Connections between the topological limit and categorical limit

The definitions of a filter and its related concepts are from Tai-Danae Bradley, Tyler Bryson, and John Terilla, *Topology: A Categorical Approach*.

> **Definition.** (filter) A **filter** on a set X is a collection $\mathcal{F}$ that is
> (i) downward directed: $A, B \in \mathcal{F}$ implies there exists $C \in \mathcal{F}$ such that $C \subseteq A \cap B$,
> (ii) nonempty: $\mathcal{F} \neq \varnothing$,
> (iii) upward closed: $A \in \mathcal{F}$ and $A \subseteq B$ implies $B \in \mathcal{F}$.
> An additional property is often useful:
> (iv) proper: there exists $A \subseteq X$ such that $A \notin \mathcal{F}$.

Remarks:
- Being downward directed and upward closed implies that filters are closed under finite intersections, and being proper is equivalent to the requirement that $\varnothing \notin \mathcal{F}$.
- 2^X is itself a filter but not proper.

- A set that is only downward directed and nonempty is called a **filterbase**. Any filterbase generates a filter simply by taking the upward closure of the base.

Examples.

- (trivial filter) For any set X there is the smallest filter $\{X\}$ called the trivial filter.
- (eventuality filter) Given a sequence $\{x_n\}$ in a space X, the set

$$\mathcal{E}_{x_n} = \{A \subseteq X \mid \text{ there exists an } N \text{ so that } x_n \in A \text{ for all } n \geq N\}$$

is a proper filter.
- (non-example, open neighborhoods of a point x) Given a topological space X and a point $x \in X$, the collection of open neighborhoods of x denoted by $\mathcal{T}_x$, is generally not a filter, but a filterbase.

> **Definition.** (convergence) A filter $\mathcal{F}$ on a topological space $(X, \mathcal{T})$ converges to x if and only if $\mathcal{F}$ refines $\mathcal{T}_x$, that is, if $\mathcal{T}_x \subseteq \mathcal{F}$. When $\mathcal{F}$ converges to x we will write $\mathcal{F} \to x$.

As an easy observation, a sequence $\{x_n\}$ converges to a point x if and only if $\mathcal{E}_{x_n} \to x$.

With the concept of filters, we now give an equivalence statement between the **categorical limit** and the **topological limit.**

Let X be a topological space and $\mathcal{F}_X$ be the collection of all filters on X.
Given $x \in X$ and $F \in \mathcal{F}_X$, let $\mathcal{U}_X(x)$ be the neighborhood filter of x, which is the filter generated by $\mathcal{T}_x$, and let

$$\mathcal{F}_{x,F} = \{G \in \mathcal{F}_X \mid F \cup \mathcal{U}_X(x) \subseteq G\}$$

be the collection of all the filters containing both F and the neighborhood filter of x.

We observed that $F \subseteq F \cup \mathcal{U}_X(x) \subseteq \bigcap_{G \in \mathcal{F}_{x,F}} G$, and the latter two sets are filters. Also since $\bigcap_{G \in \mathcal{F}_{x,F}} G$ is the smallest filter containing $F \cup \mathcal{U}_X(x)$ and $F \cup \mathcal{U}_X(x)$ is itself a filter, we have $\bigcap_{G \in \mathcal{F}_{x,F}} G \subseteq F \cup \mathcal{U}_X(x)$ so that $F \cup \mathcal{U}_X(x) = \bigcap_{G \in \mathcal{F}_{x,F}} G$.

View the inclusion of sets "$\subseteq$" as a partial order on $\mathcal{F}_X$ and $\mathcal{F}_{x,F}$. Then $\mathcal{F}_X$ becomes a small category and $\mathcal{F}_{x,F}$ is its subcategory. Let $E : \mathcal{F}_{x,F} \longrightarrow \mathcal{F}_X$ be the embedding functor. Now we have a theorem indicating the connection between the categorical limit and the topological limit.

> **Theorem.** x is a topological limit of the filter F if and only if F is a categorical limit of the functor E.

Proof.

$$\begin{gathered} x \text{ is a topological limit of the filter } F, \text{ (i.e., } F \to x \text{)} \\ \Leftrightarrow\ (\mathcal{T}_x \subseteq)\, \mathcal{U}_X(x) \subseteq F \Leftrightarrow F \cup \mathcal{U}(x) \subseteq F \\ \Leftrightarrow\ F = \bigcap_{G \in \mathcal{F}_{x,F}} G \text{ (the universal object)} \\ \Leftrightarrow\ \left(F \xrightarrow{p_G} E(G)\right)_{G \in \mathcal{F}_{x,F}} \text{ is a limit cone of } E \\ \text{(i.e., } F \text{ is a categorical limit of } E) \end{gathered}$$