Exercise 2 Chapter 2.

Symmetries of functions imply certain properties of functions. Let f be a 2π-periodic Riemann integrable function defined on $\mathbb{R}$.

(a) Show that the Fourier series of the function f can be written as $f(\theta) \sim \hat{f}(0) + \sum_{n\geq 1} [\hat{f}(n) + \hat{f}(-n)] \cos n\theta + i[\hat{f}(n) - \hat{f}(-n)] \sin n\theta$.

(b) Prove that if f is even, then $\hat{f}(n) = \hat{f}(-n)$ and we get a cosine series.

(c) Prove that if f is odd, then $\hat{f}(n) = -\hat{f}(-n)$ and we get a sine series

(d) Suppose that $f(\theta+\pi) = f(\theta)$ for all $\theta \in \mathbb{R}$. Show that $\hat{f}(n) = 0$ for all odd n

(e) Show that f is real-valued if and only if $\overline{\hat{f}(n)} = \hat{f}(-n)$ for all n.

Proof: We denote $a_n = \hat{f}(n) = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(\theta) e^{-in\theta} d\theta$

(a) The Fourier series of the function f is

$$f(\theta) \sim \sum_{n=-\infty}^{\infty} a_n e^{in\theta} = a_0 + \sum_{n=1}^{\infty} a_n e^{in\theta} + a_{-n} e^{-in\theta}$$

$$= a_0 + \sum_{n=1}^{\infty} a_n(\cos n\theta + i \sin n\theta) + a_{-n}(\cos n\theta - i \sin n\theta)$$

$$= a_0 + \sum_{n=1}^{\infty} (a_n + a_{-n}) \cos n\theta + i(a_n - a_{-n}) \sin n\theta$$

(b) If f is even, $f(\theta) = f(-\theta)$

$$a_{-n} = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(\theta) e^{-i(-n)\theta} d\theta$$

$$= \frac{1}{2\pi}\int_{-\pi}^{\pi} f(\theta) e^{-in(-\theta)} d\theta$$

Let $-\theta = t$, $\theta = -t$ $d\theta = -dt$.

$$= \frac{1}{2\pi}\int_{-t=-\pi}^{t=\pi} f(-t) e^{-int} d(-t)$$

$$= \frac{1}{2\pi}\int_{-\pi}^{\pi} f(t) e^{-int} dt.$$

$$= \frac{1}{2\pi}\int_{-\pi}^{\pi} f(\theta) e^{-in\theta} d\theta$$

$$= a_n.$$

(c) Similarly, if f is odd $f(\theta) = -f(-\theta)$

$$a_{-n} = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(\theta) e^{-i(-n)\theta} d\theta$$

$$= \frac{1}{2\pi}\int_{-\pi}^{\pi} f(\theta) e^{-in(-\theta)} d\theta$$

$$= \frac{1}{2\pi}\int_{-t=-\pi}^{-t=\pi} f(-t) e^{-int} d(-t)$$

$$= \frac{1}{2\pi}\int_{\pi}^{-\pi} f(t) e^{-int} dt$$

$$= -\frac{1}{2\pi}\int_{-\pi}^{\pi} f(\theta) e^{-in\theta} d\theta$$

$$= -a_n.$$

(d) $a_n = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(\theta) e^{-in\theta} d\theta$ If $f(\theta+\pi) = f(\theta)$. for all odd n,

$$a_n = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(\theta+\pi) e^{-in(\theta+\pi)} \cdot e^{in\pi} d\theta = -\frac{1}{2\pi}\int_{-\pi}^{\pi} f(\theta) e^{-in\theta} d\theta = -a_n$$

$a_n + a_n = 2a_n = 0$. Thus $a_n = 0$.

(e) The Fourier coefficients of f is

$$a_n = \hat{f}(n) = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(\theta) e^{-in\theta}\, d\theta.$$

$$a_{-n} = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(\theta) e^{in\theta}\, d\theta$$

If f is real valued, then

$$\bar{a}_n = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(\theta) e^{in\theta}\, d\theta = a_{-n}$$

If f is not real valued, $f = u + iv$ $v \neq 0$

$$a_n = \frac{1}{2\pi}\left(\int_{-\pi}^{\pi} u(\theta) e^{-in\theta}\, d\theta + i\int_{-\pi}^{\pi} v(\theta) e^{-in\theta}\, d\theta\right)$$

$$a_n = \frac{1}{2\pi}\left(\int_{-\pi}^{\pi} u(\theta)\, e^{in\theta}\, d\theta + i\int_{-\pi}^{\pi} v(\theta) e^{in\theta}\, d\theta\right)$$

$$\bar{a}_n = \frac{1}{2\pi}\left(\int_{-\pi}^{\pi} u(\theta) e^{in\theta}\, d\theta - i\int_{-\pi}^{\pi} v(\theta) e^{in\theta}\, d\theta\right)$$

They are not equal.

Therefore, f is real-valued if and only if

$$\bar{a}_n = a_{-n}$$

Exercise 6 Chapter 2

Let f be the function defined on $[-\pi, \pi]$ by $f(\theta) = |\theta|$

(a) Draw the graph of f.

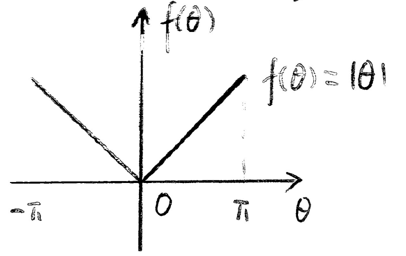
(b) Calculate the Fourier coefficients of f, and show that

$$\hat{f}(n) = \begin{cases} \frac{\pi}{2} & \text{if } n = 0 \\ \frac{-1+(-1)^n}{\pi n^2} & \text{if } n \neq 0. \end{cases}$$

(c) What is the Fourier series of f in terms of sines and cosines.

(d) Taking $\theta = 0$, prove that

$$\sum_{n \text{ odd} \geq 1} \frac{1}{n^2} = \frac{\pi^2}{8} \quad \text{and} \quad \sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}.$$



(b). $$\hat{f}(0) = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(\theta)\, d\theta = \frac{1}{2\pi}\left(\int_{-\pi}^{0} -\theta\, d\theta + \int_{0}^{\pi} \theta\, d\theta\right) = \frac{1}{2\pi}\left(-\frac{1}{2}\theta^2\Big|_{-\pi}^{0} + \frac{1}{2}\theta^2\Big|_{0}^{\pi}\right)$$
$$= \frac{\pi}{2}$$

$n \neq 0$

$$\hat{f}(n) = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(\theta) e^{-in\theta}\, d\theta = \frac{1}{2\pi}\left(\int_{-\pi}^{0} -\theta e^{-in\theta}\, d\theta + \int_{0}^{\pi} \theta e^{-in\theta}\, d\theta\right)$$

$$\int \theta e^{-in\theta}\, d\theta = \frac{1}{-in} \theta e^{-in\theta} - \int \frac{1}{-in} e^{-in\theta}\, d\theta = \frac{1}{-in}\theta e^{-in\theta} - \frac{1}{(-in)^2} e^{-in\theta} + C$$
$$= \frac{1}{-in}\theta e^{-in\theta} + \frac{1}{n^2} e^{-in\theta}$$
$$= \frac{e^{-in\theta}(1 + in\theta)}{n^2} + C.$$

$$\int_0^{\pi} \theta e^{-in\theta}\, d\theta = \frac{e^{-in\pi}(1+in\pi)-1}{n^2}$$

$$\int_{-\pi}^{0} \theta e^{-in\theta}\, d\theta = \frac{1-e^{in\pi}(1-in\pi)}{n^2}$$

$$\left(\int_0^{\pi} - \int_{-\pi}^{0}\right)\theta e^{in\theta}\, d\theta = \frac{2}{n^2}(\cos(\pi n)-1)$$

$$n \neq 0, \quad \hat{f}(n) = \frac{-1+(-1)^n}{\pi n^2}$$

(c) The Fourier series of f in the exponential form is

$$f(\theta) \sim \sum_{n=\cdots -2,-1,1,2,\cdots} \frac{-1+(-1)^n}{\pi n^2} e^{in\theta} + \frac{\pi}{2}$$

$$= \sum_{n\in\mathbb{Z}\setminus\{0\}} \frac{-1+(-1)^n}{\pi n^2} (\cos n\theta + i \sin n\theta) + \frac{\pi}{2}$$

$$= \frac{\pi}{2} + \sum_{n\in\mathbb{Z}\setminus\{0\}} \frac{-1+(-1)^n}{\pi n^2} \cos n\theta + i\left(\frac{-1+(-1)^n}{\pi n^2} \sin n\theta\right)$$

$$= \frac{\pi}{2} + \sum_{n\in\mathbb{Z}\setminus\{0\}} \frac{-1+(-1)^n}{\pi n^2} \left(\cos n\theta + i \cos\left(n\theta - \frac{\pi}{2}\right)\right)$$

(cosine form)

$$= \frac{\pi}{2} + \sum_{n\in\mathbb{Z}\setminus\{0\}} \frac{-1+(-1)^n}{\pi n^2} \left(\sin\left(n\theta + \frac{\pi}{2}\right) + i \sin n\theta\right)$$

(sine form)

(d) Taking $\theta = 0$ we obtain (assuming convergent)

$$0 = \frac{\pi}{2} + \sum_{n=-\infty}^{-1} \frac{-1+(-1)^n}{\pi n^2} + \sum_{n=1}^{\infty} \frac{-1+(-1)^n}{\pi n^2}$$

when n is even, the terms vanish

then
$$2\sum_{n \text{ odd} \geq 1} \frac{-2}{n^2\pi} + \frac{\pi}{2} = 0$$

$$\frac{4}{\pi} \sum_{n \text{ odd} \geq 1} \frac{1}{n^2} = \frac{\pi}{2} \quad \Rightarrow \quad \sum_{n \text{ odd} \geq 1} \frac{1}{n^2} = \frac{\pi^2}{8}$$

when n is even,

$$\sum_{n \text{ even} \geq 1} \frac{1}{n^2} = \frac{1}{4} \sum_{n=1}^{\infty} \frac{1}{n^2}.$$

and
$$\sum_{n \text{ even} \geq 1} \frac{1}{n^2} + \sum_{n \text{ odd} \geq 1} \frac{1}{n^2} = \sum_{n=1}^{\infty} \frac{1}{n^2}$$

$$\Rightarrow \quad \frac{1}{4} \sum_{n=1}^{\infty} \frac{1}{n^2} + \frac{\pi^2}{8} = \sum_{n=1}^{\infty} \frac{1}{n^2}.$$

$$\Rightarrow \quad \sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}$$

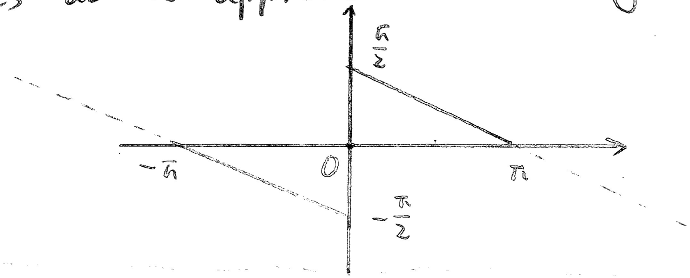
Remark: We may use the Fourier series to calculate some infinite numerical series, but the argument is not strict until we obtain the convergence theorems.

Exercise 8 Chapter 2.

Verify that $\frac{1}{2i}\sum_{n\neq 0}\frac{e^{inx}}{n}$ is the Fourier series of the 2π-periodic sawtooth function illustrated in the Figure, defined by $f(0)=0$ and

$$f(x) = \begin{cases} -\frac{\pi}{2}-\frac{x}{2} & \text{if } -\pi < x < 0 \\ \frac{\pi}{2}-\frac{x}{2} & \text{if } 0 < x < \pi \end{cases}$$

Note that this function is not continuous. Show that nevertheless, the series converges for every x (by which we mean, as usual, that the symmetric partial sums of the series converge) In particular, the value of the series at the origin, namely 0, is the average of the values as x approaches the origin from the left and the right.



Proof: The Fourier coefficients

$$a_n = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)e^{-inx}dx \qquad (n\neq 0)$$

$$= \frac{1}{2\pi}\int_{-\pi}^{0}\left(\frac{\pi}{2}-\frac{x}{2}\right)e^{-inx}dx + \int_0^{\pi}\left(\frac{\pi}{2}-\frac{x}{2}\right)e^{-inx}dx$$

$$= \frac{1}{2\pi}\left[\int_{-\pi}^{\pi} -\frac{x}{2}e^{-inx}dx + \left(\int_{-\pi}^{0} -\frac{\pi}{2}e^{-inx}dx + \int_0^{\pi} \frac{\pi}{2}e^{-inx}dx\right)\right]$$

$$= \frac{1}{2ni} \qquad \text{Since } f(x) \text{ is odd, } a_0 = 0$$

Therefore

$$\frac{1}{2i}\sum_{n\neq 0}\frac{e^{inx}}{n} \text{ is the Fourier series of } f.$$

Now we verify the convergence of $\frac{1}{2i}\sum_{n\neq 0}\frac{e^{inx}}{n}$ using Dirichlet's test. First, the partial sum of $\sum_{n\neq 0} e^{inx}$ is bounded

Since $\sum_{n=1}^{N} e^{inx}+e^{-inx} = \sum_{n=-N}^{N} e^{-inx} - 1. = \frac{\sin(N+\frac{1}{2})x}{\sin\frac{x}{2}} - 1$ (Similar to the calculation of Dirichlet kernels)

then $\left|\sum_{n=1}^{N} e^{inx}+e^{-inx}\right| \leq \frac{1}{|\sin\frac{x}{2}|}+1$ bounded

Second, $\{\frac{1}{n}\}$ is a sequence of real numbers that decreases monotonically to 0. Therefore $\frac{1}{2i}\sum_{n\neq 0} e^{inx}$ converges for every x. In particular $\frac{1}{2i}\sum_{n\neq 0}\frac{e^{in\cdot 0}}{n} = \frac{1}{2i}\sum_{n\neq 0}\frac{1}{n} = 0$ is the average of $\frac{\pi}{2}$ and $-\frac{\pi}{2}$

Exercise 10

Chapter 2.

Suppose f is a periodic function of period 2π which belongs to the class C^k. Show that $\hat{f}(n) = O(1/|n|^k)$ as $|n|\to\infty$. This notation means that there exists a constant C such that $|\hat{f}(n)| \leq C/|n|^k$. We could also write this as $|n|^k\hat{f}(n) = O(1)$, where $O(1)$ means bounded.

Proof: $\hat{f}(n) = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(\theta)e^{-in\theta}\,d\theta$

$$= \frac{1}{2\pi}\left[\frac{1}{-in} f(\theta)e^{-in\theta}\Big|_{-\pi}^{\pi} + \frac{1}{in}\int_{-\pi}^{\pi} f'(\theta)\,e^{-in\theta}d\theta\right]$$

The first term is 0 because

$$-\frac{1}{in} f(\theta)e^{-in\theta}\Big|_{-\pi}^{\pi} = -\frac{1}{in}\left[f(\pi)e^{-in\pi} - f(-\pi)e^{in\pi}\right] = 0$$

by the periodicity of $f(\theta)e^{-in\theta}$.

Therefore $\hat{f}(n) = \frac{1}{2\pi}\,\frac{1}{in}\int_{-\pi}^{\pi} f'(\theta)\,e^{-in\theta}d\theta$

or $2\pi\hat{f}(n) = \frac{1}{in}\int_{-\pi}^{\pi} f'(\theta)\,e^{-in\theta}d\theta$. Since f belongs to the class C^k, by induction, suppose $2\pi\hat{f}(n) = \frac{1}{(in)^m}\int_{-\pi}^{\pi} f^{(m)}(\theta)e^{-in\theta}d\theta$ $\quad 0\leq m\leq k-1$, then $2\pi\hat{f}(n) = -\frac{1}{in}f^{(m)}(\theta)e^{-in\theta}\Big|_{-\pi}^{\pi} + \frac{1}{(in)^{m+1}}\int_{-\pi}^{\pi} f^{(m+1)}(\theta)\,d\theta$, and $\left|\int_{-\pi}^{\pi} f^{(m+1)}(\theta)e^{-in\theta}d\theta\right| \leq \int_{-\pi}^{\pi}|f^{(m+1)}(\theta)|d\theta < C$ where C is independent of n and we conclude that $\hat{f}(n)$.

Exercise 2

Prove that $\boxed{\text{the vector space } l^2(\mathbb{Z}) \text{ is complete}}$

Proof: Suppose $A_K = (\cdots a_{K,-n}, \cdots a_{K,-1}, a_{K,0}, a_{K,1}, \cdots, a_{K,n}, \cdots)$

For any $\varepsilon > 0$, there exists an integer $N_\varepsilon \in \mathbb{N}^*$ such that

$$\|A_k - A_l\| = \left(\sum_{n\in\mathbb{Z}} |a_{k,n} - a_{l,n}|^2\right)^{1/2} < \varepsilon/\sqrt{2} \text{ whenever } k > l \geq N$$

$\sum_{n\in\mathbb{Z}} |a_{k,n} - a_{l,n}|^2 < \varepsilon^2/2$ Then for each n,

$$|a_{k,n} - a_{l,n}|^2 \leq \sum_{n\in\mathbb{Z}} |a_{k,n} - a_{l,n}|^2 < \varepsilon^2/2$$

and $$|a_{k,n} - a_{l,n}| < \varepsilon/\sqrt{2} < \varepsilon$$

Thus, $\{a_{k,n}\}_{k=1}^{\infty}$ is a Cauchy sequence, which converges to a limit, say, b_n. Let $B = (\cdots, b_{-n}, \cdots, b_{-1}, b_0, b_1, \cdots, b_n, \cdots)$

$$\sum_{n=-M}^{M} |a_{k,n} - a_{l,n}|^2 \leq \|A_k - A_l\|^2 < \varepsilon^2/2 \text{ for all } M \in \mathbb{N}^*$$

For each $k > N_\varepsilon$ and M, let $l \to \infty$:

$$\sum_{n=-M}^{M} |a_{k,n} - b_n|^2 \leq \varepsilon^2/2 < \varepsilon^2.$$

Since M is arbitrary, one has shown that

$$\|A_k - B\| < \varepsilon. \quad \text{as } k > N_\varepsilon$$

Finally we prove $B \in l^2(\mathbb{Z})$.

$$\|B\| \leq \|A_k - B\| + \|A_k\| < \infty \quad \text{for } k > N_\varepsilon.$$

The proof is completed.

注意：不能通过说明各分量收敛来得出$\{A_k\}$有极限．例如考虑$\{e_k\}_{k=1}^{\infty}$，$e_k = (\cdots 0, \cdots, 1, 0, \cdots 0, \cdots 1, 0, \cdots)$ 第k和第$-k$分量为1其余为0．$\{e_k\}_{k=1}^{\infty}$的极限不存在并且$\|e_k - e_l\| = 4$ $k \neq l$，不是Cauchy序列