The Fourier Transform on $\mathbb{R}^d$

Given $x = (x_1, \dots, x_d) \in \mathbb{R}^d$, define

$$|x| = (x_1^2 + \dots + x_d^2)^{\frac{1}{2}}$$

$$x \cdot y = x_1 y_1 + \dots + x_d y_d$$

Given a d-tuple $\alpha = (\alpha_1, \dots, \alpha_d)$ of d nonnegative integers, the monomial x^α is defined by

$$x^\alpha = x_1^{\alpha_1} x_2^{\alpha_2} \cdots x_d^{\alpha_d}$$

The differential operator $\left(\frac{\partial}{\partial x}\right)^\alpha$ is defined by

$$\left(\frac{\partial}{\partial x}\right)^\alpha = \left(\frac{\partial}{\partial x_1}\right)^{\alpha_1} \left(\frac{\partial}{\partial x_2}\right)^{\alpha_2} \cdots \left(\frac{\partial}{\partial x_d}\right)^{\alpha_d} = \frac{\partial^{\alpha_1 + \dots + \alpha_d}}{\partial x_1^{\alpha_1} \partial x_2^{\alpha_2} \cdots \partial x_d^{\alpha_d}}$$

Schwartz space $S(\mathbb{R}^d)$

The Schwartz space $S(\mathbb{R}^d)$ consists of all indefinitely differentiable functions f on $\mathbb{R}^d$ such that

$$\sup_{x \in \mathbb{R}^d} \left| x^\alpha \left(\frac{\partial}{\partial x}\right)^\beta f(x) \right| < \infty$$

for every α and β.

Fourier transform on a Schwartz function f is defined by

$$\hat{f}(\xi) = \int_{\mathbb{R}^d} f(x) e^{-2\pi i x \cdot \xi} \, dx, \quad \text{for } \xi \in \mathbb{R}^d$$

Note that the product in 1-dimensional case is replaced by the inner product of two vectors in d-dimensional case.

Proposition Let $f \in S(\mathbb{R}^d)$ ($F(x) \to G(\xi)$ means $\hat{F}(\xi) = G(\xi)$)

(i) $f(x+h) \longrightarrow \hat{f}(\xi) e^{2\pi i \xi \cdot h}$, whenever $h \in \mathbb{R}^d$

(ii) $f(x) e^{-2\pi i x \cdot h} \longrightarrow \hat{f}(\xi + h)$ whenever $h \in \mathbb{R}^d$.

(iii) $f(\delta x) \longrightarrow \delta^{-d} \hat{f}(\delta^{-1}\xi)$ whenever $\delta > 0$

(iv) $\left(\frac{\partial}{\partial x}\right)^\alpha f(x) \longrightarrow (2\pi i \xi)^\alpha \hat{f}(\xi)$

(v) $(-2\pi i x)^\alpha f(x) \longrightarrow \left(\frac{\partial}{\partial \xi}\right)^\alpha \hat{f}(\xi)$.

(vi) $f(Rx) \longrightarrow \hat{f}(R\xi)$, whenever R is a rotation (orthogonal transformation)

Proof of (iv) Let $g(x) = \left(\frac{\partial}{\partial x}\right)^\alpha f(x)$, where $\alpha = (\alpha_1, \cdots, \alpha_d)$

$x = (x_1, \cdots, x_d)$

$\xi = (\xi_1, \cdots, \xi_d)$

$$\int_{\mathbb{R}^d} g(x) e^{-2\pi i x \cdot \xi} dx$$

$$= \int_{\mathbb{R}} \cdots \int_{\mathbb{R}} \left(\frac{\partial}{\partial x_1}\right)^{\alpha_1} \cdots \left(\frac{\partial}{\partial x_d}\right)^{\alpha_d} f(x_1, \cdots, x_d) e^{-2\pi i (x_1\xi_1 + \cdots + x_d \xi_d)} dx_1 \cdots dx_d$$

$$= \int_{\mathbb{R}^{d-1}} \left(\frac{\partial}{\partial x_2}\right)^{\alpha_2} \cdots \left(\frac{\partial}{\partial x_d}\right)^{\alpha_d} (\cdot) e^{-2\pi i (x_2 \xi_2 + \cdots + x_d \xi_d)} dx_2 \cdots dx_d$$

$$\int_{\mathbb{R}} \left(\frac{\partial}{\partial x_1}\right)^{\alpha_1} f(x_1, \cdots x_d) e^{-2\pi i x_1 \xi_1} dx_1$$

Now, using integration by parts iteratively

$$\int_{\mathbb{R}} \left(\frac{\partial}{\partial x_1}\right)^{\alpha_1} f(x_1, \cdots, x_d) e^{-2\pi i x_1 \xi_1}$$

$$= (2\pi i \xi_1)^{\alpha_1} \int_{\mathbb{R}} f(x_1, \cdots, x_d) e^{-2\pi i x_1 \xi_1} dx_1$$

Inductively, we have

$$\int_{\mathbb{R}^d} g(x) e^{-2\pi i x \cdot \xi} dx$$

$$= (2\pi i \xi_1)^{\alpha_1} (2\pi i \xi_2)^{\alpha_2} \cdots (2\pi i \xi_d)^{\alpha_d} \cdot \int_{\mathbb{R}^d} f(x_1, \cdots, x_d) e^{-2\pi i (x_1 \xi_1 + \cdots + x_d \xi_d)} dx_1 \cdots dx_d$$

$$= (2\pi i \xi)^{\alpha} \int_{\mathbb{R}^d} f(x) e^{-2\pi i x \cdot \xi} dx$$

$$= (2\pi i \xi)^{\alpha} \hat{f}(\xi).$$

$\square$

Corollary 2.2. The Fourier transform maps $\mathcal{S}(\mathbb{R}^d)$ to itself.

Proof: $\xi^{\alpha} \left(\frac{\partial}{\partial \xi}\right)^{\beta} \hat{f}(\xi)$ is the Fourier transform of: $\underbrace{\frac{1}{(2\pi i)^{|\alpha|}} \left(\frac{\partial}{\partial x}\right)^{\alpha} [(-2\pi i x)^{\beta} f(x)]}$, so

$\int_{\mathbb{R}^d} \quad dx$ is finite since the integrand is of moderate decrease.

$\square$

A function f is radical if it depends only on $|x|$. In other words, f is radical if there is a function $f_0(u)$, defined for $u \geq 0$, such that $f(x) = f_0(|x|)$

Note that

f is radical if and only if $f(Rx) = f(x)$ for every rotation R.

Corollary The Fourier transform of a radical function is radical.

The Gaussian $e^{-\pi|x|^2}$ is an example of a radial function.

Theorem 2.4. Suppose $f \in S(\mathbb{R}^d)$. Then

$$f(x) = \int_{\mathbb{R}^d} \hat{f}(\xi) e^{2\pi i x \cdot \xi} \, d\xi$$

This is called the Fourier inversion formula

More over $$\int_{\mathbb{R}^d} |\hat{f}(\xi)|^2 \, d\xi = \int_{\mathbb{R}^d} |f(x)|^2 dx$$

or $$\|\hat{f}\|_{L^2(\mathbb{R}^d)} = \|f\|_{L^1(\mathbb{R}^d)}$$ (Plancherel formula)

Proof: Step 1. The Fourier transform of $e^{-\pi|x|^2}$ is $e^{-\pi|\xi|^2}$, because $e^{-\pi|x|^2} = e^{-\pi x_1^2} \cdots e^{-\pi x_d^2}$

$$e^{-2\pi i x \cdot \xi} = e^{-2\pi x_1 \xi_1} \cdots e^{-2\pi x_d \xi_d}.$$

So $$\int_{\mathbb{R}^d} e^{-\pi|x|^2} e^{-2\pi i x \cdot \xi} dx$$

$$= \int_{\mathbb{R}^{d-1}} e^{-\pi x_2^2} e^{-\pi i x_2 \xi_2} \cdots e^{-\pi x_d^2} e^{-\pi i x_d \xi_d} \left(\int_{\mathbb{R}} e^{-\pi x_1} e^{-\pi i x_1 \xi_1} dx_1 \right) dx_2 \cdots dx_d$$

inductively

$$= e^{-\pi(\xi_1^2 + \cdots + \xi_d^2)} = e^{-\pi|\xi|^2}$$

As a consequence of the previous proposition

$$\widehat{(e^{-\pi\delta|x|^2})} = \delta^{-d/2} e^{-\pi|\xi|^2/\delta}$$

Step 2. The family $\{K_\delta(x)\}$ where $K_\delta(x) = \delta^{d} e^{-\pi|x|^2/\delta}$

is a family of good kernels. By this we mean that

(i) $\int_{\mathbb{R}^d} K_\delta(x)\,dx = 1$

(ii) $\int_{\mathbb{R}^d} |K_\delta(x)|\,dx \leq M$ for some M

(in fact $K_\delta(x) \geq 0$)

(iii) For every $\eta > 0$, $\int_{|x|\geq\eta} |K_\delta(x)|\,dx \to 0$ as $\delta \to 0$.

The proofs are almost identical to the case $d=1$

As a result:

$$\int_{\mathbb{R}^d} K_\delta(x) F(x)\,dx \to F(0) \quad \text{as } \delta \to 0.$$

($K_\delta(x)$ "tends to" be a Dirac delta function as $\delta \to 0$)

where F is a Schwartz function, or more generally when F is bounded and continuous at the origin.

Step 3. The multiplication formula

$$\int_{\mathbb{R}^d} f(x)\hat{g}(x)\,dx = \int_{\mathbb{R}^d} \hat{f}(y) g(y)\,dy$$

holds whenever f and g are in $\mathcal{S}(\mathbb{R})$

Let $F(x,y) = f(x)g(y)e^{-2\pi i x\cdot y}$ over $(x,y)\in\mathbb{R}^{2d} = \mathbb{R}^d\times\mathbb{R}^d$

It aims to show that

$$\int_{\mathbb{R}^d} f(x)\left(\int_{\mathbb{R}^d} g(y)\, e^{-2\pi i x\cdot y}\, dy\right) dx = \int_{\mathbb{R}^d}\left(\int_{\mathbb{R}^d} f(x) e^{-2\pi i x\cdot y}\, dx\right) g(y)\, dy.$$

The justification is similar to that in the proof of two dimensional case, with Fubini theorem (for functions of moderate decrease).

The Fourier inversion is a simple consequence of the multiplication formula and the family of good kernels as in Chapter 5.

First we claim $f(0) = \int_{-\infty}^{\infty} \hat{f}(\xi)\, d\xi$

Let $G_\delta(x) = e^{-\pi\delta|x|^2}$. $\hat{G}_\delta(\xi) = K_\delta(\xi)$. By multiplication formula,

$$\int_{-\infty}^{\infty} f(x) K_\delta(x)\, dx = \int_{-\infty}^{\infty} \hat{f}(\xi) G_\delta(\xi)\, d\xi.$$

Taking the limit on both sides as $\delta\to 0$.

$$f(0) = \int_{-\infty}^{\infty} \hat{f}(\xi)\, d\xi$$

In general, let $F(y) = f(y+x)$

$$f(x) = F(0) = \int_{-\infty}^{\infty} \hat{F}(\xi)\, d\xi = \int_{-\infty}^{\infty} \hat{f}(\xi) e^{2\pi i x\cdot\xi}\, d\xi.$$

It also follows that the Fourier transform $\mathcal{F}$ is a bijective map of $\mathcal{S}(\mathbb{R}^d)$ to itself, whose inverse is

$$\mathcal{F}^*(g)(x) = \int_{\mathbb{R}^d} g(\xi) e^{2\pi i x\cdot\xi} \, d\xi.$$

Step 4. The convolution on $\mathbb{R}^d$ is defined by

$$(f * g)(x) = \int_{\mathbb{R}^d} f(y) g(x-y) \, dy, \quad f, g \in \mathcal{S}(\mathbb{R}^d)$$

Easy consequences. $f * g \in \mathcal{S}(\mathbb{R}^d)$

$$f * g = g * f$$

$$\widehat{(f * g)}(\xi) = \hat{f}(\xi)\hat{g}(\xi).$$

Define $f^b(x) = \overline{f(-x)}$. Then $\hat{f^b}(\xi) = \overline{\hat{f}(\xi)}$. Now let $h = f * f^b$. Clearly we have

$$\hat{h}(\xi) = |\hat{f}(\xi)|^2. \qquad h(0) = \int_{\mathbb{R}^d} |f(x)|^2 dx.$$

The Fourier inversion formula implies that (taking $x = 0$)

$$\int_{\mathbb{R}^d} |\hat{f}(\xi)|^2 d\xi = \int_{\mathbb{R}^d} \hat{h}(\xi) \, d\xi = h(0) = \int_{\mathbb{R}^d} |f(x)|^2 dx \qquad \square$$