C*-algebras of Left Cancellative Small Categories with Garside Families

a quick tour

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端午安康!

Wish you health and a peaceful Dragon Boat Festival.

2023 年 6 月 22 日

农历五月初五

June 22, 2023

The 5th Day of the 5th Lunar Month

Outline

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1. Left Cancellative Small Categories, Inverse semigroups and Characters
2. Garside Theory
3. Main Results
4. Application: Higher-rank graphs

Small Categories

A **small category** $\mathbf{C}$ is a category where the collection of objects $\mathrm{Ob}(\mathbf{C})$ and the collection of morphisms $\mathrm{Hom}_{\mathbf{C}}(A, B)$ between any two objects $A, B \in \mathrm{Ob}(\mathbf{C})$ are sets.

Notations:

$\mathbf{C} = \mathrm{Mor}(\mathbf{C})$ (the set of morphisms);

$\mathbf{C}^0 = \mathrm{Ob}(\mathbf{C})$ (the set of objects);

$\mathbf{C}^* =$ the set of isomorphisms.

By identifying each $\mathbf{v} \in \mathbf{C}^0$ as the identity morphism $\mathrm{id}_{\mathbf{v}}$, we regard $\mathbf{C}^0$ as a subset of $\mathbf{C}^*$ and hence a subset of $\mathbf{C}$. Then we have two maps $\mathbf{d} : \mathbf{C} \to \mathbf{C}^0$ and $\mathbf{t} : \mathbf{C} \to \mathbf{C}^0$ indicating the source and range of morphisms.

For every $c, d \in \mathbf{C}$, the composition cd is defined if and only if $\mathbf{t}(d) = \mathbf{d}(c)$, that is, $\mathbf{d}(d) \xrightarrow{d} \mathbf{t}(d) = \mathbf{d}(c) \xrightarrow{c} \mathbf{t}(c)$. For every $c \in \mathbf{C}$, we also define $c\mathbf{C} = \{cd : d \in \mathbf{C}, \mathbf{t}(d) = \mathbf{d}(c)\}$.

Left Cancellative Small Categories and Divisibility

A small category $\mathbf{C}$ is **left cancellative** if for all $c, x, y \in \mathbf{C}$ with $\mathbf{t}(x) = \mathbf{t}(y) = \mathbf{d}(c)$,

$$cx = cy \Longrightarrow x = y.$$

Let $\mathbf{C}$ be a left cancellative small category.

For a given $b \in \mathbf{C}$, we say an element $a \in \mathbf{C}$ is a **left divisor** of b if $b = ac$ for some $c \in \mathbf{C}$ (with $\mathbf{t}(c) = \mathbf{d}(a)$), written as $a \preceq b$.

If $a \preceq b$ but $a \neq b$, we say that a is a strict (or proper) left divisor of b, written as $a \prec b$.

Let S be a subfamily of $\mathbf{C}$. Given $a \in \mathbf{C}$, an element $s \in S$ is a **greatest left divisor** of a in S if $s \preceq a$ and is the greatest in S in the sense that every $r \in S$ with $r \preceq a$ satisfies $r \preceq s$.

$=^*$-equivalence

Let $a, b \in \mathbf{C}$ be two elements in a left cancellative small category. We say that $a =^* b$ if there exists an element $c \in \mathbf{C}^*$ such that $a = bc$.

$=^*$ is indeed an equivalence relation.

Proposition

Let $\mathbf{C}$ be a left cancellative small category. Then we have

$$a =^* b \iff a \in b\mathbf{C}^* \iff a\mathbf{C} = b\mathbf{C} \iff b \in a\mathbf{C}^*,$$

and hence $a \preceq b, b \preceq a \iff a =^* b$.

Inverse semigroups

A **semigroup** is a set S equipped with a binary operation $S \times S \to S$ called multiplication, satisfying associativity, that is, for all $a, b, c \in S$, $(xy)z = x(yz)$.

An **inverse semigroup** is a semigroup S with the property that for every $x \in S$, there is a unique $y \in S$ with

$$x = xyx \text{ and } y = yxy.$$

We write $y = x^{-1}$ and call y the **inverse** of x.

An inverse semigroup S is called an inverse semigroup **with zero** if there is a distinguished element $0 \in S$ satisfying $0 \cdot x = x \cdot 0 = 0$ for all $x \in S$.

Idempotents

Let S be a semigroup. An element $e \in S$ is called an **idempotent** if $e^2 = e$.

Lemma (Basic properties of idempotents)

In an inverse semigroup S,

1. $e = e^{-1}$ for every idempotent e.
2. Format: An element e is an idempotent if and only if it is of the form $e = x^{-1}x$.
3. Closeness under multiplication: The product of two idempotents is again an idempotent.
4. Commutativity: Any two idempotents commute.

Partial order on idempotents

Let S be an inverse semigroup.

Let $E = \{x^{-1}x : x \in S\} = \{e \in S : e^2 = e\}$ be the set of idempotents of S.

Define an **order relation** "$\leq$" on E: $e \leq f \iff e = ef$.

$\leq$ is a partial order because if $e \leq f$ and $f \leq e$, then by commutativity of idempotents,

$$e = ef = fe = f.$$

Thus E becomes a **semilattice**.

Fundamental example

Partial bijections on a set

Let X be a set. A **partial bijection** on X is a bijection between some subsets of X.

The inverse semigroup of partial bijections

Define

$$I(X) = \{\text{partial bijections on } X\}.$$

Then $I(X)$ becomes an inverse semigroup.
Multiplication is given by composition of partial bijections.
Inverses are given by the usual inverse functions of partial bijections.

Note: Let $s : \mathrm{dom}(s) \to \mathrm{im}(s)$ and $t : \mathrm{dom}(t) \to \mathrm{im}(t)$ be two partial bijections. The domain of $s \circ t$ is given by

$$\mathrm{dom}(s \circ t) = \mathrm{dom}(t) \cap t^{-1}(\mathrm{dom}(s)) = t^{-1}(\mathrm{dom}(s) \cap \mathrm{im}(t)).$$

Fundamental Example

Identity functions on subsets

The semilattice of idempotents of partial bijections

The semilattice of idempotents is given by

$$E(X) = \{s^{-1}s : s \in I(X)\} = \{\mathrm{Id}_{\mathrm{dom}(s)} : s \in I(X)\}.$$

We have the following identifications in the semilattice of idempotents in an inverse semigroup of partial bijections:

$$\begin{aligned} s^{-1}s &\quad\leftrightarrow\quad \mathrm{dom}(s), \\ \leq &\quad\leftrightarrow\quad \subseteq, \\ ef &\quad\leftrightarrow\quad \mathrm{dom}(e) \cap \mathrm{dom}(f). \end{aligned}$$

Characters

Definition (Character)

Let S be an inverse semigroup and E be its semilattice of idempotents. A **character** on E is a nonzero, multiplicative map

$$\chi : E \longrightarrow \{0,1\}$$

which sends $0 \in E$ (if it exists) to $0 \in \{0,1\}$.

By definition we can see that a character χ is completely determined by the set $\chi^{-1}(1) = \{e \in E : \chi(e) = 1\}$.

The set of characters on E is denoted by $\hat{E}$. $\hat{E}$ is called the character space when given the pointwise-convergence topology.

Left inverse hull of induced partial bijections

Let $\mathbf{C}$ be a left cancellative small category. Every $c \in \mathbf{C}$ induces a partial bijection $\sigma_c : \mathbf{d}(c)\mathbf{C} \longrightarrow c\mathbf{C}$ given by $x \mapsto cx$ (very often we may denote the induced partial bijection σ_c by c again).

The **left inverse hull** I_l of $\mathbf{C}$ is defined to be the inverse semigroup generated by the set of all the partial bijections $\{\sigma_c : c \in \mathbf{C}\}$.

The semilattice of idempotents of I_l is denoted by J The space of characters with pointwise-convergence topology on J is denoted by $\hat{J}$.

The Character from an element

Definition (The Character from an element)

Given $x \in \mathbf{C}$, we define $\chi_x : J \longrightarrow \{0,1\}$ by

$$\chi_x(e) = \begin{cases} 1, & \text{if } x\mathbf{C} \subseteq e, \\ 0, & \text{otherwise.} \end{cases}$$

Observation: $\chi_x = \chi_y$ if and only if $x\mathbf{C} = y\mathbf{C}$.

The subspace Ω of $\hat{J}$

and the inverse semigroup action on Ω

The subspace Ω of $\hat{J}$ is defined as follows:

Ω consists of characters $\chi : J \longrightarrow \{0,1\}$ with the property that whenever $e, f_1, \ldots, f_n \in J$ satisfy $e = \bigcup_{i=1}^{n} f_i$ as subsets of $\mathbf{C}$, then $\chi(e) = 1$ imples that $\chi(f_i) = 1$ for some index i.

The topology on Ω is the subspace topology from $\hat{J}$.

Lemma

$\{\chi_x : x \in \mathbf{C}\}$ is dense in Ω with respect to the pointwise-convergence topology.

Given $s \in I_l$ and $\chi \in \hat{J}$ with requirement that $\chi(s^{-1}s) = 1$, we define **the action of** s **on** χ as another character $s.\chi : J \longrightarrow \{0,1\}$ by $(s.\chi)(e) = \chi(s^{-1}es)$.

Groupoid models for the left reduced C*-algebras

The transformation groupid and its variation

The **transformation groupoid** $I_l \ltimes \Omega$ and its variation $I_l \bar{\ltimes} \Omega$ are defined to be the collection of equivalence classes on the set

$$I_l * \Omega := \{(s, \chi) \in I_l \times \Omega : \chi(s^{-1}s) = 1\}.$$

For $I_l \ltimes \Omega$, the equivalence relation $\sim$ is given by

$(s, \chi) \sim (t, \psi) \iff \chi = \psi$ and there exists an $e \in J$ with $\chi(e) = 1$ and $se = te$.

For $I_l \bar{\ltimes} \Omega$, the equivalence relation $\bar{\sim}$ is given by

$(s, \chi) \bar{\sim} (t, \psi) \iff \chi = \psi$ and there exists an $\varepsilon \in \bar{J}$ with $\chi(\mathrm{Id}_\varepsilon) = 1$ and $s\, \mathrm{Id}_\varepsilon = t\, \mathrm{Id}_\varepsilon$.

Groupoid structure:

- the source map $\mathrm{s}([s, \chi]) = \chi$ and the range map $\mathrm{r}([s, \chi]) = s.\chi$;
- multiplication $[s, t.\chi][t, \chi] = [st, \chi]$;
- inversion $[s, \chi]^{-1} = [s^{-1}, s.\chi]$.

Groupoid models for the left reduced C*-algebras

Left reduced C*-algebras for the small category $\mathbf{C}$

Let $\mathbf{C}$ be a left cancellative small category and $\mathbb{C}$ denotes the space of complex numbers. Define the $\ell^2(\mathbf{C})$ space to be $\ell^2(\mathbf{C}) = \left\{ f : \mathbf{C} \to \mathbb{C} \;\middle|\; \sum_{c\in\mathbf{C}} |f(c)|^2 < \infty \right\}$ with the "well-known" inner product. The standard orthonormal basis of $\ell^2(\mathbf{C})$ is given by $\{\delta_x\}_{x\in\mathbf{C}}$, where $\delta_x : \mathbf{C} \to \mathbb{C}$, $\delta_x(y) = \begin{cases} 1 \text{ if } y = x, \\ 0 \text{ if } y \neq x. \end{cases}$

For each $c \in \mathbf{C}$, we define a partial isometry λ_c by assigning $\delta_x \mapsto \delta_{cx}$ if $\mathbf{t}(x) = \mathbf{d}(c)$ and $\delta_x \mapsto 0$ if $\mathbf{t}(x) \neq \mathbf{d}(c)$ and extending by linearity on $\ell^2(\mathbf{C})$.

Definition (Left reduced C*-algebra of a left cancellative small category)

The left reduced C*-algebra of $\mathbf{C}$, denoted by $C^*_\lambda(\mathbf{C})$, is defined by the C*-algebra generated by the partial isometries $\{\lambda_c\}_{c\in\mathbf{C}}$.

Theorem (Spielberg 2020)

Let $\mathbf{C}$ be a left cancellative small category. The groupoid $I_l \bar{\ltimes} \Omega$ is a groupoid model for $C^*_\lambda(\mathbf{C})$, meaning that $C^*_\lambda(\mathbf{C}) \cong C^*_r(I_l \bar{\ltimes} \Omega)$.

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Garside theory

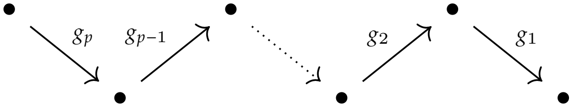
Paths

In the following, let $\mathbf{C}$ be always a left cancellative small category.

A **finite $\mathbf{C}$-path** is a finite sequence $g_1|\cdots|g_p$ such that $\mathbf{t}(g_{k+1}) = \mathbf{d}(g_k)$ for all $k = 1, 2, \ldots, p-1$.

An **infinite $\mathbf{C}$-path** is an infinite sequence $g_1|g_2|\cdots$ such that $\mathbf{t}(g_{k+1}) = \mathbf{d}(g_k)$ for all $k \in \mathbb{N}_+$

If $f_1|\cdots|f_p$ and $g_1|\cdots|g_q$ are two $\mathbf{C}$-paths with $\mathbf{t}(f_1) = \mathbf{d}(g_q)$, the **concatenation** of these is a new path $g_1|\cdots|g_q|f_1|\cdots|f_p$.



Greediness

Let S be a subfamily of $\mathbf{C}$.

Definition (S-greedy)

A length-two path $g_1|g_2$ is said to be S-**greedy** if each relation $s \preceq f g_1 g_2$ with $s \in S$ and $f \in \mathbf{C}$ implies that $s \preceq f g_1$.
A path $g_1|\cdots|g_p$ is said to be S-greedy if $g_i|g_{i+1}$ is S-greedy for each $i = 1, 2, \ldots, p-1$.

Garside Theory

Closeness under $=^*$, Closure

Let S be a subfamily of $\mathbf{C}$. We say that S is **closed under** $=^*$ or $=^*$**-closed** if for every $g' \in \mathbf{C}^*$, $g' =^* g$ for some $g \in S$ implies that $g' \in S$.

Definition ($=^*$-closure)

Let $S \subseteq \mathbf{C}$ be a subfamily, we define

$$S^\sharp = S\mathbf{C}^* \cup \mathbf{C}^*.$$

$S^\sharp$ is called the $=^*$**-closure** of S.

Normality

Let S be a subfamily of $\mathbf{C}$. Remember $S^\sharp = S\mathbf{C}^* \cup \mathbf{C}^*$.

Definition (S-normal)

A finite or infinite $\mathbf{C}$-path is S-**normal** if it is S-greedy and every entry lies in $S^\sharp$.

We say that a path $s_1|\cdots|s_p$ is an S-**normal decomposition** for an element g if $s_1|\cdots|s_p$ is an S-normal path and $g = s_1 \cdots s_p$.

Garside families

Definition (Garside family)

Let $\mathbf{C}$ be a left cancellative small category. A subfamily $\mathbf{G}$ of $\mathbf{C}$ is called a **Garside family** if every element of $\mathbf{C}$ admits at least one $\mathbf{G}$-normal decomposition.

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A slight generalization to infinite paths

Let $\mathbf{C}$ be a left cancellative small category. Recall that we defined previously $\chi_x : J \longrightarrow \{0,1\}$ by

$$\chi_x(e) = \begin{cases} 1, & \text{if } x\mathbf{C} \subseteq e, \\ 0, & \text{otherwise.} \end{cases}$$

We also have that $\chi_x \in \Omega$ for all $x \in \mathbf{C}$.

If $\mathbf{G}$ is a Garside family of $\mathbf{C}$, then in particular, $\mathbf{G}^\sharp$ generates $\mathbf{C}$, in the sense that every $x \in \mathbf{C}$ is a product of finite elements in $\mathbf{G}^\sharp$, say, $x = g_1 \cdots g_n$. This gives a finite $\mathbf{C}$-path $g_1 | \cdots | g_n$.

For infinite $\mathbf{C}$-paths and characters in Ω outside $\{\chi_x : x \in \mathbf{C}\}$, do we have a similar conclusion?

Let $w = s_1|s_2|\cdots$ be an infinite $\mathbf{C}$-path with every $s_i, (i \in \mathbb{N}_+)$ in S. In the case that every s_i lies in S, we also say that w is an infinite S-**path**.

Note that an S-normal path is an $S^\sharp$-path. We write

- $w_{\leq n} := s_1|\cdots|s_n$ for the finite path formed by the first n elements of the infinite path w,
- $w_n := s_1 \cdots s_n$ for the product of elements of $w_{\leq n}$,
- $w_{=n} := s_n$ for the n-th element of w, and
- $w_{>n} := s_{n+1}|s_{n+2}|\cdots$ for the path obtained by deleting first n-elements from w.

Let $\Omega_\infty = \Omega \setminus \{\chi_x : x \in \mathbf{C}\}$.

The character from an infinite path

Definition (Character from an infinite path)

Let S be a subfamily of $\mathbf{C}$ which generates $\mathbf{C}$. For an (infinite) S-path w, we define a map $\chi_w : J \longrightarrow \{0,1\}$ by

$$\chi_w(e) = \begin{cases} 1, & \text{if } w_n \in e \text{ for some } n \in \mathbb{N}_+, \\ 0, & \text{otherwise.} \end{cases}$$

We see that for every $x \in \mathbf{C}$, χ_x is actually the character from a finite path.

Interlude: Finite alignment

A small category $\mathbf{C}$ is said to be **finitely aligned** if for all $a, b \in \mathbf{C}$ there exists a finite subset $F \subseteq \mathbf{C}$ such that $a\mathbf{C} \cap b\mathbf{C} = \bigcup_{c \in F} c\mathbf{C}$.

Lemma

Let $\mathbf{C}$ be a finitely aligned left cancellative small category. Then the following statements hold.

(i) Every $e \in J$ is a union of finitely many principal ideals. That is, $e = \bigcup_{x \in F} x\mathbf{C}$ for some finite subset $F \subseteq \mathbf{C}$.

(ii) Every $\chi \in \Omega$ is determined by the family of principal ideals where the value of χ is 1, that is, $\mathcal{F}_p^\chi := \{x\mathbf{C} : x \in \mathbf{C} \text{ with } \chi(x\mathbf{C}) = 1\}$, in the sense that for every $e \in J$, $\chi(e) = 1$ if and only if there is an $x\mathbf{C} \in \mathcal{F}_p^\chi$ such that $x\mathbf{C} \subseteq e$.

Theorem (Li 2022)

The transformation groupoid $I_l \ltimes \Omega$ is isomorphic to its variation $I_l \bar{\ltimes} \Omega$ if either of the following conditions holds:

(1) $\mathbf{C}$ is finitely aligned.

(2) $I_l \ltimes \Omega$ is Hausdorff.

Recall that the groupoid $I_l \bar{\ltimes} \Omega$ is a groupoid model for $C^*_\lambda(\mathbf{C})$. Then we have the following corollary.

Corollary

The transformation groupoid $I_l \ltimes \Omega$ is a groupoid model for $C^*_\lambda(\mathbf{C})$ if either $\mathbf{C}$ is finitely aligned or $I_l \ltimes \Omega$ is a Hausdorff space.

Lemma ♣

Let $\mathbf{C}$ be a finitely aligned left cancellative small category which is also countable. Let S be a subfamily of $\mathbf{C}$ generating $\mathbf{C}$.
Then every $\chi \in \Omega_\infty$ is of the form χ_w for some infinite S-path.

Standard assumption

$\mathbf{C}$ is a finitely aligned, countable, left cancellative small category;
$\mathbf{G}$ is a Garside family of $\mathbf{C}$ which is $=^*$-transverse, locally bounded, and $\mathbf{G} \cap \mathbf{C}^* = \varnothing$.

Definitions:

Let $\mathbf{C}$ be a small category.

A subfamily S of $\mathbf{C}$ is said to be **$=^*$-transverse** if $a =^* b$ implies that $a = b$ for all $a, b \in S$.

A subfamily S of $\mathbf{C}$ is said to be

- **locally finite** if $\mathbf{v}S$ is finite for all $\mathbf{v} \in \mathbf{C}^0$;
- **locally bounded** if for every $\mathbf{v} \in \mathbf{C}^0$ there is no infinite sequence $s_1, s_2, \ldots$ in $\mathbf{v}S$ with $s_1 \prec s_2 \prec \cdots$.

Let S be a subfamily of $\mathbf{C}$. Given $a \in \mathbf{C}$, an element $s \in S$ is known as an S-**head** of a if s is a greatest left divisor of a is S.

Lemma (Dehornoy et al. 2015)

If $\mathbf{G}$ is a Garside family of $\mathbf{C}$, then every non-invertible element a admits a $\mathbf{G}$-head.

In the case that S is $=^*$-transverse, the S-head is unique if it exists. In this case, the S-head of an element $a \in \mathbf{C}$ is denoted as $H_S(a)$. We may also omit S and write $H(a)$ instead when it is clear in the context.

Given two S-paths $x = s_1|s_2|\cdots$ and $y = t_1|t_2|\cdots$ we mean $x = y$ by requiring $s_i = t_i$ for all indices i. In the case of finite paths, we also require that their lengths are the same.

Lemma ♠

Let $\mathbf{C}$ be a finitely aligned countable left cancellative small category. Let $\mathbf{G}$ be a Garside family of $\mathbf{C}$ which is $=^*$-transverse, locally bounded, and $\mathbf{G} \cap \mathbf{C}^* = \varnothing$. Then every $\chi \in \Omega \setminus \{\chi_\mathbf{v} : \mathbf{v} \in \mathbf{C}^0\}$ is of the form χ_p for some $\mathbf{G}$-normal path p. Moreover, for two normal paths p and q, $\chi_p = \chi_q$ if and only if $p = q$.

Let $\mathcal{W}$ be the collection of all $\mathbf{G}$-normal paths, then the above lemma gives a one-to-one correspondence between paths in $\mathcal{W} \sqcup \mathbf{C}^0$ and characters in Ω given by $w \mapsto \chi_w, \mathbf{v} \mapsto \chi_\mathbf{v}$.

Admissible pairs, H-invariance, $\max^{\infty}_{\leq}$-closeness

Let $\mathbf{C}$ be a left cancellative small category and $\mathbf{G}$ be a (nontrivial) Garside family.

For a sequence $\{s^{(i)}\}$ in $\mathbf{G}$ and an element $s \in \mathbf{G} \cup \mathbf{C}^0$, we write $\lim_i s^{(i)} = s$ if s is the greatest element with respect to $\preceq$ among the set $\{r \in \mathbf{G} \cup \mathbf{C}^0 : r \preceq s^{(i)}$ for all but finitely many $i\}$ in the sense that $s \preceq s^{(i)}$ for all but finitely many i, and every element r left dividing $s^{(i)}$ is also a left divisor of s.

Also let $\mathbf{I}$ be a subfamily of $\mathbf{G}$ and $\mathbf{D}$ be a subfamily of $\mathbf{C}^0$.

(i) The pair $(\mathbf{I}, \mathbf{D})$ is called **admissible** if for all $t \in \mathbf{I}$, either there is a $t' \in \mathbf{I}$ such that the path $t|t'$ is $\mathbf{G}$-normal or $\mathbf{d}(t) \in \mathbf{D}$.

(ii) $(\mathbf{I}, \mathbf{D})$ is called H-**invariant** if for all $a \in \mathbf{C} \backslash \mathbf{C}^*$ and $x \in \mathbf{I} \cup \mathbf{D}$ with $\mathbf{d}(a) = \mathbf{t}(x)$, $H(ax)$ lies in $\mathbf{I}$.

(iii) $(\mathbf{I}, \mathbf{D})$ is called $\max^{\infty}_{\leq}$-**closed** if for every sequence $\{t_i\}_i$ in $\mathbf{I}$, if $\lim_i t_i$ exists in $\mathbf{G}$, then $\lim_i t_i \in \mathbf{I} \cup \mathbf{D}$.

Lemma ♠ implies that there is a bijective correspondence between subsets of Ω and subsets of $\mathcal{W} \sqcup \mathbf{C}^0$:

Given $X \subseteq \Omega$, the corresponding subset of $\mathcal{W} \sqcup \mathbf{C}^0$ is $\mathcal{V}(X) := \{w \in \mathcal{W}, \mathbf{v} \in \mathbf{C}^0 : \chi_w, \chi_\mathbf{v} \in X\}$.

Given $\mathcal{V} \subseteq \mathcal{W} \sqcup \mathbf{C}^0$, the corresponding subset of Ω is $X(\mathcal{V}) := \{\chi_w, \chi_\mathbf{v} \in \Omega : w \in \mathcal{V} \cap \mathcal{W}, \mathbf{v} \in \mathcal{V} \cap \mathbf{C}^0\}$.

Definitions

- Given $X \subseteq \Omega$, let $\mathcal{V}(X) = \{w \in \mathcal{W}, \mathbf{v} \in \mathbf{C}^0 : \chi_w, \chi_\mathbf{v} \in X\}$. We define

$$\mathbf{I}(X) = \{t \in \mathbf{G} : t = v_{=i} \text{ for some } v \in \mathcal{V}(X) \cap \mathcal{W} \text{ and } i \in \mathbb{N}_+\}$$

and

$$\mathbf{D}(X) = \mathcal{V}(X) \cap \mathbf{C}^0 = \{\mathbf{v} \in \mathbf{C}^0 : \chi_\mathbf{v} \in X\}.$$

- Let $\mathbf{I}$ be a subfamily of $\mathbf{G}$ and $\mathbf{D}$ be a subfamily of $\mathbf{C}^0$. Define

$$X(\mathbf{I}, \mathbf{D}) = \{\chi_v : v_{=i} \in \mathbf{I}, \forall i \in \mathbb{N}_+\} \cup \{\chi_\mathbf{v} : \mathbf{v} \in \mathbf{D}\}.$$

Main theorem

Lemma

We have the following two necessary and sufficient statements:

(1) The pair $(\mathbf{I}, \mathbf{D})$ is admissible if and only if there is a subset $X \subseteq \Omega$ such that $\mathbf{I} = \mathbf{I}(X)$ and $\mathbf{D} = \mathbf{D}(X)$.

(2) $(\mathbf{I}(X), \mathbf{D}(X))$ is H-invariant and $\max_{\leq}^{\infty}$-closed if and only if X is $(I_l \ltimes \Omega)$-invariant and closed.

Theorem

There is an inclusion preserving one-to-one correspondence:

$$\{(I_l \ltimes \Omega)\text{-invariant closed subspaces of } \Omega\} \longrightarrow \left\{\text{admissible, } H\text{-invariant } \max_{\leq}^{\infty}\text{-closed pairs}\right\}$$

$$X \longmapsto (\mathbf{I}(X), \mathbf{D}(X))$$

$$X(\mathbf{I}, \mathbf{D}) \longleftarrow\!\shortmid (\mathbf{I}, \mathbf{D})$$

with $\mathbf{I} \subseteq \mathbf{G}$ and $\mathbf{D} \subseteq \mathbf{C}^0$.

If further $\mathbf{G}$ is locally finite, then every pair $(\mathbf{I}, \mathbf{D})$ is automatically $\max^{\infty}_{\leq}$-closed.

Theorem

Let $\mathbf{C}$ be a finitely aligned countable left cancellative small category and $\mathbf{G}$ is a Garside family of $\mathbf{C}$ which is $=^*$-transverse, locally bounded and $\mathbf{G} \cap \mathbf{C}^* = \varnothing$. Then we have the following:

- The transformation groupoid $I_l \ltimes \Omega$ is a groupoid model for the left reduced C*-algebra $C^*_\lambda(\mathbf{C})$.
- There is an inclusion preserving one-to-one correspondence between $I_l \ltimes \Omega$-invariant closed subspaces of Ω and admissible, H-invariant $\max^{\infty}_{\leq}$-closed pairs $(\mathbf{I}, \mathbf{D})$ with $\mathbf{I} \subseteq \mathbf{G}$ and $\mathbf{D} \subseteq \mathbf{C}^0$.
- If further $\mathbf{G}$ is locally finite, then the $\max^{\infty}_{\leq}$-closeness condition can be removed.

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Higher-rank graphs

Definition (Graph of rank k)

Let k be an nonnegative integer. A **graph of rank** k (also called a k-graph) is a countable small category $\mathbf{E}$ equipped with a functor $\mathrm{d} : \mathbf{E} \to \mathbb{N}^k$ satisfying the following unique factorization property: For all $e \in \mathbf{E}$ and $m, n \in \mathbb{N}^k$ with $\mathrm{d}(e) = m + n$, there are unique elements $u \in \mathrm{d}^{-1}(m)$ and $v \in \mathrm{d}^{-1}(n)$ such that $e = vu$.

We often call a graph of rank k a higher-rank graph when $k \geq 2$.

Basic properties of higher-rank graphs:

(i) $\mathrm{d}^{-1}(0) = \{\mathrm{id}_{\mathbf{v}} : \mathbf{v} \in \mathbf{E}^0\} = \mathbf{E}^0$.

(ii) $\mathbf{E}^* = \mathbf{E}^0$.

(iii) Let $a, b \in \mathbf{E}$. Then $a =^* b$ if and only if $a = b$.

Now we need to answer the following questions:

(Q1) Does a higher-rank graph possess left cancellative property?

(Q2) If so, what can be its Garside family?

(Q3) What do the results obtained previously mean for higher-rank graphs?

Answer to **(Q1)**

A higher-rank graph indeed possesses left cancellative property.

Proposition

Let $\mathbf{E}$ be a graph of rank k. Then $\mathbf{E}$ is both left and right cancellative.

Proof. Let $v, u, w \in \mathbf{E}$ such that $vu = vw$, we set out to verify that $u = w$. Actually we have $\mathrm{d}(vu) = \mathrm{d}(v) + \mathrm{d}(u)$ and $\mathrm{d}(vw) = \mathrm{d}(v) + \mathrm{d}(w)$. Then the identity $\mathrm{d}(v) + \mathrm{d}(u) = \mathrm{d}(v) + \mathrm{d}(w)$ with unique factorization property implies that $u = w$. This means $\mathbf{E}$ is indeed left cancellative. The same argument shows that $\mathbf{E}$ is indeed right cancellative.

Answer to **(Q2)**

Let $\mathbf{E}$ be a graph of rank k. Let $S_p = \{0,1\}^k \setminus \{(0,\dots,0)\}$ be the set k-tuples whose components are only 0 or 1, without the zero tuple. Then

$$\mathbf{G} := \mathrm{d}^{-1}(S_p)$$

is a Garside family of $\mathbf{E}$.

Moreover, $\mathbf{G}$ has the following properties:

- $\mathbf{G}$ is $=^*$-transverse;

  By basic property (iii), $\mathbf{E}$ itself is already $=^*$-transverse.

- $\mathbf{G}$ is locally bounded;

  For any $\mathbf{v} \in \mathbf{E}^0$, if there were an infinite strictly increasing sequence $\mathrm{id}_{\mathbf{v}}\, s_1 \prec \mathrm{id}_{\mathbf{v}}\, s_2 \prec \cdots$ in $\mathrm{id}_{\mathbf{v}}\, \mathbf{G}$ then $\mathrm{d}(\mathrm{id}_{\mathbf{v}}\, s_1) \prec \mathrm{d}(\mathrm{id}_{\mathbf{v}}\, s_2) \preceq \cdots$ is an infinite strictly increasing sequence in S_p since $\mathrm{d}(\mathrm{id}_{\mathbf{v}}\, s_i) = \mathrm{d}(\mathrm{id}_{\mathbf{v}}) + \mathrm{d}(s_i) = \mathrm{d}(s_i)$ and $\mathrm{d}(\mathrm{id}_{\mathbf{v}}) = 0$. However, there cannot exist any infinitely increasing sequence in S_p because the greatest element in it is $(1,1,\dots,1)$.

- $\mathbf{G} \cap \mathbf{E}^* = \varnothing$.

  By basic properties (i) and (ii), $\mathrm{d}(\mathbf{E}^*) = \{0\}$.

Characterization of admissible pairs, H-invariance and $\max^{\infty}_{\leq}$-closeness in a higher-rank graph

Lemma

Let $\mathbf{E}$ be a graph or a higher-rank graph with the Garside family $\mathbf{G}$ defined above. Let $\mathbf{I}$ be a subfamily of $\mathbf{G}$ and $\mathbf{D}$ be a subfamily of $\mathbf{E}^0$.

- The pair $(\mathbf{I}, \mathbf{D})$ is admissible if and only if
  (A) for every $t \in \mathbf{I}$ there exists a $t' \in \mathbf{I}$ with $\mathrm{d}(t') \leq \mathrm{d}(t)$ or $\mathbf{d}(t) \in \mathbf{D}$.
- $(\mathbf{I}, \mathbf{D})$ is H-invariant if and only if
  (I) for every $t \in \mathbf{I} \cup \mathbf{D}$ and every atom a with $\mathbf{d}(a) = \mathbf{t}(t)$ if $\mathrm{d}(a) \not\leq \mathrm{d}(t)$ then $at \in \mathbf{I}$, and if $\mathrm{d}(a) \leq \mathrm{d}(t)$ and $t = rs$ with $\mathrm{d}(s) = \mathrm{d}(a)$ then $ar \in \mathbf{I}$.
- $(\mathbf{I}, \mathbf{D})$ is $\max^{\infty}_{\leq}$-closed if and only if
  (C) for every sequence $\{az_i\}_i$ with a fixed $a \in \mathbf{G} \cup \mathbf{E}^0$ and $\mathrm{d}(z_i) = d \in \mathbb{N}^k$ a constant tuple, if whenever $e \leq d$ is a standard basis element of $\mathbb{N}^k$ and $s_i \leq z_i$ satisfies $\mathrm{d}(s_i) = e$ we must have $s_i \neq s_j$ for all $i \neq j$, then $a \in \mathbf{I} \cup \mathbf{D}$.

Answer to (Q3)

Theorem (Farthing et al. 2005)

The transformation groupoid of a higher-rank graph $\mathbf{E}$ is a groupoid model for the Toeplitz-Cuntz-Kriger algebra $\mathcal{T}C^*(\mathbf{E})$ of $\mathbf{E}$.

Theorem

Let $\mathbf{E}$ be a countable finitely aligned higher-rank graph, with the Garside family $\mathbf{G} = \mathrm{d}^{-1}(S_p)$ and let $I_l \ltimes \Omega$ be the corresponding transformation groupoid. Then $I_l \ltimes \Omega$ is the groupoid model for the Toeplitz-Cuntz-Kriger algebra of $\mathbf{E}$, and there is an inclusion preserving one-to-one correspondence:

$$\{(I_l \ltimes \Omega)\text{-invariant closed subspaces of } \Omega\} \longrightarrow \{\text{pairs satisfying conditions } \textbf{(A)}, \textbf{(I)} \text{ and } \textbf{(C)}\}$$

$$X \longmapsto (\mathbf{I}(X), \mathbf{D}(X))$$

$$X(\mathbf{I}, \mathbf{D}) \longleftarrow\!\shortmid (\mathbf{I}, \mathbf{D})$$

with $\mathbf{I} \subseteq \mathbf{G}$ and $\mathbf{D} \subseteq \mathbf{C}^0$.

If further $\mathbf{G}$ is locally finite, then the condition **(C)** can be removed.

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