Proposition If $f, g \in S(\mathbb{R})$, then

(i) $f * g \in S(\mathbb{R})$

(ii) $f * g = g * f$

(iii) $\widehat{(f * g)}(\xi) = \hat{f}(\xi)\,\hat{g}(\xi)$

Proof: (i) First observe that for any $\ell \geq 0$, we have

$\sup_x |x|^\ell |g(x-y)| \leq A_\ell (1+|y|)^\ell$. We check this observation:

$$|x|^\ell |g(x-y)| = |x|^\ell \frac{1}{|x-y|^\ell} |x-y|^\ell |g(x-y)|$$
$$\leq |x|^\ell \frac{1}{|x-y|^\ell} B_\ell, \quad \text{where } B_\ell = \sup_z |z|^\ell |g(z)|$$

For $|x| > 2|y|$, we have $|x-y| \geq |x| - |y| \geq |x| - \frac{1}{2}|x| = \frac{1}{2}|x|$.

$|x|^\ell |g(x-y)| \leq [|x|^\ell / (\frac{1}{2}|x|)^\ell] B = 2^\ell B_\ell \leq 2^\ell B_\ell (1+|y|^\ell)$

For $|x| \leq 2|y|$, we have $|x|^\ell |g(x-y)| \leq 2^\ell |y|^\ell |g(x-y)| \leq 2^\ell |y|^\ell C$

where $C = \sup_z |g(z)|$, and thus $|x|^\ell |g(x-y)| \leq 2^\ell C (1+|y|^\ell)$

Therefore there is always an A_ℓ such that $\sup_x |x|^\ell |g(x-y)| \leq A_\ell (1+|y|)^\ell$.

From this we see that $\sup_x |x^\ell (f*g)(x)| \leq A_\ell \int_{-\infty}^{\infty} |f(y)| (1+|y|)^\ell dy < \infty$

because f is rapidly decreasing, so that $x^\ell (f*g)(x)$ is a bounded function for every $\ell \geq 0$.

Now we claim, which will be proved later, that

$$\frac{d}{dx}(f*g)(x) = (f * \frac{d}{dx} g)(x)$$

and thus by iteration

$$\left(\frac{d}{dx}\right)^k (f*g)(x) = \left(f * \left(\frac{d}{dx}\right)^k g\right)(x)$$

Since $\left(\frac{d}{dx}\right)^k g \in S(\mathbb{R})$, the estimates above carry over to k-th derivatives of $f*g$ by the above identity, and thus (i) is proved.

(ii) Recall the definition of the convolution

$$(f*g)(x) = \int_{-\infty}^{\infty} f(y)g(x-y)\,dy$$

Let $x-y=u$, $y=x-u$ (change of variables)

$$(f*g)(x) = \int_{x-u=-\infty}^{x-u=\infty} f(x-u)g(u)\,d(x-u)$$

$$= -\int_{\infty}^{-\infty} f(x-u)g(u)\,du$$

$$= \int_{-\infty}^{\infty} f(x-u)g(u)\,du = (g*f)(x)$$

(iii) $$\widehat{(f*g)}(\xi) = \int_{-\infty}^{\infty} (f*g)(x)e^{-2\pi i x\xi}\,dx$$

$$= \int_{-\infty}^{\infty}\left(\int_{-\infty}^{\infty} f(y)g(x-y)\,dy\right)e^{-2\pi i x\xi}\,dx$$

$$= \int_{-\infty}^{\infty}\left(\int_{-\infty}^{\infty} f(y)g(x-y)e^{-2\pi i x\xi}\,dy\right)dx$$

Consider $F(x,y) = f(y)g(x-y)e^{-2\pi i x\xi}$

If we can change the order of the iterated integral of $F(x,y)$, i.e. $\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} F(x,y)\,dy\,dx = \int_{-\infty}^{\infty} F(x,y)\,dx\,dy$, then

$$\widehat{(f*g)}(\xi) = \int_{-\infty}^{\infty}\left(\int_{-\infty}^{\infty} f(y)g(x-y)e^{-2\pi i x\xi}\,dx\right)dy$$

$$= \int_{-\infty}^{\infty}\left(\int_{-\infty}^{\infty} f(y)e^{-2\pi i y\xi}g(x-y)e^{-2\pi i(x-y)\xi}\,dx\right)dy$$

$$= \int_{-\infty}^{\infty} f(y)e^{-2\pi i y\xi}\left(\int_{-\infty}^{\infty} g(x-y)e^{-2\pi i(x-y)\xi}\,dx\right)dy$$

Now, by letting $t = x-y$, ranging from $-\infty$ to ∞.

$$\int_{-\infty}^{\infty} g(x-y) e^{-2\pi i (x-y)\xi} dx = \int_{-\infty}^{\infty} g(t) e^{-2\pi i t \xi} dt$$

$$= \hat{g}(\xi)$$

and

$$\widehat{(f * g)}(\xi) = \left(\int_{-\infty}^{\infty} f(y) e^{-2\pi i y \xi} dy\right) \cdot \hat{g}(\xi)$$

$$= \hat{f}(\xi)\, \hat{g}(\xi)$$

Next we verify the validity of changing the order of the iterated integral to complete the proof.

Since $g \in S(\mathbb{R})$, then g is of moderate decrease.

$$|g(x-y)| \leq \frac{C}{1+(x-y)^2} \quad \text{for some } C \geq 0.$$

Noting that $||x|-|y|| \leq |x-y|$, we have

$$|g(x-y)| \leq \frac{C}{1+(|x|-|y|)^2} = \frac{C}{1+x^2-2|x||y|+y^2}$$

leading to the discussion of two cases:

① $y^2-2|x||y| \geq 0$, $|y| \geq 2|x|$. then $|g(x-y)| \leq \frac{C}{1+x^2}$ immediately.

② $y^2-2|x||y| < 0$, $|y| < 2|x|$, let $|y| = k|x|$ for $0 \leq k < 2$

$$g(x-y) \leq \frac{C}{1+x^2+2kx^2+k^2x^2} = \frac{C}{1+(k-1)^2x^2}$$

$$= \frac{C/(k-1)^2}{\frac{1}{(k-1)^2}+x^2} \leq \frac{C(k-1)^2}{1+x^2} \quad \text{since } 0 \leq k-1 \leq 1$$

Therefore in both cases, we have

$$|F(x,y)| \leq \frac{A}{(1+y^2)(1+x^2)} \quad \text{for some } A > 0.$$

By the previous discussion in the multiplication formula, the order of the integration can be changed safely. $\square$

Hermitian inner product on the Schwartz space

$$\langle f, g \rangle = \int_{-\infty}^{\infty} f(x)\overline{g(x)}\, dx ,$$

whose associate norm is

$$\|f\| = \left(\int_{-\infty}^{\infty} |f(x)|^2 dx \right)^{\frac{1}{2}} .$$

The Plancherel formula states that the Fourier transform is a unitary transformation on $S(\mathbb{R})$

Theorem (Plancherel) If $f \in S(\mathbb{R})$, then $\|\hat{f}\| = \|f\|$

Proof: If $f \in S(\mathbb{R})$, define $f^{b}(x) = \overline{f(-x)}$. Then $\widehat{f^{b}}(\xi) = \overline{\hat{f}(\xi)}$.

Now let $h = f * f^{b}$. Clearly we have (Convolution theorem)

$$\hat{h}(\xi) = |\hat{f}(\xi)|^2 \quad \text{and} \quad h(0) = \int_{-\infty}^{\infty} |f(x)|^2 dx$$

The theorem now follows from the inversion formula applied with $x = 0$, that is

$$\int_{-\infty}^{\infty} \hat{h}(\xi)\, d\xi = h(0)$$

$$\int_{-\infty}^{\infty} |\hat{f}(\xi)|^2 d\xi = \|\hat{f}\|^2 = \int_{-\infty}^{\infty} |f(x)|^2 dx = \|f\|^2$$

Periodization

Given $f \in S(\mathbb{R})$ on the real line, we can construct a new function on the circle

$$F_1(x) = \sum_{n=-\infty}^{\infty} f(x+n)$$

The sum converges absolutely and uniformly on every compact subset of $\mathbb{R}$, so F_1 is continuous. and

$$F_1(x+1) = F_1(x).$$

The function F_1 is called periodization of f.

Let $\hat{f}(\xi)$ be the Fourier transform of f. Define

$$F_2(x) = \sum_{n=-\infty}^{\infty} \hat{f}(n) e^{2\pi i n x}$$

The sum converges absolutely and uniformly, since $\hat{f}$ belongs to the Schwartz space, hence F_2 is continuous.

Moreover $F_2(x+1) = F_2(x)$.

Theorem 3.1. (Poisson summation formula)
If $f \in S(\mathbb{R})$, then

$$\sum_{n=-\infty}^{\infty} f(x+n) = \sum_{n=-\infty}^{\infty} \hat{f}(n) e^{2\pi i n x}$$

In particular, setting $x=0$ we have

$$\sum_{n=-\infty}^{\infty} f(n) = \sum_{n=-\infty}^{\infty} \hat{f}(n)$$

In other words, the Fourier coefficients of the periodization of f are given precisely by the values of the Fourier transform of f on the integers

Proof: It suffices to show that both sides (which is continuous) have the same Fourier coefficients (viewed as functions on the circle). Clearly, the m-th Fourier coefficient of the right-hand side is $\hat{f}(m)$. For the left-hand side we have

$$\int_0^1 \left(\sum_{n=-\infty}^{\infty} f(x+n)\right) e^{-2\pi i m x}\,dx = \sum_{n=-\infty}^{\infty} \int_0^1 f(x+n)\,e^{-2\pi i m x}\,dx$$

$$= \sum_{n=-\infty}^{\infty} \int_n^{n+1} f(y)\,e^{-2\pi i m y}\,dy$$

(change of variables)

$$= \int_{-\infty}^{\infty} f(y)\,e^{-2\pi i m y}\,dy$$

$$= \hat{f}(m)$$

where the interchange of the sum and the integral is permissible since f is rapidly decreasing. $\square$