Remark of Theorem 1.1.

1. Integrability (mean square convergence) does not guarantee that the Fourier series converges for any θ. (Exercise 3)

2. Differentiability at θ_0 guarantees that the Fourier series converges at θ_0. (Theorem 2.1)

3. Continuous functions may have diverging Fourier series at one point.

Theorem Let f be an integrable function on the circle which is differentiable at a point θ_0. Then $S_N(f)(\theta_0) \to f(\theta_0)$ as N tends to infinity.

Proposition Let f be a bounded function on the compact interval $[a,b]$. If $c \in (a,b)$ and if for all $\delta > 0$, the function f is integrable on the intervals $[a, c-\delta]$ and $[c+\delta, b]$, then f is integrable on $[a,b]$.

Proof: Suppose $|f| \le M$ and let $\varepsilon > 0$. Choose a (small) $\delta > 0$ so that $4\delta M \le \varepsilon/3$. Now let P_1, P_2 be partitions of $[a, c-\delta]$ and $[c+\delta, b]$, so that for each $i = 1, 2$, we have

$$\mathcal{U}(P_i, f) - \mathcal{L}(P_i, f) < \varepsilon/3.$$

This is possible since f is integrable on each one of the intervals. Then by taking as a partition $P = P_1 \cup \{c-\delta\} \cup \{c+\delta\} \cup P_2$ we immediately see that $\mathcal{U}(P, f) - \mathcal{L}(P, f) < \varepsilon$.

Proof of the Theorem.

Define $F(t) = \begin{cases} \dfrac{f(\theta_0 - t) - f(\theta_0)}{t} & \text{if } t \neq 0,\ |t| < \pi \\ -f'(\theta_0) & \text{if } t = 0. \end{cases}$

First, F is bounded near 0 since f is differentiable there. Second, for all small δ the function F is integrable on $[-\pi, -\delta] \cup [\delta, \pi]$ because f has this property and $|t| > \delta$ there.

As a consequence of the previous proposition, the function F is integrable on all of $[-\pi, \pi]$.

Now $S_N(f)(\theta_0) = (f * D_N)(\theta_0)$ where D_N is the Dirichlet kernel. Since $\frac{1}{2\pi}\int D_N = 1$, we find that

$$S_N(f)(\theta_0) - f(\theta_0) = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(\theta_0 - t) D_N(t)\, dt - f(\theta_0)$$

$$= \frac{1}{2\pi}\int_{-\pi}^{\pi} [f(\theta_0 - t) - f(\theta_0)] D_N(t)\, dt.$$

$$= \frac{1}{2\pi}\int_{-\pi}^{\pi} F(t)\, t D_N(t)\, dt.$$

Recall that $D_N(t) = \dfrac{\sin((N+\frac{1}{2})t)}{\sin(t/2)}$

then $t D_N(t) = \dfrac{t}{\sin\frac{t}{2}} \sin((N+\tfrac{1}{2})t)$

where the quotient $\dfrac{t}{\sin(\frac{t}{2})}$ is continuous in the interval $[-\pi, \pi]$. Since we can write $\sin((N+\frac{1}{2})t) = \sin(Nt)\cos\frac{t}{2} + \cos(Nt)\sin\frac{t}{2}$

$$S_N f(\theta_0) - f(\theta_0) = \frac{1}{2\pi}\int_{-\pi}^{\pi} F(t) \frac{t}{\sin\frac{t}{2}} [\sin(Nt)\cos\tfrac{t}{2} + \cos(Nt)\sin\tfrac{t}{2}]$$

$$= \frac{1}{2\pi}\int_{-\pi}^{\pi} F(t) \frac{t\cos\frac{t}{2}}{\sin\frac{t}{2}} \sin(Nt)\, dt + \int_{-\pi}^{\pi} t F(t) \cos(Nt)\, dt$$

we apply the Riemann-Lebesgue lemma to conclude that $S_N(f)(\theta_0) - f(\theta_0) \longrightarrow 0$ and thus $S_N(f)(\theta_0) \longrightarrow f(\theta_0)$

The proof is complete. $\square$

Remark: The conclusion of the theorem still holds if we only assume that f satisfies a Lipschitz condition at θ_0, that is $|f(\theta) - f(\theta_0)| \leq M|\theta - \theta_0|$. for some M and all θ_0.

The verification is similar as the proof of the theorem. just to modify the definition of $F(t)$:

$$F(t) = \begin{cases} \dfrac{f(\theta - t) - f(\theta_0)}{t} & t \neq 0 \text{ and } |t| < \pi. \\ -D^{\circ} F(\theta_0) & t = 0 \end{cases}$$

where $D^{\circ} F(\theta_0) = \limsup\limits_{t \to 0} \dfrac{f(\theta_0 + t) - f(\theta_0)}{t}$ Obviously, $F(t)$ is an integrable function on $[-\pi, \pi]$.

The Lipschitz condition is the same as saying that f satisfies a Hölder condition of order $\alpha = 1$.

The convergence of $S_N(f)(\theta_0)$ depends only on the behavior of f near θ_0.

Theorem (Localization Lemma)

Suppose f and g are two integrable functions defined on the circle, and for some θ_0 there exists an open interval I containing θ_0 such that $f(\theta) = g(\theta)$ for all $\theta \in I$. Then $S_N(f)(\theta_0) - S_N g(\theta_0) \longrightarrow 0$ as N tends to infinity.

Proof: The function $f-g$ is 0 in I, so it is differentiable at θ_0. Apply the previous theorem,

$$S_N(f-g)(\theta_0) = S_N(f)(\theta_0) - S_N(g)(\theta_0) \to 0 \qquad N \to \infty$$

(Here we observe the linearity of Fourier series) and we complete the proof. $\square$

Theorem (Dini's Test Version 1).

Let f be an integrable function on the circle.

For a real number s, let

$$\varphi(t) = f(\theta_0 - t) - s$$

If there exists a $\delta > 0$ such that $\varphi(t)/t$ is an integrable function on $[-\delta, \delta]$, then the Fourier series of f converges to s.

Proof: $$S_N(f)(\theta_0) - s = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(\theta_0 - t) D_N(t)\, dt - s$$

$$= \frac{1}{2\pi}\int_{-\pi}^{\pi} [f(\theta_0 - t) - s] D_N(t)\, dt$$

$$= \frac{1}{2\pi}\int_{-\pi}^{\pi} \frac{\varphi(t)}{t} \cdot t\, D_N(t)\, dt.$$

Since $\frac{\varphi(t)}{t}$ is integrable on $[-\delta, \delta]$ and on the set $\{t \mid \delta \le |t| \le \pi\}$ naturally, $\varphi(t)/t$ is integrable on $[-\pi, \pi]$.

Next, $$t D_N(t) = \frac{t}{\sin\frac{t}{2}} \sin((N+\tfrac{1}{2})t)$$

where $\frac{t}{\sin(\frac{t}{2})}$ is continuous on $[-\pi, \pi]$. Since we can write $\sin((N+\frac{1}{2})t) = \sin(Nt)\cos\frac{t}{2} + \cos(Nt)\sin\frac{t}{2}$, we can apply the Riemann-Lebesgue lemma to the Riemann integrable function $(\varphi(t)/t) \cdot t\cos\frac{t}{2} / \sin\frac{t}{2} = \varphi(t)\cos\frac{t}{2}/\sin\frac{t}{2}$ and $\varphi(t)/t \cdot t = \varphi(t)$ to finish the proof.

Theorem (Dini's Test Version 2).

Let f be an integrable function on the circle.

For a fixed real number S, let

$$\varphi(t) = f(\theta_0+t) + f(\theta_0-t) - 2S.$$

If there exists a $\delta > 0$ such that $\varphi(t)/t$ is an integrable function on $[0,\delta)$, then the Fourier series of f converges to S.

Proof:

Write $S_N(f)(\theta_0) = (f * D_N)(\theta_0) = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(\theta_0 - t) D_N(t)\,dt$, where D_N is the N-th Dirichlet kernel.

Let $s = -t$.

$$\int_{-\pi}^{\pi} f(\theta_0-t) D_N(t)\,dt$$

$$= \int_{-s=-\pi}^{-s=\pi} f(\theta_0+s) D_N(-s)\,d(-s)$$

$$= \int_{\pi}^{-\pi} -f(\theta_0+s) D_N(s)\,ds \qquad (D_N(s) = D_N(-s))$$

$$= \int_{-\pi}^{\pi} f(\theta_0+s) D_N(s)\,ds$$

Therefore $S_N(f)(\theta_0) - f(\theta_0) = \frac{1}{2}\left(\frac{1}{2\pi}\int_{-\pi}^{\pi}[f(\theta_0+t) + f(\theta_0-t) - 2S] D_N(t)\,dt\right)$

$$= \frac{1}{4\pi}\int_{-\pi}^{\pi} (\varphi(t)/t).\ t\, D_N(t)\,dt.$$

Similar as the argument in Theorem 2.1, we conclude that $S_N(f)(\theta_0) - f(\theta_0) \to 0$ as $N \to \infty$, and hence we finished the proof. $\square$

Theorem (Riemann's Localization Lemma formal version)

Let f be an integrable function on the circle. If

$$\lim_{N\to\infty} \frac{1}{\pi}\int_0^\delta \frac{f(\theta_0-t)+f(\theta_0+t)}{2} D_N(t)\,dt = S$$

exists for some $0<\delta<\pi$, then the Fourier series of f converges at θ_0 to S.

Proof: Write: $S_N(f)(\theta_0) = (f * D_N)(\theta_0) = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(\theta_0-t) D_N(t)\,dt$

$$= \frac{1}{2\pi}\int_0^\pi \left(f(\theta_0-t)+f(\theta_0+t)\right) D_N(t)\,dt$$

($D_N(t)$ is the N-th Dirichlet kernel which is an even function)

Now, divide the above expressions into two integrals $\int_0^\delta + \int_\delta^\pi$.

$$\frac{1}{2\pi}\int_\delta^\pi [f(\theta_0-t)+f(\theta_0+t)] D_N(t)\,dt \quad \left(D_N(t) = \frac{\sin[(N+\frac{1}{2})t]}{\sin\frac{t}{2}}\right)$$

$$= \frac{1}{2\pi}\int_\delta^\pi \left(\frac{f(\theta_0-t)+f(\theta_0+t)}{\sin\frac{t}{2}}\right) \sin(N+\tfrac{1}{2})t\,dt \quad (*)$$

We see that $\dfrac{f(\theta_0-t)+f(\theta_0+t)}{\sin\frac{t}{2}}$ is (bounded and hence) integrable on $[\sigma,\pi]$. By Riemann-Lebesgue Lemma, the expression $(*)$ converges to zero as $N\to\infty$

Thus $\lim_{N\to\infty} S_N(f)(\theta) = \lim_{N\to\infty}\frac{1}{2\pi}\left(\int_0^\delta + \int_\delta^\pi\right)[f(\theta_0-t)+f(\theta_0+t)] D_N(t)\,dt$

$$= \lim_{N\to\infty}\frac{1}{2\pi}\int_0^\delta [f(\theta_0-t)+f(\theta_0+t)] D_N(t)\,dt$$

$= S$ by the given condition.

The following is partially from 《数学分析教程》下册 常庚哲 史济怀

Definition (Lipschitz condition)

Let f be a function defined near x_0. If there exists a $\delta > 0$, $L > 0$ and $\alpha > 0$, such that

$$|f(x_0+t) - f(x_0^+)| \le L t^\alpha, \text{ and}$$

$$|f(x_0-t) - f(x_0^-)| \le L t^\alpha$$

whenever $t \in [0, \delta)$, then we say f satisfies the Lipschitz condition of order α near x_0

Theorem Let f be an integrable function on the circle. If f satisfies Lipschitz condition of order $\alpha > 0$, near θ_0, then the Fourier series of f converges to $\frac{1}{2}(f(\theta_0^+) + f(\theta_0^-))$.

Proof: Take $S = \frac{1}{2}(f(\theta_0^+) + f(\theta_0^-))$ then

$$\frac{\varphi(t)}{t} = \frac{f(\theta_0+t) - f(\theta_0^+) + f(\theta_0-t) - f(\theta_0^-)}{t}.$$

Since f satisfies Lipschitz condition of order α near θ_0,

$$\left|\frac{\varphi(t)}{t}\right| \le \frac{2L}{t^{1-\alpha}} \qquad 0 < t \le \delta$$

If $\alpha \ge 1$, $\varphi(t)/t$ is bounded and integrable.

$0 < \alpha < 1$ $\varphi(t)/t$ is (absolutely) improper integrable on $[0, \delta)$. We apply Dini's Test to finish the proof of the theorem

Corollary. Let f be an integrable function on the circle. If f has two one-sided generalized derivatives at θ_0:

$$f_+'(\theta_0) = \lim_{t\to 0^+} \frac{f(\theta_0+t)-f(\theta_0^+)}{t}$$

$$f_-'(\theta_0) = \lim_{t\to 0^+} \frac{f(\theta_0-t)-f(\theta_0^-)}{-t}$$

Then the Fourier series of f at θ_0 converges to $\frac{1}{2}(f(\theta_0^+)+f(\theta_0^-))$

Proof: Under the condition we may conclude that f satisfies the Lipschitz condition of order $\alpha=1$ near θ_0. By the previous theorem, we conclude the proof.

The above Corollary is called the <u>convergence theorem</u> of Fourier series, which can also be proved by Riemann's Localization lemma formal version above.