NOTES on Fourier Analysis an introduction (Stein)

Fourier Series

If f is an integrable function on $[a,b]$ of length L. $(b-a=L)$, then the n-th Fourier coefficient of f is defined by

$$\hat{f}(n) = \frac{1}{L}\int_a^b f(x)\, e^{-2\pi i n x/L}\, dx \qquad n \in \mathbb{Z}$$

The Fourier series is given formally by

$$\sum_{n=-\infty}^{\infty} \hat{f}(n)\, e^{2\pi i n x/L}$$

The N-th partial sum of f is given by

$$S_N(f)(x) = \sum_{n=-N}^{N} \hat{f}(n)\, e^{2\pi i n x/L}.$$

The discussions below are about functions on a circle, that is functions defined 2π-periodically

Problem: In what sense does $S_N(f)$ converges to f as $N \to \infty$?

Kernels.

① N-th Dirichlet kernel

$$D_N(x) = \sum_{n=-N}^{N} e^{inx}, \qquad x \in [-\pi, \pi]$$

A second formula for the Dirichlet kernel is

$$D_N(x) = \frac{\sin((N+\frac{1}{2})x)}{\sin(x/2)}$$

Proof of the equivalence: Let $w = e^{ix}$

$$D_N(x) = \sum_{n=-N}^{N} w^n = \sum_{n=0}^{N} w^n + \sum_{n=-N}^{-1} w^n$$

$$= \frac{1-w^{N+1}}{1-w} + \frac{w^{-N}-1}{1-w}$$

$$= \frac{w^{-N}-w^{N+1}}{1-w}$$

Multiply $\omega^{-\frac{1}{2}}$ on the numerator and denominator,

$$D_N(x) = \frac{\omega^{-N-\frac{1}{2}} - \omega^{N+\frac{1}{2}}}{\omega^{-\frac{1}{2}} - \omega^{\frac{1}{2}}} = \frac{\sin((N+\frac{1}{2})x)}{\sin(x/2)}$$

The last equality is directly from Euler's formula

② Poisson kernel

$$P_r(\theta) = \sum_{n=-\infty}^{\infty} r^{|n|} e^{in\theta} \qquad \theta \in [-\pi, \pi], \; 0 < r < 1$$

A second formula for Poisson kernel is

$$P_r(\theta) = \frac{1-r^2}{1-2r\cos\theta + r^2}$$

Proof of the equivalence: Let $\omega = re^{i\theta}$

$$P_r(\theta) = \sum_{n=0}^{\infty} \omega^n + \sum_{n=1}^{\infty} \bar{\omega}^n$$

$$= \frac{1}{1-\omega} + \frac{\bar{\omega}}{1-\bar{\omega}}$$

$$= \frac{1-|\omega|^2}{|1-\omega|^2}$$

$$= \frac{1-r^2}{1-2r\cos\theta + r^2}$$

③ Nth Fejér kernel

$$F_N(x) = \frac{D_0(x) + \cdots + D_{N-1}(x)}{N} = \frac{1}{N}\sum_{n=0}^{N-1} D_n(x)$$

The equivalt form of the Fejér kernel is

$$F_N(x) = \frac{1}{N}\frac{\sin^2(Nx/2)}{\sin^2(x/2)} \qquad \text{(Lemma 5.1, Exercise 15)}$$

Abel means and summartion. Poisson kernel.

A series of complex numbers $\sum_{k=0}^{\infty} C_k$ is said to be Abel summable to s if for every $0 \le r < 1$, the series $A(r) = \sum_{k=0}^{\infty} C_k r^k$ converges, and $\lim_{r\to 1} A(r) = s$. The quantities $A(r)$ are called Abel means of the series.

Define the Abel means of the function $f(\theta) \sim \sum_{n=-\infty}^{\infty} a_n e^{in\theta}$ by

$$A_r(f)(\theta) = \sum_{n=-\infty}^{\infty} r^{|n|} a_n e^{in\theta}$$

Since f is integrable, $|a_n|$ is uniformly bounded in n, so that $A_r(f)$ converges absolutely and uniformly for each $0 \le r < 1$.

The Abel means can be written as convolutions.

$$A_r(f)(\theta) = (f * P_r)(\theta)$$

where $P_r(\theta)$ is the Poisson kernel given by

$$P_r(\theta) = \sum_{n=-\infty}^{\infty} r^{|n|} a_n e^{in\theta}$$

Verification: In fact

$$A_r f(\theta) = \sum_{n=-\infty}^{\infty} r^{|n|} a_n e^{in\theta}$$

$$= \sum_{n=-\infty}^{\infty} r^{|n|} \left(\frac{1}{2\pi} \int_{-\pi}^{\pi} f(\varphi) e^{-in\varphi} d\varphi \right) e^{in\theta}$$

$$= \frac{1}{2\pi} \int_{-\pi}^{\pi} f(\varphi) \left(\sum_{n=-\infty}^{\infty} r^{|n|} e^{-in(\varphi-\theta)} \right) d\varphi$$

where the interchange of the integral and infinite sum is justified by the uniform convergence of the series.

Lemma If $0 \leq r < 1$, then

$$P_r(\theta) = \frac{1-r^2}{1-2r\cos\theta + r^2}$$

The Poisson kernel is a good kernel, as r tends to 1 from below.

Proof: $P_r(\theta) = \sum_{n=-\infty}^{\infty} r^{|n|} e^{in\theta}$

$$= \sum_{n=0}^{\infty} \omega^n + \sum_{n=1}^{\infty} \bar{\omega}^n, \text{ where } \omega = re^{i\theta}$$

$$= \frac{1}{1-\omega} + \frac{\bar{\omega}}{1-\bar{\omega}}$$

$$= \frac{1-\bar{\omega} + (1-\omega)\bar{\omega}}{(1-\omega)(1-\bar{\omega})}$$

$$= \frac{1-|\omega|^2}{|1-\omega|^2} = \frac{1-r^2}{1-2r\cos\theta + r^2}$$

Note that $1-2r\cos\theta + r^2 = (1-r)^2 + 2r(1-\cos\theta)$

If $\frac{1}{2} \leq r \leq 1$, and $\delta \leq |\theta| \leq \pi$, then

$$1-2r\cos\theta + r^2 \geq C_\delta > 0$$

Thus $P_r(\theta) \leq (1-r^2)/C_\delta$ when $\delta \leq |\theta| \leq \pi$ and the third property of good kernels is verified.

Clearly $P_r(\theta) \geq 0$. Since $1-2r\cos\theta + r^2 \geq 1-2r+r^2 = (1-r)^2 \geq 0$.

By integrating $P_r(\theta) = \sum_{n=-\infty}^{\infty} r^{|n|} e^{in\theta}$ term by term (which is justified by the absolute convergence of the series) yields

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} P_r(\theta)\, d\theta = 1.$$

thereby concluding the proof that $P_r(\theta)$ is a good kernel. □

Theorem. The Fourier series of an integrable function on the circle is Abel summable to f at every point of continuity. Moreover, if f is continuous on the circle, then the Fourier series of f is uniformly Abel summable to f.

Properties of Fourier series

Let f, g be 2π-periodic integrable functions defined on $\mathbb{R}$.

If the Fourier coefficients of $f(\theta)$ are denoted by a_n,

we use the notation $f(\theta) \overset{FS}{\longleftrightarrow} a_n$, where $a_n = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(\theta) e^{-in\theta} d\theta$

Property 1 (Linearity)

If $f(\theta) \overset{FS}{\longleftrightarrow} a_n$, $g(\theta) \overset{FS}{\longleftrightarrow} b_n$,

Then $Af(\theta) + Bg(\theta) \overset{FS}{\longleftrightarrow} Aa_n + Bb_n$, for $A, B \in \mathbb{C}$.

Proof: This is directly checked by definition

$$a_n = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(\theta) e^{-in\theta} d\theta$$

$$b_n = \frac{1}{2\pi}\int_{-\pi}^{\pi} g(\theta) e^{-in\theta} d\theta$$

Let $Af(\theta) + Bg(\theta) \overset{FS}{\longleftrightarrow} C_n$, then

$$C_n = \frac{1}{2\pi}\int_{-\pi}^{\pi} (Af(\theta) + Bg(\theta)) e^{-in\theta} d\theta = Aa_n + Bb_n$$

by the linearity of Riemann integral.

Property 2 (Shifting property)

$$f(\theta-\theta_0) \overset{FS}{\longleftrightarrow} e^{-in\theta_0} a_n$$

Proof Let $f(\theta-\theta_0) \overset{FS}{\longleftrightarrow} C_n$ then

$$C_n = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(\theta-\theta_0) e^{-in\theta} d\theta$$

$$= \frac{1}{2\pi}\int_{-\pi}^{\pi} f(\theta-\theta_0) e^{-in(\theta-\theta_0)} e^{-in\theta_0} d\theta$$

$$= e^{-in\theta_0} \cdot \frac{1}{2\pi}\int_{-\pi}^{\pi} f(\theta-\theta_0) e^{-in(\theta-\theta_0)} d\theta$$

By periodicity $= e^{-in\theta_0} \frac{1}{2\pi}\int_{-\pi}^{\pi} f(\theta) e^{-in\theta} d\theta$

$$= e^{-in\theta_0} a_n$$

Property 3 (Reversal)

$$f(-\theta) \overset{FS}{\longleftrightarrow} a_{-n}$$

Proof Let $f(-\theta) \overset{FS}{\longleftrightarrow} c_n$ then

$$c_n = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(-\theta)\, e^{-in\theta}\, d\theta \quad \text{Let } t=-\theta$$

$$c_n = \frac{1}{2\pi}\int_{t=\pi}^{t=-\pi} f(t)\, e^{int}\, d(-t)$$

$$= \frac{1}{2\pi}\int_{-\pi}^{\pi} f(t)\, e^{-i(-n)t}\, dt$$

$$= a_{-n}.$$

Remark If $f(\theta)$ is an even function, i.e. $f(\theta)=f(-\theta)$

$$a_{-n} = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(\theta)\, e^{-in\theta}\, d\theta = a_n, \quad a_n \text{ is real in } n.$$

If $f(\theta)$ is an odd function, i.e. $f(-\theta)=-f(\theta)$

$$a_{-n} = \frac{1}{2\pi}\int_{-\pi}^{\pi} -f(\theta)\, e^{-in\theta}\, d\theta = -a_n, \quad a_n \text{ is odd in } n.$$

$$f(\theta) \sim a_0 + \sum_{n=1}^{\infty} [a_n + a_{-n}]\cos n\theta + i[a_n - a_{-n}]\sin n\theta$$

Property 4. (Conjugation)

$$\overline{f(\theta)} \overset{FS}{\longleftrightarrow} \bar{a}_{-n}$$

If $f(\theta)$ is even, then the Fourier series is a cosine series

If $f(\theta)$ is odd, the the Fourier series is a sine series

Proof Let $\overline{f(\theta)} \overset{FS}{\longleftrightarrow} c_n$, then

$$c_n = \frac{1}{2\pi}\int_{-\pi}^{\pi} \overline{f(\theta)}\, e^{-in\theta}\, d\theta = \frac{1}{2\pi}\int_{-\pi}^{\pi} \overline{f(\theta)\, e^{in\theta}}\, d\theta$$

$$= \overline{\frac{1}{2\pi}\int_{-\pi}^{\pi} f(\theta)\, e^{-i(-n)\theta}\, d\theta}$$

$$= \bar{a}_{-n}$$

Remark: If $f(\theta)$ is real, i.e. $\overline{f(\theta)} = f(\theta)$.

then $\underline{\bar{a}_{-n}} = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(\theta)\, e^{-in\theta}\, d\theta = \underline{a_n}$, moreover, $|a_{-n}| = |a_n|$.

If f is real and even, then we have both of the properties
$$\begin{cases} a_{-n} = a_n \\ \bar{a}_{-n} = a_n \end{cases} \Rightarrow a_n = \bar{a}_n \text{ and } a_n = a_{-n}$$
The Fourier coefficients of f are real and even

If f is real and odd, then we have both of the properties:
$$\begin{cases} a_{-n} = -a_n \\ \bar{a}_{-n} = a_n \end{cases} \Rightarrow \bar{a}_n = -a_n \text{ and } a_{-n} = -a_n$$
The Fourier coefficients of f are purely imaginary and odd.

Property 5. (Differentiation)
$$\frac{df(\theta)}{d\theta} \overset{FS}{\longleftrightarrow} in\, a_n$$
Proof: Let $\frac{df(\theta)}{d\theta} \overset{FS}{\longleftrightarrow} C_n$
$$C_n = \frac{1}{2\pi}\int_{-\pi}^{\pi} f'(\theta) e^{-in\theta} d\theta$$
$$= \frac{1}{2\pi}\left(f(\theta) e^{-in\theta}\Big|_{-\pi}^{\pi} + in\int_{-\pi}^{\pi} f(\theta) e^{in\theta}\right)$$
$$= in \cdot \frac{1}{2\pi}\int_{-\pi}^{\pi} f(\theta) e^{-in\theta} d\theta$$
$$= in\, a_n$$
Property 6 (Convolution)
$$(f * g)(\theta) \overset{FS}{\longleftrightarrow} a_n b_n$$
This is proved in Proposition 3.1 (vi) of Chapter 2.

Good kernels and approximation to Identity.

A family of kernels $\{K_n(x)\}_{n=1}^{\infty}$ on the circle is said to be a family of good kernels if it satisfies the following properties:

(a) For all $n \geq 1$,

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} K_n(x)\,dx = 1.$$

(b) There exists $M > 0$ such that for all $n \geq 1$,

$$\int_{-\pi}^{\pi} |K_n(x)|\,dx \leq M$$

(c) For every $\delta > 0$

$$\int_{\delta \leq |x| \leq \pi} |K_n(x)|\,dx \to 0 \quad \text{as} \quad n \to \infty.$$

Theorem Let $\{K_n\}_{n=1}^{\infty}$ be a family of good kernel, and f an ineegrable function on the circle. Then

$$\lim_{n\to\infty} (f * K_n)(x) = f(x)$$

whenever f is continuous at x. If f is continuous everywhere, the the above limit is uniform.

The good kernel is sometimes refered to as an approximation to the identity.

Example of good kernels: Fejér kernel, Poisson kernel

Theorem. Suppose that f is an integrable function on the circle with $\hat{f}(n) = 0$ for all $n \in \mathbb{Z}$. Then $f(\theta_0) = 0$ whenever f is continuous at the point θ_0.

Corollary 5.4. Continuous functions on the circle can be uniformly approximated by trigonometric polynomials.

Proof. By Theorem 4.1, since Fejér kernels $\{F_n(x)\}_{n=1}^{\infty}$ is a family of good kernels. Then $\lim_{N\to\infty}(f * F_N)(x) = f(x)$ uniformly when f is a continuous function.

The N-th Cesàro mean of the Fourier series is

$$\sigma_N(f)(x) = \frac{S_0(f)(x) + \cdots + S_{N-1}(f)(x)}{N}$$

where $S_n(f)(x) = \sum_{k=-n}^{n} a_k e^{ikx}$ $n = 0, \ldots, N-1$ is a trigonometric polynomial

and we have $\sigma_N(f)(x) = (f * F_N)(x)$

Therefore given any $\varepsilon > 0$, there exists a sufficiently large N such that $|\sigma_N(f)(x) - f(x)| < \varepsilon$. for all $-\pi \le x \le \pi$

Corollary 5.3. If f is integrable on the circle and $\hat{f}(n)$ for all n, then $f = 0$ at all point of continuity of f.

Proof: Recall Theorem 4.1. $f(x) = \lim_{n\to\infty} (f * F_n)(x)$ where $F_n(x)$ is the Fejér kernel. $\hat{f}(n) = 0$ implies that $(f * F_n)(x) = 0$ for all $n \in \mathbb{N}^*$. Hence $f(x) = 0$ at all the point of continuity of f.

Theorem 5.7. Let f be an integrable function defined on the unit circle. Then the function u defined in the unit disc by the Poisson integral

$$u(r,\theta) = (f * P_r)(\theta)$$

has the following properties

(i) u has two continuous derivatives in the unit disc and satisfies $\Delta u = 0$.

(ii) If θ is any point of continuity of f, then

$$\lim_{r \to 1^-} u(r,\theta) = f(\theta)$$

If f is continuous everywhere, then this limit is uniform

(iii) If f is continuous, then $u(r,\theta)$ is the unique solution to the steady-state heat equation in the disc which satisfies conditions (i) and (ii).