin{aligned} M(\pi_j)^H &= \{F \in M(\pi_j) \mid \forall h \in H :\ F(\cdot h) = F\} \\ &= \{\nu_j(x) \mid x \in V_j \otimes V_j',\ \forall h \in H :\ \mathrm{Reg}_{\pi_j}(h,1)\nu_j(x) = \nu_j(x)\} \\ &= \{\nu_j(x) \mid x \in V_j \otimes V_j',\ \forall h \in H :\ \nu_j(\pi_j \times \pi_j'(h,1)x) = \nu_j(x)\} = \nu_j(V_j^H \otimes V_j') \end{aligned}$$

which is exactly what we were looking for. □

Corollary 6.10. *Let the notation be as in Theorem 6.9. Then*

$$\pi_H \simeq \pi_1^{m_1^H} \oplus \ldots \oplus \pi_r^{m_r^H}$$

where $m_j^H = \dim V_j^H$.

[2] We've skipped this section in class for now, but we'll come back to it later

Proof. It follows as in the proof of Corollary 5.10 that for each j the homomorphism ν_j gives an equivalence between the representations $1_H \otimes \pi_j'$ and $L_{M(\pi_j)^H}$ where 1_H denotes the trivial representation of G on V_j^H. Hence

$$\pi_H \simeq L_{\mathbb{C}[G]^H} = L_{M(\pi_1)^H} \oplus \ldots \oplus L_{M(\pi_r)^H} \simeq (\pi_1')^{\dim V_1^H} \oplus \ldots \oplus (\pi_r')^{\dim V_r^H}$$
$$= \pi_1^{\dim (V_1')^H} \oplus \ldots \oplus \pi_r^{\dim (V_r')^H}$$

where the last equality follows from the fact that each π_j' is irreducible, hence must be equal to one of $\pi_1, \ldots, \pi_r$, say $\pi_j' \simeq \pi_k$, so that then $\dim V_j^H = \dim (V_k')^H$.

It will therefore suffice to show that $\dim V_j^H = \dim (V_j')^H$. We will in fact show the following: If (σ, W) is a completely reducible representation of H, then $\dim W^H = \dim (W')^H$. To see this let $\sigma_1, \ldots, \sigma_l$ denote a complete list of representatives for the equivalence classes of irreducible representations of H with σ_1 being the trivial representation. Write $\sigma \simeq \sigma_1^{n_1} \oplus \ldots \oplus \sigma_l^{n_l}$ for suitable $n_1, \ldots, n_l \in \mathbb{N}_0$. Then $\dim W^H = n_1$. On the other hand, we can also write $\sigma' \simeq \sigma_1^{n_1'} \oplus \ldots \oplus \sigma_l^{n_l'}$ for suitable $n_1', \ldots, n_r' \in \mathbb{N}_0$. As discussed before, the numbers $n_1', \ldots, n_l'$ agree with $n_1, \ldots, n_l$ up to permutation and $n_i = n_j'$ if $\sigma_i \simeq \sigma_j'$. Since the trivial representation is equivalent to its dual, it therefore follows that $n_1 = n_1'$, that is, $\dim W^H = \dim (W')^H$. ☐

7 Character Theory

For now we don't need to assume anything about our group G, but all our representations are finite dimensional over $\mathbb{R}$ or $\mathbb{C}$ even if not explicitly mentioned.

Definition 7.1. Let (π, V) be a finite dimensional real or complex representation of G. Then

$$\chi_\pi : G \longrightarrow \mathbb{C}, \; \chi_\pi(g) := \operatorname{tr} \pi(g)$$

is called the *character* of π.

Note that $\chi_\pi \in M(\pi)$.

Example 7.2.

- If π has dimension 1, that is, $\pi : G \longrightarrow K^\times$, then $\chi_\pi = \pi$. Hence if G is abelian and finite, we sometimes call its irreducible complex representations its characters.
- If $1_V : G \longrightarrow \mathrm{GL}(V)$ denotes the trivial representation on a finite dimensional vector space V, then $\chi_{1_V}(g) = \dim V$ for all $g \in G$.
- Recall the representation $\pi : \mathbb{R} \longrightarrow \mathrm{GL}_2(\mathbb{R})$, $\pi(t) = \begin{pmatrix} \cos(t) & \sin(t) \\ -\sin(t) & \cos(t) \end{pmatrix}$ from Example 1.10. Then $\chi_\pi(t) = 2\cos(t)$.
- Recall the representation $\pi : S_n \longrightarrow \mathrm{GL}_n(\mathbb{C})$ which sends $\sigma \in S_n$ to the corresponding permutation matrix from Example 1.11. Then

$$\chi_\pi(\sigma) = |\{m \in \{1, \ldots, n\} \mid \sigma(m) = m\}| = \# \text{ number of fixed points of } \sigma.$$

7.1 Basic properties of characters

We collect some basic properties of characters for π, σ finite dimensional representations of G (1 denotes the unit of G):

(i) $\chi_\pi(1) = \dim \pi$

(ii) $\pi \simeq \sigma \Rightarrow \chi_\pi = \chi_\sigma$

(iii) $\chi_{\pi \oplus \sigma} = \chi_\pi + \chi_\sigma$

(iv) $\forall g \in G: \ \chi_{\pi'}(g) = \chi_\pi(g^{-1})$

(v) $\chi_{\pi \otimes \sigma} = \chi_\pi \chi_\sigma$

(vi) $\forall g, h \in G: \ \chi_\pi(g^{-1}hg) = \chi_\pi(h)$

Moreover, if H is another group, π a finite dimensional representation of G, and σ a finite dimensional representation of H, then

(vii) $\chi_{\pi \times \sigma}(g, h) = \chi_\pi(g)\chi_\sigma(h)$

for $g \in G$, $h \in H$.

All those properties are immediate from the definition.

Remark 7.3. *If we apply (i) to the right regular representation R of a finite group G, we get*

$$|G| = \dim R = \chi_R(1) = \sum_{i=1}^{r} \dim \pi_i \chi_{\pi_i}(1) = \sum_{i=1}^{r} (\dim \pi_i)^2$$

where $\pi_1, \ldots, \pi_r$ are representatives for the equivalence classes of irreducible complex representations of G.

Definition 7.4. A function $f \in \mathbb{C}[G]$ is called a *class function* or *central function* if it is constant on conjugacy classes of G, that is, for all $g, h \in G$, we have $f(g^{-1}hg) = f(h)$. We also write

$$\mathbb{C}[G]^\# = \{\text{class functions } f \in \mathbb{C}[G]\}.$$

Characters are therefore class functions, that is, $\chi_\pi \in \mathbb{C}[G]^\#$.

7.2 Characters of finite groups

Theorem 7.5. *Suppose G is finite and $\pi_1, \ldots, \pi_r$ is a complete list of representatives for the equivalence classes of irreducible complex representations of G. Then*

$$\mathbb{C}[G]^\# = \mathbb{C}\chi_{\pi_1} \oplus \ldots \oplus \mathbb{C}\chi_{\pi_r}$$

Proof. Because of (4) it will suffice to show that $\mathbb{C}[G]^{\#} \cap M(\pi_j) = \mathbb{C}\chi_{\pi_j}$.

Recall that $M(\pi_j) = \{\mathrm{tr}_A \mid A \in \mathrm{Hom}(V,V)\}$ with $\mathrm{tr}_A(g) = \mathrm{tr}(\pi_j(g)A)$. Suppose that tr_A is a class function. Then for all $g, h \in G$,

$$\mathrm{tr}_A(h) = \mathrm{tr}(\pi_j(h)A) = \mathrm{tr}(\pi_j(g)^{-1}\pi_j(h)\pi_j(g)A) = \mathrm{tr}(\pi_j(h)\pi_j(g)A\pi_j(g)^{-1}) = \mathrm{tr}_{\pi_j(g)A\pi_j(g)^{-1}}(h).$$

Since π_j is irreducible, the map $\mathrm{Hom}(V_j, V_j) \longrightarrow M(\pi_j)$, $A \mapsto \mathrm{tr}_A$ defines an isomorphism, hence it follows that $A = \pi_j(g)A\pi_j(g)^{-1}$. Hence A is an intertwining operator of π_j so that by Schur's Lemma, there exists $\alpha \in \mathbb{C}$ such that $A = \alpha \mathrm{id}_{V_j}$. Hence

$$\mathrm{tr}_A(h) = \mathrm{tr}_{\alpha \mathrm{id}_{V_j}}(h) = \mathrm{tr}(\alpha\pi_j(h)) = \alpha\chi_{\pi_j} \in \mathbb{C}\chi_{\pi_j}.$$

□

Corollary 7.6. *Suppose G is finite. Then the number of equivalence classes of irreducible complex representations of G equals the number of conjugacy classes in G.*

Corollary 7.7. *Suppose G is finite. Then any finite dimensional complex representation of G is uniquely determined (up to equivalence) by its character.*

Proof. Let π be a finite dimensional complex representation of G, and let $m_1, \ldots, m_r \in \mathbb{N}_0$ be such that $\pi \simeq \pi_1^{m_1} \oplus \ldots \oplus \pi_r^{m_r}$. Then $\chi_\pi = m_1\chi_{\pi_1} + \ldots + m_r\chi_{\pi_r}$, and the m_j are uniquely determined since $\chi_{\pi_1}, \ldots, \chi_{\pi_r}$ are a basis of $\mathbb{C}[G]^{\#}$. □

8 Schur orthogonality

In this section G is a finite group. We denote by $(\pi_1, V_1), \ldots, (\pi_r, V_r)$ a complete list of representatives for the equivalence classes of irreducible complex representations of G, and by $\chi_1, \ldots, \chi_r$ the corresponding characters.

Recall the two-sided regular representation $\mathrm{Reg} : G \times G \longrightarrow \mathrm{GL}(\mathbb{C}[G])$.

Definition 8.1. We equip $\mathbb{C}[G]$ with the inner product

$$\langle f_1, f_2 \rangle = \frac{1}{|G|}\sum_{g\in G} f_1(g)\overline{f_2(g)}$$

for $f_1, f_2 \in \mathbb{C}[G]$.

Lemma 8.2. $\langle \cdot, \cdot \rangle$ *is* Reg*-invariant, that is, for all $g, h \in G$, and all $f_1, f_2 \in \mathbb{C}[G]$ we have*

$$\langle \mathrm{Reg}(g,h)f_1, \mathrm{Reg}(g,h)f_2 \rangle = \langle f_1, f_2 \rangle.$$

Proof. We compute:

$$\langle \mathrm{Reg}(g,h)f_1, \mathrm{Reg}(g,h)f_2 \rangle = \frac{1}{|G|}\sum_{x\in G} f_1(h^{-1}xg)\overline{f_2(h^{-1}xg)} = \frac{1}{|G|}\sum_{y\in G} f_1(y)\overline{f_2(y)} = \langle f_1, f_2 \rangle$$

where we used the bijection $G \longrightarrow G$, $x \mapsto y := h^{-1}xg$ in the second equality. □

8.1 Invariant inner products

Theorem 8.3. *Suppose (π, V) is a fintie dimensional complex representation of G, and B_1, B_2 are π-invariant inner products on V. Then there exists $\alpha \in \mathbb{R}_{>0}$ such that $B_1 = \alpha B_2$*

Proof. See Exercise 18. □

Theorem 8.4. *Suppose (π, V) is a finite dimensional complex representation of G. Let $U, W \subseteq V$ be minimal invariant subspaces such that $\pi_U \not\simeq \pi_V$. Let B be a π-invariant inner product on V. Then U and W are orthogonal with respect to B.*

Proof. Let $p : U \hookrightarrow V \longrightarrow W$ denote the orthogonal projection with respect to B. Then p is clearly an intertwining operator from π_U to π_W, thus $\equiv 0$ since π_U and π_W are not equivalent. Now for all $u \in U$, $w \in W$ we also have

$$B(u, w) = B(u - p(u), w) + B(p(u), w) = B(p(u), w)$$

hence, $B(u, w) = 0$ since $p \equiv 0$. □

An immediate consequence of this is the following:

Corollary 8.5. *The subspaces $M(\pi_1), \ldots, M(\pi_r)$ of $\mathbb{C}[G]$ are orthogonal with respect to $\langle \cdot, \cdot \rangle$.*

We can be even more precise: Fix a π_i-invariant inner product B_i on V_i for each i, and let $\mathcal{B}_i$ denote an orthonormal (with respect to B_i) basis of V_i. Write

$$\pi_{i,\mathcal{B}_i}(g) = (a^{(i)}_{\kappa\mu}(g))_{\kappa,\mu=1,\ldots,n_i} \in \mathrm{GL}_{n_i}(\mathbb{C})$$

where $n_i = \dim \pi_i$.

Theorem 8.6. *The set*

$$\{a^{(i)}_{\kappa,\mu} \mid i = 1, \ldots, r,\ \mu, \kappa = 1, \ldots, n_i\}$$

is an orthogonal basis for $\mathbb{C}[G]$ with respect to $\langle \cdot, \cdot, \rangle$. More precisely,

$$\langle a^{(i)}_{\kappa,\mu}, a^{(j)}_{\lambda,\nu} \rangle = \begin{cases} \frac{1}{n_i} & \text{if } i = j,\ (\kappa, \mu) = (\lambda, \nu), \\ 0 & \text{else.} \end{cases}$$

Proof. We already know that $a^{(i)}_{\kappa\mu}$ and $a^{(j)}_{\lambda\nu}$ are orthogonal if $i \neq j$. We therefore fix $i = j$ and drop this index from the notation for the remainder of the proof, that is, we write $n = n_i$, $V = V_i$ etc. Define an inner product $(\cdot, \cdot)$ on $M_n(\mathbb{C}) \simeq \mathrm{End}(V)$ (where the isomorphism is with respect to the fixed basis) by

$$(A, B) = \mathrm{tr}(A\overline{B}^t).$$

Consider the representation $\rho : G \times G \longrightarrow \mathrm{GL}(M_n(\mathbb{C}))$ given by

$$\rho(g, h)A = \pi_{\mathcal{B}}(h)^{-1} A \pi_{\mathcal{B}}(g).$$

Then $(\cdot,\cdot)$ is invariant with respect to ρ since $\pi_{\mathcal{B}}(h)^{-1} = \overline{\pi_{\mathcal{B}}(h)}^t$ since our chosen basis is orthonormal with respect to the π-invariant inner product. Moreover, the isomorphism $M_n(\mathbb{C}) \longrightarrow M(\pi)$, $A \mapsto \mathrm{tr}_A$, gives the equivalence $\rho \simeq \mathrm{Reg}_{M(\pi)}$. Hence via this isomorphism we obtain another $\mathrm{Reg}_{M(\pi)}$ invariant inner product on $M(\pi)$, which therefore coincides with $\langle\cdot,\cdot\rangle$ up to some positive real constant, that is, there exists $\alpha > 0$ such that for all $A, B \in M_n(\mathbb{C})$ we have $(A, B) = \alpha\langle \mathrm{tr}_A, \mathrm{tr}_B\rangle$.

Recall the definition of the matrix $E_{ij} \in M_n(\mathbb{C})$ which has entry 1 at position (i,j) and 0s else. Note that $\mathrm{tr}_{E_{ij}}(g) = \mathrm{tr}(\pi_{\mathcal{B}}(g)E_{ij}) = a_{ij}(g)$. Hence we get

$$\alpha\langle a_{ij}, a_{\mu\nu}\rangle = \alpha\langle \mathrm{tr}_{E_{ij}}, \mathrm{tr}_{E_{\mu\nu}}\rangle = (E_{ij}, E_{\mu\nu}) = \begin{cases} 1 & \text{if } (i,j) = (\mu,\nu), \\ 0 & \text{else.} \end{cases}$$

We are left to show that $\alpha = n$. For that first note that the fact that π is unitary with respect to the fixed inner product on V implies that

$$\sum_{j=1}^{n} a_{ij}(g)a_{kj}(g) = \begin{cases} 1 & \text{if } i = k, \\ 0 & \text{else.} \end{cases}$$

This therefore implies that

$$1 = \frac{1}{|G|}\sum_{g\in G}\sum_{j=1}^{n} a_{ij}(g)a_{kj}(g) = \sum_{j=1}^{n}\langle a_{ij}, a_{ij}\rangle = n/\alpha,$$

hence $\alpha = n$. ☐

Corollary 8.7. *$\chi_1, \dots, \chi_r$ is an orthonormal basis of $\mathbb{C}[G]^{\#}$.*

Proof. For each i we have $\chi_i(g) = \sum_{k=1,\dots,n_i} a_{kk}^{(i)}(g)$. Hence by Theorem 8.6 we have $\langle\chi_i,\chi_j\rangle = 0$ if $i \neq j$. If $i = j$, we get also by Theorem 8.6

$$\langle\chi_i,\chi_i\rangle = \sum_{k,l=1,\dots,n_i}\langle a_{kk}^{(i)}, a_{ll}^{(i)}\rangle = \sum_{k=1,\dots,n_i}\langle a_{kk}^{(i)}, a_{kk}^{(i)}\rangle = n_i\frac{1}{n_i} = 1.$$

☐

Corollary 8.8. *If (π, V) is a finite dimensional complex representation of G, then*

$$\pi \simeq \bigoplus_{i=1,\dots,r}\pi_i^{\langle\chi_\pi,\chi_i\rangle}.$$

Proof. We already know that $\pi \simeq \bigoplus_{i=1,\dots,r}\pi_i^{m_i}$ for suitable uniquely determined $m_i \in \mathbb{N}_0$. The assertion then follows from $\chi_\pi = m_1\chi_1 + \dots m_r\chi_r$ together with Corollary 8.7. ☐

Corollary 8.9. *Let (π, V) be a finite dimensional complex representation of G. Then π is irreducible if and only if $\langle\chi_\pi,\chi_\pi\rangle = 1$.*

Proof. Write $\pi \simeq \bigoplus_{i=1,\dots,r}\pi_i^{m_i}$ for suitable uniquely determined $m_i \in \mathbb{N}_0$. Then $\langle\chi_\pi,\chi_\pi\rangle = m_1^2 + \dots + m_r^2$ so that π is irreducible if and only if $\langle\chi_\pi,\chi_\pi\rangle = 1$. ☐

8.2 Orthogonality Relations

We still assume in this section that G is finite, $\pi_1, \dots, \pi_r$ is a complete list of representatives for the equivalence classes of irreducible complex representations for G, and $\chi_1, \dots, \chi_r$ are their characters.

Recall the inner product on $\mathbb{C}[G]$ we introduced in the last section: For $F_1, F_2 \in \mathbb{C}[G]$,

$$\langle F_1, F_2 \rangle = \frac{1}{|G|} \sum_{g \in G} F_1(g)\overline{F_2(g)}.$$

Recall that $\chi_1, \dots, \chi_r$ is an orthonormal basis of $\mathbb{C}[G]^\#$ with respect to $\langle \cdot, \cdot \rangle$.

The *character table* of G collects information on the characters and irreducible representations of G in a table. Let $g_1 = 1, g_2, \dots, g_r \in G$ be representatives for the conjugacy classes $C_1 = \{1\}, C_2, \dots, C_r$ of G (recall from Corollary 7.6 that the number of conjugacy classes in G coincides with r), and let $m_j = |C_j|$, $j = 1, \dots, r$. Then the character table of G looks as follows:

	1	m_2	m_3	$\dots$	m_r
	1	g_2	g_3	$\dots$	g_r
$1 = \pi_1$	1	1	1	$\dots$	1
π_2	$\chi_2(1)$	$\chi_2(g_2)$	$\chi_2(g_3)$	$\dots$	$\chi_2(g_r)$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
π_r	$\chi_r(1)$	$\chi_r(g_2)$	$\chi_r(g_3)$	$\dots$	$\chi_r(g_r)$

Note that we in fact know the entries in the first column: $\chi_i(1) = \dim \pi_i$, $i = 1, \dots, r$. We might also view this as an $r \times r$-matrix $(\chi_i(g_j))_{i,j=1,\dots,r} \in M_r(\mathbb{C})$. Because of the orthonormality of the basis $\chi_1, \dots, \chi_r$ of $\mathbb{C}[G]$ we get

$$\frac{1}{|G|} \sum_{j=1}^{r} m_j \chi_\mu(g_j)\overline{\chi_\nu(g_j)} = \begin{cases} 1 & \text{if } \mu = \nu, \\ 0 & \text{else,} \end{cases} \tag{5}$$

that is the rows of the matrix

$$C := \left(\frac{\sqrt{m_j}}{\sqrt{|G|}} \chi_\mu(g_\nu) \right)_{\mu,\nu=1,\dots,r}$$

constitute an orthonormal basis of $\mathbb{C}^r$ so that $C\overline{C}^t = 1_r$ (where 1_r is the $r \times r$ identity matrix). Hence C is a unitary matrix so that we immediately also obtain orthonormality of the columns:

Corollary 8.10. *For every $g, h \in G$ we have*

$$\sum_{j=1}^{r} \chi_j(g)\overline{\chi_j(h)} = \begin{cases} \frac{|G|}{m_i} & \textit{if } g, h \textit{ are conjugate, say } g, h \in C_i, \\ 0 & \textit{else.} \end{cases} \tag{6}$$

The two equalities (5) and (6) together are usually called the *Schur orthogonality relations*

Example 8.11. Suppose G is the cyclic group of order m. We take $G = \mathbb{Z}/m\mathbb{Z}$ for concreteness. Then G is abelian, thus has m-conjugacy classes and m (equivalence classes of) irreducible complex representations, each of dimension 1. In fact, they are given by

$$\chi_j(k) = \zeta^{k(j-1)},\ j = 1, \ldots, m,$$

where ζ is a primitive mth root of unity (for example $\zeta = e^{2\pi i/m}$). The character table of G then looks as follows:

	1	1	1	$\ldots$	1
	0	1	2	$\ldots$	$m-1$
$1 = \chi_1$	1	1	1	$\ldots$	1
χ_2	1	ζ	ζ^2	$\ldots$	ζ^{m-1}
χ_3	1	ζ^2	ζ^4	$\ldots$	$\zeta^{2(m-1)}$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
χ_m	1	ζ^{m-1}	$\zeta^{2(m-1)}$	$\ldots$	$\zeta^{(m-1)^2}$

Example 8.12. Take $G = S_3$. Then G has three conjugacy classes, represented by 1, (12), and (123). The classes have cardinality 1, 3, and 2, respectively. We already know the irreducible representations in this case: We have two one-dimensional ones, namely the trivial representation $1 : S_3 \longrightarrow \mathbb{C}^\times$, and $\mathrm{sgn} : S_3 \longrightarrow \mathbb{C}^\times$, and we have one two-dimensional irreducible representation π. The character table of S_3 therefore looks as follows:

	1	3	2
	1	(12)	(123)
1	1	1	1
sgn	1	-1	1
π	2	0	1

Example 8.13. Let $G = S_4$. This group has five conjugacy classes, represented by 1, (12), (123), (1234), (12)(34), with cardinality of 1, 6, 8, 6, and 3, respectively. We already found representatives for the irreducible complex representations of S_4 in Example 6.7. From that we can read of the character table of S_4 as follows:

	1	6	8	6	3
	1	(12)	(123)	(1234)	(12)(34))
1	1	1	1	1	1
sgn	1	-1	1	-1	1
π	3	1	0	-1	-1
$\pi \otimes \mathrm{sgn}$	3	-1	0	1	-1
$\tilde{\sigma}$	2	0	-1	0	2

9 Induced Representations

9.1 Induction on characters

Let G be a finite group and $H \leq G$ a subgroup. Given $f \in \mathbb{C}[G]^{\#}$, we can obtain a class function on H just by restricting f to H. We thus can define the *restriction map*

$$\mathrm{Res}_H^G : \mathbb{C}[G]^{\#} \longrightarrow \mathbb{C}[H]^{\#},\ f \mapsto f_{|H}.$$

We would like to define a map in the other direction as well.

Definition/Lemma 9.1. *For $f \in \mathbb{C}[H]$ define $\tilde{f} \in \mathbb{C}[G]$ by*

$$\tilde{f}(g) = \begin{cases} f(g) & \text{if } g \in H, \\ 0 & \text{else.} \end{cases}$$

Then the map

$$\mathrm{Ind}_H^G : \mathbb{C}[H]^{\#} \longrightarrow \mathbb{C}[G]^{\#},\ \mathrm{Ind}_H^G f(g) := \frac{1}{|H|} \sum_{x \in G} \tilde{f}(x^{-1}gx)$$

is called the induction *from H to G. If $f = \chi$ is a character of H, then* $\mathrm{Ind}_H^G \chi$ *is called an* induced character*.*

Note that a priori $\mathrm{Ind}_H^G \chi$ is not necessarily the character of a representation of G even if χ was a character of a representation of H. (But we'll see later that this is in fact the case.)

Proof. We need to check that Ind_H^G is well-defined, that is, that $\mathrm{Ind}_H^G f$ is indeed a class function on G. For that let $g, h \in G$, $f \in \mathbb{C}[H]^{\#}$. Then

$$\mathrm{Ind}_H^G f(h^{-1}gh) = \frac{1}{|H|} \sum_{x \in G} \tilde{f}(x^{-1}h^{-1}ghx) = \frac{1}{|H|} \sum_{y \in G} \tilde{f}(y^{-1}gy) = \mathrm{Ind}_H^G f(g)$$

where we made the substitution $y = hx$. Hence $\mathrm{Ind}_H^G f$ is indeed a class function on G. $\square$

Remark 9.2.

- Ind_H^G *is a linear map* $\mathbb{C}[H]^{\#} \longrightarrow \mathbb{C}[G]^{\#}$*.*
- *If $x_1, \ldots, x_l \in G$ are representatives for the right cosets G/H, then*

$$\mathrm{Ind}_H^G f(g) = \sum_{x \in G/H} \tilde{f}(x^{-1}gx) = \sum_{i=1}^{l} \tilde{f}(x_i^{-1}gx_i).$$

Theorem 9.3 (Frobenius Reciprocity)**.** *Let $f \in \mathbb{C}[H]^{\#}$ and $F \in \mathbb{C}[G]^{\#}$. Then by unfolding the definitions we get*

$$\langle \mathrm{Ind}_H^G f, F \rangle_G = \langle f, \mathrm{Res}_H^G F \rangle_H,$$

where $\langle \cdot, \cdot \rangle_G$ denotes the inner product on $\mathbb{C}[G]$ as introduced at the beginning of §8.2, and $\langle \cdot, \cdot \rangle_H$ is the analogous inner product on $\mathbb{C}[H]$.

Proof. Let $f \in \mathbb{C}[H]^{\#}$ and $F \in \mathbb{C}[G]^{\#}$. Then

$$\langle \mathrm{Ind}_H^G f, F\rangle_G = \frac{1}{|G|}\sum_{g\in G}\mathrm{Ind}_H^G f(g)\overline{F(g)} = \frac{1}{|G|}\frac{1}{|H|}\sum_{g\in G}\sum_{x\in G}\tilde{f}(x^{-1}gx)\overline{F(g)}.$$

Now note that $\tilde{f}(x^{-1}gx) \neq 0$ if and only if $x^{-1}gx \in H$, that is, if and only if there exists $h \in H$ such that $g = xhx^{-1}$, and this h determines g uniquely. Hence we can continue our calculation as follows:

$$\begin{aligned}\langle \mathrm{Ind}_H^G f, F\rangle_G = \frac{1}{|G|}\frac{1}{|H|}\sum_{x\in G}\sum_{h\in H}\tilde{f}(h)\overline{F(xhx^{-1})} &= \frac{1}{|G|}\frac{1}{|H|}\sum_{x\in G}\sum_{h\in H}f(h)\overline{F(h)}\\ &= \frac{1}{|H|}\sum_{h\in H}f(h)\overline{F(h)} = \langle f, \mathrm{Res}_H^G F\rangle_H\end{aligned}$$

where we used that F is a class function on G in the second equality. □

9.2 Induced representations

G still denotes a finite group and $H \leq G$ a subgroup. If (π, V) is a representation of G, we obtain a representation of H on V by simply restricting π to H. We denote the resulting representation by $\mathrm{Res}_H^G\pi : H \longrightarrow \mathrm{GL}(V)$.

Now suppose (σ, W) is a complex finite dimensional representation of H. Define

$$C(G, H, W) := \{f : G \longrightarrow W \mid \forall h \in H, g \in G:\ f(hg) = \sigma(h)f(g)\}.$$

Definition 9.4. The map

$$\mathrm{Ind}_H^G\pi : G \longrightarrow \mathrm{GL}(C(G,H,W)),\ \left(\mathrm{Ind}_H^G\pi(g)f\right)(x) := f(xg)$$

defines a representation of G on $C(G, H, W)$, called the *induced representation*.

Example 9.5. Suppose $H = \{1\}$ is the trivial subgroup of G, and let $1 : H \longrightarrow \mathbb{C}^\times$ denote the trivial representation of H. Then $\mathrm{Ind}_H^G 1$ is isomorphic to the right regular representation of G on $\mathbb{C}[G]$. (See exercises)

Remark 9.6. *If σ is a representation of H of dimension m, then* $\dim \mathrm{Ind}_H^G\sigma = m \cdot |G/H|$.

We now obtain a second version of Frobenius reciprocity:

Theorem 9.7 (Frobenius Reciprocity for representations)**.** *Let π be a finite dimensional complex representation of G, and σ a finite-dimensional complex representation of H. Then*

$$\langle \chi_\pi, \chi_{\mathrm{Ind}_H^G\sigma}\rangle_G = \langle \chi_{\mathrm{Res}_H^G\pi}, \chi_\sigma\rangle_H.$$

An immediate consequence of this is the following:

Corollary 9.8. *Let π and σ be as in Theorem 9.7 and both irreducible. Then π occurs in $\mathrm{Ind}_H^G\sigma$ with the same multiplicity as σ occurs in $\mathrm{Res}_H^G\pi$.* □

In view of Theorem 9.3 to prove Theorem 9.7, it will suffice to show the following lemma:

Lemma 9.9. *Let π and σ be as in Theorem 9.7. Then*

(i) $\mathrm{Res}_H^G\chi_\pi = \chi_{\mathrm{Res}_H^G\pi}$,

(ii) $\mathrm{Ind}_H^G\chi_\sigma = \chi_{\mathrm{Ind}_H^G\sigma}$.

Proof. The first assertion is clear. To prove the second, let $x_1, \ldots, x_l \in G$ denote representatives for the cosets G/H, and recall that

$$\mathrm{Ind}_H^G\chi_\sigma(g) = \sum_{i=1}^{l} \widetilde{\chi_\sigma}(x_i^{-1}gx_i).$$

Let W denote the representation space of σ and write $W^G = C(G,H,W)$. Note that $x_1^{-1}, \ldots, x_l^{-1}$ is then a set of representatives for $H\backslash G$. Hence the map

$$W^G \longrightarrow \bigoplus_{i=1}^{l} W =: \bigoplus_{i=1}^{l} W^{(i)},\ f \mapsto (f(x_1^{-1}), \ldots, f(x_l^{-1}))$$

defines an isomorphism of vector spaces.

Let $g \in G$. Then for every $i \in \{1, \ldots, l\}$ we can find unique $j_i \in \{1, \ldots, l\}$ and $h \in H$ such $x_i^{-1}g = hx_{j_i}^{-1}$. This means that $\mathrm{Ind}_H^G\sigma(g)$ maps $W^{(i)}$ into $W^{(j_i)}$. Note that $i = j_i$ if and only if $x_i^{-1}gx_i \in H$. In that case the restriction of $\mathrm{Ind}_H^G\sigma(g)$ to $W^{(i)}$ defines an endomorphism of $W^{(i)}$ which in fact just acts as $\sigma(x_i^{-1}gx_i)$ on $W^{(i)} \simeq W$. Hence we can compute:

$$\begin{aligned}\chi_{\mathrm{Ind}_H^G\sigma}(g) = \mathrm{tr}\,\mathrm{Ind}_H^G\sigma(g) &= \sum_{i=1,\ldots,l:\, i=j_i} \mathrm{tr}\left(\mathrm{Ind}_H^G\sigma(g)_{|W^{(i)}}\right) = \sum_{i=1,\ldots,l:\, x_i^{-1}gx_i\in H} \mathrm{tr}\left(\mathrm{Ind}_H^G\sigma(g)_{|W^{(i)}}\right) \\ &= \sum_{i=1,\ldots,l:\, x_i^{-1}gx_i\in H} \mathrm{tr}\,\sigma(x_i^{-1}gx_i) = \sum_{i=1,\ldots,l} \widetilde{\chi_\sigma}(x_i^{-1}gx_i) = \mathrm{Ind}_H^G\chi_\sigma(g).\end{aligned}$$

□

Remark 9.10. *One can more generally show that*

$$\mathrm{Hom}(\pi, \mathrm{Ind}_H^G\sigma) \simeq \mathrm{Hom}(\mathrm{Res}_H^G\pi, \sigma)$$

via a canonical isomorphism (here the left and right hand side denote the vector spaces of intertwining operators between the respective representations).

9.3 Mackey Theory

Induced representations are in general not irreducible as can already be seen from Example 9.5, but they can be irreducible in some cases:

Example 9.11. Let $H = A_3 \leq S_3 = G$. Then A_3 is a cyclic group of order 3 generated by the cycle (123). Let $\rho : A_3 \longrightarrow \mathbb{C}^\times$ be the representation given by $\rho((123)) = e^{2\pi i/3}$, and let $\pi = \mathrm{Ind}_{A_3}^{S_3}\rho$. The elements id and (12) are a set of representatives for the cosets S_3/A_3 so that we can compute

$$\chi_\pi(\sigma) = \widetilde{\chi_\rho}(\sigma) + \widetilde{\chi_\rho}((12)\sigma(12)).$$

Hence the values of χ_π on the three conjugacy classes of S_3 are given as follows:

	id	(12)	(123)
χ_π	2	0	-1

which coincides with the character of the two-dimensional irreducible representation of S_3 that we found in Example 6.6 so that π must be equivalent to this representation and is therefore irreducible.

Suppose now that H, K are two subgroups. For $g \in G$ we write

$$HgK = \{hgk \mid h \in H, k \in K\}$$

for the double coset of g in the double quotient

$$H\backslash G/K = \{HgK \mid g \in G\}.$$

Note that there exists $S \subseteq G$ such that $G = \bigsqcup_{s\in S} HsK$ (disjoint union). Such S exists since $g \sim h :\Leftrightarrow h \in HgK$ is an equivalence relation. We fix such S for now.

Theorem 9.12 (Mackey's Theorem)**.** *Let H, K, S be as before. Then for every $F \in \mathbb{C}[K]^\#$ we have*

$$\mathrm{Res}_H^G \mathrm{Ind}_K^G F = \sum_{s\in S} \mathrm{Ind}_{H\cap sKs^{-1}}^H \mathrm{Res}_{H\cap sKs^{-1}}^{sKs^{-1}} F^s$$

where $F^s \in \mathbb{C}[sKs^{-1}]^\#$ is given by $F^s(x) = F(s^{-1}xs)$.

Proof. For each $s \in S$ fix a set of representatives $R_s \subseteq H$ for $H/(H \cap sKs^{-1})$. Then

$$H = \bigsqcup_{r\in R_s} r(H \cap sKs^{-1}). \tag{7}$$

We now claim that then for every $s \in S$ we have

$$HsK = \bigsqcup_{r\in R_s} rsK.$$

To prove this claim we first use (7) to get

$$HsK = \bigcup_{r\in R_s} r(H\cap sKs^{-1})sK \subseteq \bigcup_{r\in R_s} rsK \subseteq HsK$$

so that we are left to show that $rsK \cap \tilde{r}sK = \emptyset$ for $r, \tilde{r} \in R_s$, $r \neq \tilde{r}$. For that suppose that $rsK \cap \tilde{r}sK \neq \emptyset$. Then $s^{-1}\tilde{r}^{-1}rsK \cap K \neq \emptyset$ so that $\tilde{r}^{-1}r \in s^{-1}Ks \cap H$. This implies therefore $r = \tilde{r}$ so that the claim is proven.

We now write $T_s = R_s s = \{rs \mid r \in R_s\}$. By the claim that we just proved, T_s is a set of representatives for HsK/K. Write $T = \bigcup_{s\in S} T_s = \bigsqcup_{s\in S} T_s$. Then T is a set of representatives for the cosets G/K.

Now let $h \in H$ and $F \in \mathbb{C}[K]^\#$. We can then compute:

$$\mathrm{Ind}_K^G F(h) = \sum_{t\in T} \tilde{F}(t^{-1}ht) = \sum_{s\in S}\sum_{r\in R_s} \tilde{F}(s^{-1}r^{-1}hrs).$$

If $\tilde{F}(s^{-1}r^{-1}hrs) \neq 0$, then by the definition of $\tilde{F}$ we must have $r^{-1}hr \in sKs^{-1}$. Hence we can continue our computation as follows:

$$\begin{aligned}\mathrm{Ind}_K^G F(h) &= \sum_{s\in S} \sum_{\substack{r\in R_s:\\ r^{-1}hr\in sKs^{-1}}} F^s(r^{-1}hr) = \sum_{s\in S} \sum_{\substack{r\in R_s:\\ r^{-1}hr\in sKs^{-1}}} \mathrm{Ind}_{H\cap sKs^{-1}}^{sKs^{-1}} F^s(r^{-1}hr)\\ &= \sum_{s\in S}\sum_{r\in R_s} \mathrm{Ind}_{H\cap sKs^{-1}}^{\widetilde{sKs^{-1}}} F^s(r^{-1}hr) = \sum_{s\in S} \mathrm{Ind}_{H\cap sKs^{-1}}^{H} \mathrm{Res}_{H\cap sKs^{-1}}^{sKs^{-1}} F^s(h)\end{aligned}$$

which finishes the proof of the theorem. □

Definition 9.13. Let π and σ be two finite dimensional complex representations of G. We call π and σ *disjoint* if no irreducible representation of G occurs as a subrepresentation of both π and σ.

Lemma 9.14. *π and σ are disjoint if and only if $\langle \chi_\pi, \chi_\sigma\rangle = 0$.*

Proof. Let $m_1, \dots, m_r, l_1, \dots, l_r \in \mathbb{N}_0$ be such that $\pi \simeq \pi_1^{m_1} \oplus \dots \oplus \pi_r^{m_r}$ and $\sigma \simeq \pi_1^{l_1} \oplus \dots \oplus \pi_r^{l_r}$. Then $\langle \chi_\pi, \chi_\sigma\rangle = \sum_{i=1}^r m_i l_i$ so that $\langle \chi_\pi, \chi_\sigma\rangle = 0$ if and only if $m_i l_i = 0$ for all i which happens if and only if π and σ are disjoint. □

The following is a consequence of Theorem 9.12:

Corollary 9.15 (Mackey's irreducibility criterion)**.** *Suppose $H \leq G$ is a subgroup, and σ is a finite dimensional complex representation of H. Then $\mathrm{Ind}_H^G \sigma$ is an irreducible representation of G if and only if both of the following hold:*

(i) σ is irreducible

(ii) the representations $\mathrm{Res}_{sHs^{-1}\cap H}^{H} \sigma$ and $\mathrm{Res}_{sHs^{-1}\cap H}^{sHs^{-1}} \sigma^s$ of $sHs^{-1} \cap H$ are disjoint for all $s \in G$, $s \notin H$. Here $\sigma^s(x) = \sigma(s^{-1}xs)$ for $x \in sHs^{-1}$.

Proof. Write $\chi = \chi_\sigma$, and let S be a set of representatives for $H\backslash G/H$. Without loss of generality we can assume that $1 \in S$. Then by Mackey's Theorem we obtain

$$\mathrm{Res}_H^G \mathrm{Ind}_H^G \chi = \sum_{s\in S} \mathrm{Ind}_{H\cap sHs^{-1}}^H \mathrm{Res}_{H\cap sHs^{-1}}^{sHs^{-1}} \chi^s = \chi + \sum_{s\in S\smallsetminus\{1\}} \mathrm{Ind}_{H\cap sHs^{-1}}^H \mathrm{Res}_{H\cap sHs^{-1}}^{sHs^{-1}} \chi^s$$

where we used that for $s = 1$, $H \cap sHs^{-1} = H$. Hence we can compute, using Frobenius reciprocity in the first and last step:

$$\begin{aligned}\langle \mathrm{Ind}_H^G \chi, \mathrm{Ind}_H^G \chi\rangle = \langle \chi, \mathrm{Res}_H^G \mathrm{Ind}_H^G \chi\rangle &= \langle \chi, \chi\rangle + \sum_{s\in S\smallsetminus\{1\}} \langle \chi, \mathrm{Ind}_{H\cap sHs^{-1}}^H \mathrm{Res}_{H\cap sHs^{-1}}^{sHs^{-1}} \chi^s\rangle \\ &= \langle \chi, \chi\rangle + \sum_{s\in S\smallsetminus\{1\}} \langle \mathrm{Res}_{H\cap sHs^{-1}}^H \chi, \mathrm{Res}_{H\cap sHs^{-1}}^{sHs^{-1}} \chi^s\rangle.\end{aligned}$$

For $s \in S \smallsetminus \{1\}$ write $a_s := \langle \mathrm{Res}_{H\cap sHs^{-1}}^H \chi, \mathrm{Res}_{H\cap sHs^{-1}}^{sHs^{-1}} \chi^s\rangle$. Note that $\langle \chi, \chi\rangle \geq 1$, and in fact $\langle \chi, \chi\rangle = 1$ if and only if σ is irreducible. Also note that $a_s \geq 0$ for all $s \in S \smallsetminus \{1\}$.

Since $\mathrm{Ind}_H^G \sigma$ is irreducible if and only if $\langle \mathrm{Ind}_H^G \chi, \mathrm{Ind}_H^G \chi\rangle = 1$, this discussion therefore implies that $\mathrm{Ind}_H^G \sigma$ is irreducible if and only if σ is irreducible and $a_s = 0$ for all $s \in S \smallsetminus \{1\}$. But $a_s = 0$ if and only if $\mathrm{Res}_{H\cap sHs^{-1}}^H \sigma$ and $\mathrm{Res}_{H\cap sHs^{-1}}^{sHs^{-1}}$ are disjoint by Lemma 9.14. This finishes the proof of the corollary. □

Remark 9.16. *Suppose H is a normal subgroup of G. Then $H\backslash G/H = G/H$ and $sHs^{-1} = H$ for all $s \in G$. Mackey's Theorem simplifies in this case (for $K = H$) to*

$$\mathrm{Res}_H^G \mathrm{Ind}_H^G F = \sum_{s\in S} F^s$$

for all $F \in \mathbb{C}[H]^\#$, where S is a set of representatives for G/H. In this case Mackey's irreducibility criterion therefore also simplies to: $\mathrm{Ind}_H^G \sigma$ is irreducible if and only if σ is irreducible and σ and σ^s are disjoint for all $s \in S$, $s \neq 1$.

Example 9.17. Let $G = S_3$. Then $H = A_3$ is a normal subgroup of G and $S = \{\mathrm{id}, (12)\}$ is a set of representatives for G/H. Note that A_3 is a cyclic subgroup of order 3 generated by the permutation (123). Define $\sigma : A_3 \longrightarrow \mathbb{C}^\times$ by $\sigma((123)) = e^{2\pi i/3} =: \zeta$. Then $\sigma^{(12)}((123)) = \sigma((213)) = \zeta^2 \neq \zeta$ so that σ and $\sigma^{(12)}$ are disjoint. The induced representation $\mathrm{Ind}_{A_3}^{S_3}\sigma$ is therefore irreducible (and necessarily equivalent to the irreducible two-dimensional representation of S_3). We of course already proved this by direct computation in Example 9.11.

Example 9.18. Let $\mathbb{F}_p$ denote the finite field with p elements, $p \geq 3$ prime. Let

$$G = \{\left(\begin{smallmatrix} a & b \\ 0 & d\end{smallmatrix}\right) \in \mathrm{GL}_2(\mathbb{F}_p) \mid a, d \in \mathbb{F}_p^\times,\, b \in \mathbb{F}_p\}$$

which is a group where the multiplication is given by matrix mutliplication. Let

$$H = \{\left(\begin{smallmatrix} 1 & b \\ 0 & 1\end{smallmatrix}\right) \in \mathrm{GL}_2(\mathbb{F}_p) \mid b \in \mathbb{F}_p\}$$

which is a normal subgroup of G. The set $S = \{\left(\begin{smallmatrix} a & 0 \\ 0 & d \end{smallmatrix}\right) \mid a, d \in \mathbb{F}_p^\times\}$ is a set of representatives for G/H. Note that H is an abelian group of order p generated by $u_0 := \left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right)$, G has order $p(p-1)^2$, and S contains $(p-1)^2$ elements. The irreducible complex representations of H will therefore be of the form

$$\chi_\zeta(u_0^m) = \zeta^m$$

where ζ is a (not necessarily primitive) p-th root of unity. Now if $\left(\begin{smallmatrix} a & 0 \\ 0 & d \end{smallmatrix}\right) \in S$ it can easily be checked that

$$\chi^s(u_0^m) = \zeta^{md/a}.$$

Hence χ^s and χ are disjoint whenever $a \neq d$, and $\chi^s = \chi$ for all $a = d \in \mathbb{F}_p^\times$. Thus $\mathrm{Ind}_H^G \chi$ is not irreducible.

10 Compact Groups and Peter-Weyl Theorem

In this section G denotes a compact topological group (for example, G might be finite). This also means that we will assume implicitly from now on that all the finite dimensional complex representations of G that appear in this section are *continuous*, which means that if (π, V) is a finite dimensional complex representation, then π is *continuous* if the map $\pi : G \longrightarrow \mathrm{GL}(V)$ is continuous (recall that $\mathrm{GL}(V)$ is a topological group as explained in Example 12.2).

Recall that we fixed a Haar measure on G such that G gets measure 1, and we denote the corresponding integration by $\int_G f(g)\, dg$ for $f : G \longrightarrow \mathbb{C}$ continuous (see §12.2). Recall that the Haar measure is in fact left and right invariant by Remark 12.10. We thus obtain an inner product on the vector space $C(G) := \{f : G \longrightarrow \mathbb{C} \text{ continuous}\}$ defined by

$$\langle f_1, f_2 \rangle = \int_G f_1(g)\overline{f_2(g)}\, dg,$$

and this turns $C(G)$ into a pre-Hilbert space which we denote by $C_2(G)$.

Example 10.1. If G is finite, then $C_2(G) = \mathbb{C}[G]$ together with the previously defined inner product from Definition 8.1.

More generally, if X is a compact topological space with a given measure dx which is normalized such that $\int_X dx = 1$, we denote by $C_2(X)$ the pre-Hilbert space consisting of the set $C(X)$ of all complex valued continuous functions on X together with the inner product

$$\langle f_1, f_2 \rangle = \int_X f_1(x)\overline{f_2(x)}\, dx,\ f_1, f_2 \in C(X).$$

Definition 10.2. We denote by $\mathrm{Reg} : G \times G \longrightarrow \mathrm{GL}(C_2(G))$ the two-sided regular representation on the vector space $C_2(G)$ given by

$$\mathrm{Reg}(g, h)f(x) = f(h^{-1}xg).$$

Note that $\langle \cdot, \cdot \rangle$ is invariant under Reg, that is, Reg is unitary with respect to $\langle \cdot, \cdot \rangle$.

Note that we previously defined the two-sided regular representation of G on the space $\mathbb{C}[G]$ of all complex valued functions (see Definition 5.7) so that the representation defined here is in fact a subrepresentation of that representation if we want to be completely precise. We will, however, just write Reg for the two-sided regular representation of G on $C_2(G)$ from now on.

Convention: If (π, V) is a finite dimensional complex representation of G, we fix a π-invariant inner product on V and a basis $\mathcal{B}$ of V which is orthonormal with respect to this inner product. We then write

$$\pi_{\mathcal{B}}(g) = (a^{\pi}_{ij}(g))_{i,j=1,\ldots,n_\pi}$$

where $n_\pi = \dim V$.

We denote by $\mathcal{R}$ a complete set of representatives for the equivalence classes of finite dimensional complex representations of G.

We will prove the following theorem only for the case that G is a subgroup of $\mathrm{GL}_n(\mathbb{C})$ for some $n \in \mathbb{N}$:

Theorem 10.3 (Peter-Weyl)**.** *The collection of all the functions a^{π}_{ij}, $i, j = 1, \ldots, n_\pi$, $\pi \in \mathcal{R}$, constitutes an orthogonal basis for $C_2(G)$ with respect to $\langle \cdot, \cdot \rangle$. More precisely,*

$$\langle a^{\pi}_{ij}, a^{\tilde{\pi}}_{\mu\nu} \rangle = \begin{cases} 1/n_\pi & \text{if } (i,j,\pi) = (\mu,\nu,\tilde{\pi}), \\ 0 & \text{else} \end{cases}$$

for all $\pi, \tilde{\pi} \in \mathcal{R}$, $i, j = 1, \ldots, n_\pi$, $\mu, \nu = 1, \ldots, n_{\tilde{\pi}}$.

Remark 10.4. *Note that 'basis' here is understood in the Hilbert space sense, that is, the basis generates a dense subspace of $C_2(G)$.*

Example 10.5. We consider the circle group $G = S^1 = \{z \in \mathbb{C} \mid |z| = 1\}$ which is a compact topological group under multiplication. G is in fact homeomorphic to the compact topological group $\mathbb{R}/\mathbb{Z}$ via $\mathbb{R}/\mathbb{Z} \ni \theta \mapsto e^{2\pi i\theta} \in G$ and we can use this to write the integration as

$$\int_G f(g)\, dg = \frac{1}{2\pi} \int_0^{2\pi} f(e^{i\theta})\, d\theta.$$

We already know that every irreducible finite dimensional complex representation of G must be one-dimensional. In fact, $\mathcal{R} = \{\chi_n \mid n \in \mathbb{Z}\}$ where

$$\chi_n : G \longrightarrow \mathbb{C}^\times,\ \chi_n(e^{2\pi i\theta}) = e^{2\pi i n\theta}.$$

Those χ_n are clearly pairwise distinct one-dimensional representations of G. They are in fact also all, since $\left(e^{2\pi i/m}\right)^m = 1$ so that every one-dimensional representation of G must be of the form χ_n for some $n \in \mathbb{Z}$.

Thus by Theorem 10.3 the set $\{\chi_n \mid n \in \mathbb{Z}\}$ is an orthonormal basis of $C_2(G)$ with respect to the inner product

$$\langle f_1, f_2 \rangle = \frac{1}{2\pi} \int_0^{2\pi} f_1(e^{i\theta}) \overline{f_2(e^{i\theta})}\, d\theta.$$

This allows us to recover the Fourier expansion for continuous 2π-periodic functions $f : \mathbb{R} \longrightarrow \mathbb{C}$: If f is such a function, then there exist unique $a_n \in \mathbb{C}$, $n \in \mathbb{Z}$ such that

$$f(\theta) = \sum_{n\in\mathbb{Z}} a_n \chi_n(e^{i\theta}) = \sum_{n\in\mathbb{Z}} a_n e^{in\theta},$$

and those coefficients can be computed via

$$a_n = \langle f, \chi_n \rangle = \frac{1}{2\pi} \int_0^{2\pi} f(\theta)\overline{e^{-in\theta}}\, d\theta$$

(in the inner product we view f as a function on $G \simeq \mathbb{R}/\mathbb{Z}$).

Proof of Theorem 10.3. The orthogonality assumption of the theorem can be proven exactly as in the case that G is finite in Theorem 8.6 so that we don't repeat the argument here.

We are left to prove completeness. For this we will assume that G is a *subgroup* $\mathrm{GL}_n(\mathbb{C})$ for some $n \in \mathbb{N}$. Recall that we write $M(\pi)$ for the space of matrix coefficients of $\pi \in \mathcal{R}$.

Since $G \leq \mathrm{GL}_n(\mathbb{C})$ we can write

$$g = (a_{ij}(g))_{i,j=1,\ldots,n} \in \mathrm{GL}_n(\mathbb{C})$$

for suitable continuous functions $a_{ij} : G \longrightarrow \mathbb{C}$. In other words, the a_{ij} are the matrix coefficients of the identity representation $G \longrightarrow \mathrm{GL}_n(\mathbb{C})$, $g \mapsto g$. We write $C^0(G)$ for the vector subspace of $C(G)$ consisting of all functions on G that can be written as polynomials in a_{ij} and $\overline{a_{ij}}$, $i, j = 1, \ldots, n$. The Stone-Weierstrass Theorem implies that $C^0(G)$ is dense in $C_2(G)$ with respect to $\langle \cdot, \cdot \rangle$. The proof of Theorem 10.3 is finished once we prove Proposition 10.6 below. □

Proposition 10.6. *With the notation as in the proof of Theorem 10.3 we have*

$$C^0(G) = \bigoplus_{\pi\in\mathcal{R}} M(\pi).$$

Proof. For each $m \in \mathbb{N}_0$ we write $C^{0,m}(G)$ for the vector subspace of $C^0(G)$ consisting of functions on G that can be written as polynomials in a_{ij}, $\overline{a_{ij}}$, $i, j = 1, \ldots, n$, of degree at most m. For example, if $m = 0$, then $C^{0,0}(G)$ consists of constant functions only.

Note that $C^{0,m}(G)$ is invariant under Reg so that we can we define the right regular representation $R_m := R_{C^{0,m}(G)}$ of G on the vector space $C^{0,m}(G)$. Note that $C^{0,m}(G)$ is finite dimensional in contrast to $C^0(G)$ (which is infinite dimensional unless G is finite). By Theorem 2.11 we can therefore find a finite subset $\mathcal{R}_m \subseteq \mathcal{R}$ and $k_\pi \in \mathbb{N}_0$, $\pi \in \mathcal{R}_m$, such that

$$R_m \simeq \bigoplus_{\pi\in\mathcal{R}_m} \pi^{k_\pi}.$$

Hence

$$C^{0,m}(G) = M(R_m) = \bigoplus_{\pi\in\mathcal{R}_m} M(\pi)$$

so that

$$C^0(G) \subseteq \bigoplus_{m\in\mathbb{N}_0} M(R_m) \subseteq \bigoplus_{\pi\in\mathcal{R}} M(\pi).$$

For the other inclusion, suppose there exists $\pi_0 \in \mathcal{R}$ such that $M(\pi_0) \nsubseteq C^0(G)$. Then $M(\pi_0)$ is orthogonal to $C^0(G)$ and hence also to $C_2(G)$ which is a contradiction since $M(\pi_0)$ consists of continuous functions on G. □

Corollary 10.7. *(i) If π_1, π_2 are two irreducible finite dimensional complex representations of G, then*

$$\langle \chi_{\pi_1}, \chi_{\pi_2} \rangle = \begin{cases} 1 & \textit{if } \pi_1 \simeq \pi_2, \\ 0 & \textit{else.} \end{cases}$$

(ii) If π is a finite dimensional complex representation of G, then π is irreducible if and only if $\langle \chi_\pi, \chi_\pi \rangle = 1$. □

10.1 Invariants

Similarly as in the finite group case in §6.1, we can consider a closed subgroup $H \leq G$ and define the subspaces

$$C_2(G)^H := \{f \in C_2(G) \mid \forall h \in H, g \in G :\ f(gh) = f(g)\}$$

and $C^0(G)^H = C^0(G) \cap C_2(G)^H$. Similarly as in the finite group case we can then show that

$$L^H \simeq \bigoplus_{\pi\in\mathcal{R}} \pi^{m_\pi}$$

where L^H denotes the left regular representation of G on $C^0(G)^H$ and $m_\pi = \dim V_\pi^H$ with V_π the representation space of π.

Moreover, $C^0(G)^H$ is dense in $C_2(G)^H$. This can be seen as follows: Let $f \in C_2(G)$ and write $f = \sum_{\pi\in\mathcal{R}} f_\pi$ for suitable (and uniquely determined) $f_\pi \in M(\pi)$. Then $R(h)f = f$ for all $h \in H$ if and only if $R(h)f_\pi = f_\pi$ for all $\pi \in \mathcal{R}$ and all $h \in H$ which in turn holds if and only if $f \in \overline{\bigoplus_{\pi\in\mathcal{R}} M(\pi)^H} = \overline{C^0(G)^H}$.

11 Example: Representations of $\mathrm{SO}(3)$ and $\mathrm{SU}(2)$

11.1 Representations of $\mathrm{SO}(3)$

Recall that SO(3) consists of all matrices $g \in \mathrm{GL}_3(\mathbb{R})$ which satisfy $g^t g = 1_3$ and $\det g = 1$. We want to study representations of SO(3) via its action on the sphere

$$S^2 := \{x = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \in \mathbb{R}^3 \mid \|x\| = \sqrt{x_1^2 + x_2^2 + x_3^2} = 1\}.$$

SO(3) acts on S^2 via

$$g \cdot x := gx,$$

and this action leaves the usual Lebesgue measure on S^2 invariant. It is easily checked that this action is transitive, and if we denote by $e_0 := (0,0,1)^t$ the 'south pole' of the sphere, then the stabilizer of e_0 under the action of SO(3) equals

$$H = \{ \begin{pmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix} \mid 0 \leq \theta < 2\pi \} \simeq \mathrm{SO}(2).$$

We therefore obtain a homeomorphism of topological spaces

$$\mathrm{SO}(3)/H \longrightarrow S^2,\ gH \mapsto ge_0,$$

and a representation L of SO(3) on $C_2(S^2) \simeq C_2(\mathrm{SO}(3))^{\mathrm{SO}(2)}$ given by

$$L(g)f(x) = f(g^{-1}x) \tag{8}$$

for $f \in C_2(S^2)$, $x \in S^2$.

Remark 11.1. *One can show (see exercises) that if $V \subseteq C_2(S^2)$ is finite dimensional subspace which is L-invariant, then there exists $f \in V$ such that $f(hx) = f(x)$ for all $h \in H$, $x \in S^2$.*

Recall the Laplace operator Δ on $\mathbb{R}^3$, that is, Δ is the differential operator given by

$$\Delta = \partial_1^2 + \partial_2^2 + \partial_3^2$$

where $\partial_j = d/(dx_j)$. For $x \in \mathbb{R}^3$ we write $r(x) := \|x\|$. Further define $\mathcal{P} := \mathbb{C}[x_1, x_2, x_3]$ to be the complex vector space consisting of all polynomials in the variables x_1, x_2, x_3 with complex coefficients. Let $\mathcal{P}_n \subseteq \mathcal{P}$ denote the subspace consisting of homogeneous polynomials of degree n so that

$$\mathcal{P} = \bigoplus_{n \in \mathbb{N}_0} \mathcal{P}_n.$$

We further define an inner product on $\mathcal{P}$ by

$$\langle p, q \rangle := (p(\partial_1, \partial_2, \partial_3)q)(0).$$

It will follow from Lemma 11.3(i) below that this is indeed an inner product on $\mathcal{P}$.

Example 11.2. The definition of the inner product can look a bit strange, so we compute some examples:

- $\langle 1, 1 \rangle = 1$, $\langle 1, x_1 \rangle = 0 = \langle x_1, 1 \rangle$.
- $\langle x_1, x_1x_2x_3 + x_1^2 \rangle = \partial_1(x_1x_2x_3 + x_1^2)_{|x=0} = 0$

- $\langle x_1^2, x_1x_2x_3 + x_1^2\rangle = \partial_1^2(x_1x_2x_3 + x_1^2)_{|x=0} = 2$

Lemma 11.3. *(i) The set $\{x_1^{\kappa_1}x_2^{\kappa_2}x_3^{\kappa_3} \mid \kappa_1, \kappa_2, \kappa_3 \in \mathbb{N}_0\}$ is an orthogonal basis of $\mathcal{P}$ with respect to $\langle\cdot,\cdot\rangle$. Moreover, $\langle x_1^{\kappa_1}x_2^{\kappa_2}x_3^{\kappa_3}, x_1^{\kappa_1}x_2^{\kappa_2}x_3^{\kappa_3}\rangle = \kappa_1!\kappa_2!\kappa_3!$*

(ii) Δ and r^2 (i.e. multiplication by $r(x)^2$) are adjoint operators with respect to $\langle\cdot,\cdot\rangle$.

Proof. (i) Note that $\partial_i^k(x_j^m)_{|x=0} \neq 0$ if and only if $i = j$ and $l = m$ so that the orthogonality follows. The inner product of $x_1^{\kappa_1}x_2^{\kappa_2}x_3^{\kappa_3}$ with itself can be easily computed explicitly.

(ii) It will suffice to show that ∂_j and multiplication by x_j are adjoint. Without loss of generality $j = 1$. Then

$$\begin{aligned}\langle\partial_1(x_1^{\lambda_1}x_2^{\lambda_2}x_3^{\lambda_3}), x_1^{\kappa_1}x_2^{\kappa_2}x_3^{\kappa_3}\rangle &= \lambda_1\langle x_1^{\lambda_1-1}x_2^{\lambda_2}x_3^{\lambda_3}, x_1^{\kappa_1}x_2^{\kappa_2}x_3^{\kappa_3}\rangle \\ &= \begin{cases}\lambda_1!\lambda_2!\lambda_3! & \text{if } \lambda_1 - 1 = \kappa_1, \lambda_2 = \kappa_2, \lambda_3 = \kappa_3, \\ 0 & \text{else.}\end{cases}\end{aligned}$$

On the other hand,

$$\begin{aligned}&\langle x_1^{\lambda_1}x_2^{\lambda_2}x_3^{\lambda_3}, x_1 \cdot x_1^{\kappa_1}x_2^{\kappa_2}x_3^{\kappa_3}\rangle\langle x_1^{\lambda_1}x_2^{\lambda_2}x_3^{\lambda_3}, x_1^{\kappa_1+1}x_2^{\kappa_2}x_3^{\kappa_3}\rangle \\ &\qquad = \begin{cases}\lambda_1!\lambda_2!\lambda_3! & \text{if } \lambda_1 = \kappa_1 + 1, \lambda_2 = \kappa_2, \lambda_3 = \kappa_3, \\ 0 & \text{else}\end{cases}\end{aligned}$$

so that ∂_1 and multiplication by x_1 are indeed adjoint operators with respect to $\langle\cdot,\cdot\rangle$. □

We define a representation π of SO(3) on $\mathcal{P}$ by

$$(\pi(g)p)(x) = p(g^{-1}x).$$

Note that for each $n \in \mathbb{N}_0$, the subspace $\mathcal{P}_n$ is invariant under π and we denote the corresponding subrepresentation by $(\pi_n, \mathcal{P}_n)$ so that

$$\pi \simeq \bigoplus_{n\in\mathbb{N}_0} \pi_n.$$

Note that $\langle\cdot,\cdot\rangle$ is π-invariant. We further define the space of harmonic polynomials (i.e. the kernel of Δ in $\mathcal{P}$)

$$W = \{p \in \mathcal{P} \mid \Delta p = 0\}$$

and write $W_n = W \cap \mathcal{P}_n$.

Corollary 11.4. *For each $n \in \mathbb{N}_0$ we have*

$$\mathcal{P}_n = W_n \oplus r^2W_{n-2} \oplus r^4W_{n-4} \oplus \ldots \oplus r^{2\lfloor\frac{n}{2}\rfloor}W_{n-2\lfloor\frac{n}{2}\rfloor}$$

where $\lfloor\frac{n}{2}\rfloor$ denotes the largest integer $\leq n/2$. Moreover, each summand on the right hand side is π_n-invariant.

Proof. Note that $W_n = \ker \Delta_{|\mathcal{P}_n}$ so that by Lemma 11.3(ii) we get $\mathcal{P}_n = W_n \oplus r^2 \mathcal{P}_{n-2}$ (with $\mathcal{P}_{-2} := 0 =: \mathcal{P}_{-1}$). Hence the asserted decomposition follows by induction on n. Regarding the invariance of the summands it suffices to note that the function $r : \mathbb{R}^3 \longrightarrow \mathbb{R}$ is invariant under the action of SO(3) and that $\Delta p(g^{-1} \cdot) = 0$ if and only if $\Delta p = 0$ for all $g \in \mathrm{SO}(3)$. $\square$

Let $\Phi : \mathcal{P} \longrightarrow C_2(S^2)$ denote the linear map $\Phi(p) = p_{|S^2}$, and let $C_0(S^2)$ denote the image of Φ in $C_2(S^2)$. By Stone-Weierstrass $C_0(S^2)$ is dense in $C_2(S^2)$.

Lemma 11.5. *Φ intertwines π with L (L as defined in (8)). Moreover, if we restrict Φ to $\mathcal{P}_n$, then $\Phi_{|\mathcal{P}_n}$ intertwines π_n with the subrepresentation $L_{\Phi(\mathcal{P}_n)}$ of L, and $\Phi_{|\mathcal{P}_n} : \mathcal{P}_n \longrightarrow \Phi(\mathcal{P}_n)$ is in fact an isomorphism.*

Proof. All assertions are immediate from the definitions with the exception of the injectivity of $\Phi_{|\mathcal{P}_n} : \mathcal{P}_n \longrightarrow \Phi(\mathcal{P}_n)$. To prove injectivity let $p \in \mathcal{P}_n$ such that $p_{|S^2} \equiv 0$. Let $x \in \mathbb{R}^3$, $x \neq 0$, and write $\tilde{x} := \|x\|^{-1}x \in S^2$. Then $p(x) = \|x\|^n p(\tilde{x}) = 0$, hence $p \equiv 0$ proving injectivity. $\square$

Theorem 11.6. *We have*

$$C_0(S^2) = \bigoplus_{n \in \mathbb{N}_0} \Phi(W_n)$$

and each $\Phi(W_n)$ is a minimal SO(3)-invariant subspace of dimension $2n+1$.

Moreover, we can find an orthonormal basis $Y_{n,-n}, Y_{n,-n+1}, \ldots, Y_{n,0}, \ldots, Y_{n,n}$ of $\Phi(W_n)$ such that each $Y_{n,j}$ is an eigenfunction for the action of SO(2), that is,

$$L(k_\theta) Y_{n,j} = e^{ij\theta} Y_{n,j}$$

for all

$$k_\theta = \begin{pmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix} \in H \simeq \mathrm{SO}(2).$$

Proof. Corollary 11.4 implies that

$$C_0(S^2) = \sum_{n \in \mathbb{N}_0} \Phi(\mathcal{P}_n) = \sum_{n \in \mathbb{N}_0} \sum_{k=0}^{\lfloor n/2 \rfloor} \Phi(H_{n-2k}) = \sum_{k \in \mathbb{N}_0} \Phi(H_k).$$

We now claim that

$$\Phi(\mathcal{P}_n) = \Phi(W_n) \oplus \Phi(W_{n-2}) \oplus \ldots \oplus \Phi(W_{n-2\lfloor n/2 \rfloor}) \tag{9}$$

is a decomposition of $\Phi(\mathcal{P}_n)$ into minimal SO(3)-invariant subspaces, and that $\dim \Phi(W_{n-2k})^{\mathrm{SO}(2)} = 1$ for all k.

To prove this claim we write $V = \Phi(\mathcal{P}_n)$. Let $V = V_1 \oplus \ldots \oplus V_m$ be a decomposition of V into minimal invariant subspaces. By Remark 11.1 we have $V_j^{\mathrm{SO}(2)} \neq 0$ for all $j = 1, \ldots, m$. Hence $\lfloor n/2 \rfloor + 1 \leq m \leq \dim V^{\mathrm{SO}(2)}$. It will therefore suffice to show that in fact $\dim V^{\mathrm{SO}(2)} =$

$\lfloor n/2 \rfloor + 1$. To see this we make a change of variables: Write $u = x_1 - ix_2 \in \mathcal{P}_1$ so that $\overline{u} = x_1 + ix_2 \in \mathcal{P}_1$. The collection of monomials $\{u^\mu \overline{u}^\nu x_3^\kappa \mid \mu, \nu, \kappa \in \mathbb{N}_0,\ \mu + \nu + \kappa = n\}$ is then a basis for $\mathcal{P}_n$. Moreover, we can compute

$$L(k_\theta)u = \begin{pmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ -i \\ 0 \end{pmatrix} = \begin{pmatrix} \cos\theta - i\sin\theta \\ -\sin\theta - i\cos\theta \\ 0 \end{pmatrix} = e^{-i\theta} u$$

(where we write u in the basis x_1, x_2, x_3 for the matrix calculations), and therefore $L(k_\theta)\overline{u} = e^{i\theta}\overline{u}$. Hence $L(k_\theta)u^\mu \overline{u}^\nu x_3^\kappa = e^{(\nu-\mu)i\theta} u^\mu \overline{u}^\nu x_3^\kappa$ so that

$$L(k_\theta)u^\mu \overline{u}^\nu x_3^\kappa = u^\mu \overline{u}^\nu x_3^\kappa \quad \forall k_\theta \in \mathrm{SO}(2) \Leftrightarrow \mu = \nu. \tag{10}$$

Hence

$$\dim V = \dim \mathcal{P}_n^{\mathrm{SO}(2)} = \#\{\nu \in \mathbb{N}_0 \mid 2\nu \le n\} = \lfloor n/2 \rfloor$$

so that the decomposition in (9) must be indeed a decomposition into minimal invariant subspaces, and the space of SO(2)-invariants in each such subspace has dimension 1.

So, by the claim we can find $Y_{m,0} \in W_m$ (normalized to have length 1) such that $\Phi(W_m)^{\mathrm{SO}(2)} = \mathbb{C}Y_{m,0}$. Furthermore, viewing $\Phi(W_m)$ as an SO(2)-representation we obtain a decomposition of $\Phi(W_m)$ into one-dimensional subspaces, each invariant under SO(2). In fact we now that each irreducible complex representation of SO(2) is of the form $\chi_l : \mathrm{SO}(2) \longrightarrow \mathbb{C}^\times$, $\chi_l(k_\theta) = e^{il\theta}$ for some $l \in \mathbb{Z}$. Hence to understand the decomposition of $\Phi(W_m)$ into SO(2)-minimal subspaces it will suffice to compute the multiplicity with which each χ_l appears in $\Phi(W_m)$. We first compute its multiplicity in $\mathcal{P}_n$ using an argument analogous to (10):

$$L(k_\theta)u^\mu \overline{u}^\nu x_3^\kappa = e^{il\theta} u^\mu \overline{u}^\nu x_3^\kappa \quad \forall k_\theta \in \mathrm{SO}(2) \Leftrightarrow \mu - \nu = l$$

so that the multiplicity of χ_l in $\mathcal{P}_n$ equals

$$\begin{aligned} \#\{(\mu,\nu) \in \mathbb{N}_0 \times \mathbb{N}_0 \mid \nu - \mu = l \text{ and } \mu + \nu \le n\} &= \#\{\mu \in \mathbb{N}_0 \mid \mu \le (n-l)/2\} \\ &= \begin{cases} \lfloor \frac{n-l}{2} \rfloor + 1 & \text{if } l \le n, \\ 0 & \text{else.} \end{cases} \end{aligned}$$

Hence the multiplicity of χ_l in $\Phi(W_m)$ equals

$$\lfloor \frac{m-l}{2} \rfloor + 1 - (\lfloor \frac{m-2-l}{2} \rfloor + 1) = 1$$

for $l = -m, -m+1, \ldots, m-1, m$ (where we set $\lfloor \frac{m-2-l}{2} \rfloor + 1 = 0$ if $m - 2 - l < 0$), and we can accordingly choose normalized eigenfunctions $Y_{m,l} \in \Phi(W_m)$. Their orthogonality follows from the fact that $\Phi(W_m)$ and $\Phi(W_n)$ are orthogonal for $m \ne n$, and that $\chi_l \not\simeq \chi_k$ for $l \ne k$. $\square$

Corollary 11.7 (Fourier analysis on the sphere)**.** *For every $f \in C(S^2)$ there exist unique $a_{n,j} \in \mathbb{C}$, $j = -n, \dots, n$, $n \in \mathbb{Z}$ such that*

$$f = \sum_{n \in \mathbb{Z}} \sum_{j=-n}^{n} a_{n,j} Y_{n,j}.$$

Moreover, the coefficients equal $a_{n,j} = \int_{S^2} f(x)\overline{Y_{n,j}(x)}\, dx$. □

Remark 11.8. *The functions $Y_{n,j}$ are called* spherical harmonics*. They are in fact eigenfunctions for the spherical Laplace operator Δ_{S^2}, more precisely,*

$$\Delta_{S^2} Y_{n,j} = -n(n+1) Y_{n,j}.$$

In particular, the eigenvalues of Δ_{S^2} in $C_2(S^2)$ are exactly $-n(n+1)$, $n \in \mathbb{N}_0$, each occurring with multiplicity $2n+1$.

This spherical Laplacian Δ_{S^2} can be obtained from Δ by a change of variables: Write

$$x_1 = r \sin\theta \cos\varphi, \quad x_2 = r \sin\theta \sin\varphi, \quad x_3 = r\cos\theta.$$

Then for a smooth function $f : \mathbb{R}^3 \longrightarrow \mathbb{C}$ we have

$$\Delta f = \frac{1}{r^2 \sin\theta} \frac{\partial}{\partial r}\left(r^2 \sin\theta \frac{\partial f}{\partial r}\right) + \frac{1}{r^2} \Delta_{S^2} f$$

so that we obtain in those new variables

$$\Delta_{S^2} = \frac{1}{\sin\theta}\left(\frac{\partial}{\partial \theta}(\sin\theta \frac{\partial}{\partial\theta}) + \frac{1}{\sin\theta}\frac{\partial^2}{\partial\varphi^2}\right).$$

Remark 11.9. *One can compute the spherical harmonics $Y_{n,j}$ in terms of* Legendre polynomials*: For $n \in \mathbb{N}_0$ define*

$$P_n(x) := \frac{1}{2^n n!} \frac{d^n}{dx^n}\left[(x^2-1)^n\right]$$

and

$$P_n^j(x) := (1-x^2)^{j/2} \frac{d^j}{dx^j} P_n(x).$$

Then

$$Y_{n,j}(\theta, \varphi) = e^{ij\varphi} P_n^j(\cos\theta)$$

(with θ, φ the variables parametrizing S^2 as introduced in the previous remark).

11.2 Representations of SU(2)

We now consider the group

$$\mathrm{SU}(2) = \{g \in \mathrm{GL}_2(\mathbb{C}) \mid g\overline{g}^t = 1_2,\ \det g = 1\} = \{\begin{pmatrix} a & b \\ -\overline{b} & \overline{a} \end{pmatrix} \mid a, b \in \mathbb{C},\ |a|^2 + |b|^2 = 1\},$$

which is a compact topological group. For $n \in \mathbb{N}_0$ we define V_n to be the complex vector space consisting of homogeneous polynomials in x and y of degree n so that $\dim V_n = n + 1$. We define a representation (Φ_n, V_n) of SU(2) by

$$\Phi_n(g)p(x, y) = p((x, y)g) = p(ax + cy, bx + dy)$$

for $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{SU}(2)$.

Theorem 11.10. *Φ_n is an irreducible representation of* SU(2).

Proof. Write $T = \{\mathrm{diag}(z, z^{-1}) \mid z \in \mathbb{C}^\times,\ |z| = 1\} \subseteq \mathrm{SU}(2)$. Then for $g = \mathrm{diag}(z, z^{-1}) \in T$ and $p(x, y) = x^k y^{n-k}$ we get

$$\Phi_n(g)p(x, y) = z^{2k-n}p(x, y)$$

so that we obtain the decomposition

$$V_n = \bigoplus_{k=0}^{n} \mathbb{C}x^k y^{n-k} \tag{11}$$

into one-dimensional vector spaces which are each invariant under the representation $\mathrm{Res}_T^{\mathrm{SU}(2)} \Phi_n$ of T.

Now suppose $U \subseteq V_n$ is a Φ_n-invariant subspace, $U \neq 0$. Then U is also $\mathrm{Res}_T^{\mathrm{SU}(2)} \Phi_n$-invariant so that because of (11) we have

$$U = \bigoplus_{i=1}^{r} \mathbb{C}x^{k_i} y^{n-k_i} \tag{12}$$

for suitable $x_1, \ldots, x_r \in \{0, \ldots, n\}$. Then $r \geq 1$ since $U \neq 0$. Let $k \in \{k_1, \ldots, k_r\}$. Then for $g = \left(\begin{smallmatrix} a & b \\ -\overline{b} & \overline{a} \end{smallmatrix}\right) \in \mathrm{SU}(2)$ we get

$$\Phi_n(g)x^k y^{n-k} = (ax - \overline{b}y)^k(bx + \overline{a}y)^{n-k} = \sum_{j=0}^{n} \alpha_j x^j y^{n-j}$$

where $\alpha_n = a^k b^{n-k}$. Hence when we choose $a, b \neq 0$, then $\alpha_n \neq 0$ and therefore $x^n \in U$ because of (12). But then

$$\Phi_n(g)x^n = (ax - \overline{b}y)^n = \sum_{j=0}^{n} \beta_j x^j y^{n-j}$$

where $\beta_j = \pm \binom{n}{j} a^j \overline{b}^{n-j} \neq 0$ when $a, b \neq 0$. Hence, again by (12), $x^j y^{n-j} \in U$ for every $j = 0, \ldots, n$ so that $U = V_n$. □

Remark 11.11. *One can show (see exercises) that up to equivalence the Φ_n, $n \in \mathbb{N}_0$ are all irreducible finite dimensional complex representations of* SU(2).

11.3 Relation between representations of SO(3) and SU(2)

Our first goal in this section is to construct a continuous homomorphism of topological groups

$$\Psi : \mathrm{SU}(2) \longrightarrow \mathrm{SO}(3)$$

such that $\ker \Psi = \{\pm 1\}$.

As a first step we consider the four dimensional $\mathbb{R}$-vector space

$$\mathbb{H} = \{\begin{pmatrix} z & w \\ -\overline{z} & \overline{w} \end{pmatrix} \in M_2(\mathbb{C})\}.$$

We list some properties:

1) $\mathbb{H}$ is a four dimensional division algebra over $\mathbb{R}$ (with respect to the usual matrix addition and multiplication): It is easy to see that $\mathbb{H}$ is closed under addition and multiplication, and moreover, if $A = \left(\begin{smallmatrix} z & w \\ -\overline{z} & \overline{w} \end{smallmatrix}\right) \in \mathbb{H}$, $A \neq 0$, then

$$A^{-1} = \frac{1}{|z|^2 + |w|^2} \begin{pmatrix} \overline{z} & -w \\ \overline{w} & z \end{pmatrix} \in \mathbb{H}.$$

2) The map

$$\varphi : H \longrightarrow \mathbb{R}^4, \begin{pmatrix} z & w \\ -\overline{z} & \overline{w} \end{pmatrix} \mapsto (x, y, s, t)$$

where $z = x + iy$, $w = s + it$ is an isomorphism of $\mathbb{R}$-vector spaces.

3) The map $\mathbb{H} \times \mathbb{H} \mapsto \mathbb{R}$, $(A, B) \mapsto [A, B] := (\det(AB))^{1/2}$ defines an inner product on $\mathbb{H}$ such that $[A, A] = \|\varphi(A)\|^2$ (where $\|\cdot\|$ denotes the usual Euclidean norm on $\mathbb{R}^4$).

4) We have $\mathrm{SU}(2) = \{A \in \mathbb{H} \mid [A, A] = 1\}$. In particular, $\mathrm{SU}(2) \simeq S^3$ (the unit sphere in $\mathbb{R}^4$) via the isomorphism φ so that $\mathrm{SU}(2)$ is connected as a topological space. This isomorphism can also be used to define a measure on $\mathrm{SU}(2)$ by taking the (normalized) Lebesgue measure on S^3. It is easily seen that this measure is invariant under the action of $\mathrm{SU}(2)$, that is, it is the (normalized) left-invariant Haar measure on $\mathrm{SU}(2)$.

5) The representation $\mathrm{SU}(2) \longrightarrow \mathrm{GL}(\mathbb{H})$, $g \mapsto (X \mapsto gX)$ is orthogonal with respect to $[\cdot, \cdot]$.

Now consider the three-dimensional $\mathbb{R}$-vector space

$$E = \{\begin{pmatrix} x & s+it \\ s-it & -x \end{pmatrix} \mid x, s, t \in \mathbb{R}\} = \{iX \mid X \in \mathbb{H}, \operatorname{tr} X = 0\},$$

and define

$$\tilde{\Psi} : \mathrm{SU}(2) \longrightarrow \mathrm{GL}(E) \simeq \mathrm{GL}_3(\mathbb{R}),\ \tilde{\Psi}(g)A = gAg^{-1},$$

where the isomorphism with $\mathrm{GL}_3(\mathbb{R})$ is given by $E \ni \left(\begin{smallmatrix} x & s+it \\ s-it & -x \end{smallmatrix}\right) \mapsto (x, s, t) \in \mathbb{R}^3$.

Remark 11.12.

- *$iE \subseteq \mathbb{H}$ is the orthogonal complement of the line $\mathbb{R}1_2 \in \mathbb{H}$ with respect to $[\cdot,\cdot]$.*
- *E is a three-dimensional Euclidean vector space with inner product*

$$\langle A, B\rangle = (-\det(AB))^{1/2}.$$

Note that $\langle A, B\rangle = [iA, iB]$.

Lemma 11.13. $\operatorname{Im}\tilde{\Psi} \subseteq \mathrm{SO}(3)$. *We write $\Psi : \mathrm{SU}(2) \longrightarrow \mathrm{SO}(3)$, $\Psi(g) = \tilde{\Psi}(g)$.*

Proof. By above remarks $\tilde{\Psi}(g)$ is an orthogonal matrix for all $g \in \mathrm{SU}(2)$. Moreover, since $\tilde{\Psi}(1_2) = 1_3 \in \mathrm{SO}(3)$ and $\mathrm{SU}(2)$ is connected, we must have that $\operatorname{Im}\tilde{\Psi} \subseteq \mathrm{SO}(3)$. □

Proposition 11.14. $\Psi : \mathrm{SU}(2) \longrightarrow \mathrm{SO}(3)$ *is a continuous surjective group homomorphism with* $\ker\Psi = \{\pm 1_2\}$.

Proof. We only need to prove surjectivity and the assertion about the kernel. We start with the kernel. Let $g \in \mathrm{SU}(2)$. Then $\Psi(g) = 1_3$ if and only if $gAg^{-1} = A$ for all $A \in E$. This is the case if and only if

$$g\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}g^{-1} = \begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}, \quad \text{and} \quad g\begin{pmatrix}0 & z\\ \overline{z} & 0\end{pmatrix}g^{-1} = \begin{pmatrix}0 & z\\ \overline{z} & 0\end{pmatrix} \ \forall z \in \mathbb{C}.$$

The first property is satisfied if and only if g is a diagonal matrix, that is $g = \operatorname{diag}(w, \overline{w})$ for some $w \in \mathbb{C}^1$. The second property then becomes

$$\begin{pmatrix}0 & w^2 z\\ \overline{w^2 z} & 0\end{pmatrix} = \begin{pmatrix}0 & z\\ \overline{z} & 0\end{pmatrix} \ \forall z \in \mathbb{C}$$

which is satisfied if and only if $z = \pm 1$, that is, $g = \pm 1_2$.

The surjectivity requires some preparation:

Claim 1:

For every $A \in E$ we can find $g \in \mathrm{SU}(2)$ such that $gAg^{-1} \in \mathbb{R}_{\geq 0}\left(\begin{smallmatrix}1 & 0\\ 0 & -1\end{smallmatrix}\right)$.

To see this let $A \in E$. By the Spectral Theorem for Hermitian matrices, we now that the eigenvalues α, β of A are real and that there exists a unitary matrix B such that $BAB^{-1} = \operatorname{diag}(\alpha, \beta)$. Since $\operatorname{tr} A = 0$ we must have $\alpha = -\beta$, and we can assume $\alpha > 0$. If $\det B = 1$ we are done, if $\det B = -1$, we multiply B by $\operatorname{diag}(1, -1)$ to obtain a matrix in $\mathrm{SU}(2)$. $\square_{\text{Claim 1}}$

Claim 2:

Suppose $H \leq \mathrm{SO}(3)$ is a subgroup such that H acts transitively on S^2 (the unit sphere in $\mathbb{R}^3$), and such that there exists a line in $\mathbb{R}^3$ such that H contains all rotations around this line. Then $H = \mathrm{SO}(3)$.

To see this let $\mathbb{R}v \subseteq \mathbb{R}^3$ be the line that H stabilizes. We can assume that $v \in S^2$. Let $g \in \mathrm{SO}(3)$ so that $gv \in S^2$. By the transitivity of the action, there exists $h \in H$ such that $gv = hv$, that is, $h^{-1}gv = v$. This means that $h^{-1}g$ is a rotation around the line $\mathbb{R}v$ so that by assumption we have $h^{-1}g \in H$, and therefore $g \in H$. $\square_{\text{Claim 2}}$

We now let $H = \operatorname{Im}\Psi \leq \mathrm{SO}(3)$. Note that $S^2 \simeq \{A \in E \mid \det A = -1\}$ so that by Claim 1 H acts transitively on S^2. By Claim 2 the proof of the proposition will therefore be finished once we show the following:

Claim 3:

H contains all rotations around the line $\mathbb{R}\left(\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right) \subseteq E \simeq \mathbb{R}^3$.

To see this we compute for $t \in \mathbb{R}$, $z \in \mathbb{C}$,

$$\Psi(\begin{pmatrix} e^{it} & 0 \\ 0 & e^{-it} \end{pmatrix})\begin{pmatrix} 1 & z \\ \overline{z} & 1 \end{pmatrix} = \begin{pmatrix} 1 & e^{2it}z \\ \overline{e^{2it}z} & 1 \end{pmatrix},$$

that is, H contains all rotations around the line $\mathbb{R}\left(\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right)$ as asserted. $\square_{\text{Claim 3}}$ $\square$

Using the proposition we therefore obtain group homomorphisms

$$\mathrm{SU}(2) \longrightarrow \{\pm 1_2\}\backslash \mathrm{SU}(2) \longrightarrow \mathrm{SO}(3)$$

where the first map is just the projection $p : \mathrm{SU}(2) \longrightarrow \{\pm 1_2\}\backslash\mathrm{SU}(2)$, and the second map, which is an isomorphism of topological groups, is obtained from Ψ. We thus get a bijection

$$\{\text{irred finite dim'l complex rep'ns of SU(2) with kernel } \supseteq \{\pm 1_2\}\} \longrightarrow \{\text{irred finite dim'l complex rep'ns of SO(3)}\}$$

given by $\pi \mapsto \pi \circ p$. (Recall that for us representations of topological groups are always continuous.)

Recall the representations (Φ_n, V_n) of $\mathrm{SU}(2)$, $n \in \mathbb{N}_0$, from §11.2 By the exercises we know that $\{\Phi_n \mid n \in \mathbb{N}_0\}$ is a complete set of representatives of irreducible finite dimensional complex representations of $\mathrm{SU}(2)$.

Lemma 11.15. $\ker \Phi_n \geq \{\pm 1_2\}$ *if and only if n is even.*

Proof. We have $\Phi_n(-1_2)x^k y^{n-k} = (-1)^n x^k y^{n-k}$ so that $\Phi_n(-1_2)$ acts trivially on V_n if and only if n is even. $\square$

Remark 11.16. *One can in fact show that*

$$\ker \Phi_n = \begin{cases} \{1_2\} & n \text{ odd}, \\ \{\pm 1_2\} & n \text{ even}. \end{cases}$$

Recall the irreducible representations $(\pi_{\Phi(W_n)}, \Phi(W_n))$ of $\mathrm{SO}(3)$ on the $2n+1$-dimensional vector space $\Phi(W_n)$ from Theorem 11.6.

Corollary 11.17. *For all $n \in \mathbb{N}_0$ we have $\Phi_{2n} \circ p = \pi_{\Phi(W_n)}$. In particular, the representations $(\pi_{\Phi(W_n)}, \Phi(W_n))$, $n \in \mathbb{N}_0$, give a complete list of representatives for the equivalence classes of irreducible finite dimensional complex representations of* $\mathrm{SO}(3)$. $\square$

12 Appendix

12.1 Topological Groups

Definition 12.1. A *topological group* is a group G whose underlying set is equipped with a topology such that the two maps

$$G \times G \longrightarrow G,\ (g, h) \mapsto gh,$$

and

$$G \longrightarrow G,\ g \mapsto g^{-1}$$

are continuous with respect to this topology. (Here we equip $G \times G$ with the product topology.)

Example 12.2.

- We can obtain a topological group from any given group by equipping it with the trivial or discrete topology.

- The groups $(\mathbb{R}, +)$, $(\mathbb{R}^\times, \cdot)$, $(\mathbb{C}, +)$, and $(\mathbb{C}^\times, \cdot)$ are topological groups when equipped with the usual (Euclidean) topology.

- $\mathrm{GL}_n(\mathbb{R})$ and $\mathrm{GL}_n(\mathbb{C})$ are topological groups when we equip them with the subspace topologies coming from $M_n(\mathbb{R})$ and $M_n(\mathbb{C})$, respectively. Here we canonically identify $M_n(\mathbb{R})$ with $\mathbb{R}^{n^2}$ and $M_n(\mathbb{C})$ with $\mathbb{C}^{n^2}$ via the matrix entries and use the usual Euclidean topology on $\mathbb{R}^{n^2}$ and $\mathbb{C}^{n^2}$. To see that the multiplication is indeed a continuous map, note that if $A = (a_{ij})_{i,j=1,\ldots,n}$, $B = (b_{ij})_{i,j=1,\ldots,n} \in \mathrm{GL}_n(\mathbb{R})$ (the same argument works over $\mathbb{C}$), then the matrix entries of the product AB will be polynomials in the a_{ij} and b_{kl}, and therefore continuous maps. To see that $A \mapsto A^{-1}$ is continuous, we can use Cramer's rule to write

$$A^{-1} = \frac{1}{\det A}(p_{ij}(a_{kl}, k, l = 1, \ldots, n))_{i,j=1,\ldots,n}$$

  for suitable polynomials p_{ij}. Those polynomials are again continuous functions in the matrix entries of A, and moreover, $\det A$ is also continuous in the matrix entries, and in fact non-zero in some small neighborhood of A. Hence $A \mapsto A^{-1}$ is indeed continuous.

- If V is a finite dimensional vector space over $K = \mathbb{R}$ or $K = \mathbb{C}$, and we fix a basis of V, we obtain a group isomorphism $\mathrm{GL}(V) \longrightarrow \mathrm{GL}_n(K)$. We use this to equip $\mathrm{GL}(V)$ with a topology, and it is easily seen using the previous example that this turns $\mathrm{GL}(V)$ into a topological group. Note that if we choose a different basis, the resulting topology will be the same, so that we can consider $\mathrm{GL}(V)$ as a topological group with a canonical, well-defined topology.

Definition 12.3. The last example allows us to define the notion of a continuous representation: A representation (π, V) of a topological group G on a real or complex vector space is called *continuous* if the map $\pi : G \longrightarrow \mathrm{GL}(V)$ is a continuous map of topological spaces.

Remark 12.4. *If G is a topological group and $H \subseteq G$ a subgroup, then H is also a topological group when we equip it with the subspace topology coming from G.*

Example 12.5. The subgroups $\mathrm{SL}_n(\mathbb{R})$, $\mathrm{SO}(n)$, $\mathrm{O}(n)$ of $\mathrm{GL}_n(\mathbb{R})$ are topological groups with the subspace topology coming from $\mathrm{GL}_n(\mathbb{R})$. Similarly, the subgroups $\mathrm{SL}_n(\mathbb{C})$, $\mathrm{SU}(n)$, and $\mathrm{U}(n)$ of $\mathrm{GL}_n(\mathbb{C})$ are topological groups as well.

Example 12.6. The groups $\mathrm{O}(n)$, $\mathrm{SO}(n)$, $\mathrm{SU}(n)$, and $\mathrm{U}(n)$ are in fact *compact* topological groups.

12.2 Haar measure

Definition 12.7. A *left (resp. right) Haar measure* on a topological group G is a Radon measure μ on G such that for all $g \in G$ and all Borel sets $U \subseteq G$ we have $\mu(gU) = \mu(U)$ (resp. $\mu(Ug) = \mu(U)$).

Recall that the Borel σ-algebra of a topological space is the σ-algebra generated by all the open sets, and an element of this σ-algebra is called a Borel set.

Remark 12.8. *If μ is a left Haar measure, then $\tilde{\mu}(U) := \mu(U^{-1})$ is a right Haar measure, and vice versa. Here for a Borel set $U \subseteq G$, $U^{-1} = \{g^{-1} \mid g \in G\}$ which is again a Borel set since $g \mapsto g^{-1}$ is continuous.*

Theorem 12.9. *Suppose G is a locally compact topological group. Then there exists a left (resp. right) Haar measure on μ on G. Moreover, μ is unique up to scalara multiplication, that is, if $\tilde{\mu}$ is another left (resp. right) Haar measure, there exists $\lambda \in \mathbb{R}_{>0}$ such that $\tilde{\mu} = \lambda\mu$.*

See [Foll, Ch. 2.2] for a proof.

We will use this result only for G compact, and we again assume compactness from now on. In this case we can normalize the Haar measure to be a probability measure, that is, such that G has measure 1. The left (resp. right) Haar measure μ defines an integration on G which we will denote by

$$\int_G f(g)\,dg := \int_G f(g)\,d\mu(g)$$

for $f \in C(G)$. It satisfies the following properties:

- $C(G) \ni f \mapsto \int_G f(g)\,dg \in \mathbb{C}$ is a linear map in f.
- If $f \geq 0$, $f \neq 0$, then
$$\int_G f(g)\,dg > 0 \tag{13}$$
- For every $f \in C(G)$, $x \in G$ we have $\int_G f(xg)\,dg = \int_G f(g)\,dg$ (resp. $\int_G f(gx)\,dg = \int_G f(g)\,dg$).

The first and last property are obvious. For the second one, let $f \geq 0$, $f \neq 0$, be continuous. Then there exists $\epsilon > 0$ and open set $U \subseteq G$ such that $f \geq \epsilon$ on U. Once we show that U has positive measure, we are done. For this suppose that $\mu(U) = 0$. Then also $\mu(gU) = 0$ for every $g \in G$. The collection $\{gU \mid g \in G\}$ is an open covering of G, so since G is compact, there exist finitely many $g_1, \ldots, g_l \in G$ such that $G = g_1U \cup \ldots \cup g_lU$. But then $\mu(U) \leq \mu(g_1U) + \ldots + \mu(g_lU) = 0$ in contradiction to $\mu(G) = 1$.

Remark 12.10. *One can in fact show that every compact topological group is* unimodular*, that is, every left invariant Haar measure is also right invariant and vice versa (see [Foll, Corollary (2.28)]). Hence the integral defined by the Haar measure is in that case both left* and *right invariant.*

Example 12.11. Suppose G is a finite group. Then G is a compact topological group when equipped with the discrete topology. Define for $U \subseteq G$,

$$\mu(U) = \frac{|U|}{|G|}.$$

This is a left and right probability Haar measure on G, and integration becomes

$$\int_G f(g)\, dg = \frac{1}{|G|} \sum_{g \in G} f(g).$$

Example 12.12. The topological group $U(1) = \{A \in \mathrm{GL}_1(\mathbb{C}) \mid A\bar{A}^t = \mathbf{1}_1\} = \{z \in \mathbb{C}^\times \mid z\bar{z} = 1\}$ is just the circle group, and we have an isomorphism of topological groups given by $\mathbb{R}/\mathbb{Z} \longrightarrow U(1)$, $\theta \mapsto e^{2\pi i\theta}$. Define for $f \in C(U(1))$

$$\int_{U(1)} f(g)\, dg := \int_0^1 f(e^{2\pi i\theta})\, d\theta.$$

It can easily be seen that this indeed defines a left and right probability Haar measure on $U(1)$.

References

[Foll] G.B. Folland, *A course in Abstract Harmonic Analysis*, CRC Press, 1995.

[Hall] B.C. Hall, *Lie Groups, Lie Algebra, and Representations*, Springer, 2015.

[Stein] B. Steinberg, *Representation Theory of Finite Groups*, Springer, 2012.

[Vinb] E.B. Vinberg, *Linear Representations of Groups*, Birkhäuser, 2010.