C*-algebras of Left Cancellative Small Categories with Garside Families

a quick tour

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端午安康!

Wish you health and a peaceful Dragon Boat Festival.

2023 年 6 月 22 日

农历五月初五

June 22, 2023

The 5th Day of the 5th Lunar Month

Left cancellative small categories

A small category $\mathbf{C}$ is **left cancellative** if for all $c, x, y \in \mathbf{C}$ with $\mathbf{t}(x) = \mathbf{t}(y) = \mathbf{d}(c)$,

$$cx = cy \implies x = y.$$

Let $a, b \in \mathbf{C}$ be two elements in a left cancellative small category. If there exists an element $c \in \mathbf{C}$ such that $a = bc$, then we say that b is a **left divisor** of a, written as $b \leq a$.

If $c \in \mathbf{C}^*$, we say that $a =^* b$ and $=^*$ is indeed an equivalence relation.

Proposition

Let $\mathbf{C}$ be a left cancellative small category. Then we have

$$a =^* b \iff a \in b\mathbf{C}^* \iff a\mathbf{C} = b\mathbf{C} \iff b \in a\mathbf{C}^*,$$

and hence $a \leq b, b \leq a \iff a =^* b$.

Garside theory

Garside theory

Proposition (An equivalent definition for greediness)

Let $\mathbf{C}$ be a left cancellative small category and $S \subseteq \mathbf{C}$ be a subfamily. A length-two path $g_1|g_2$ is S-greedy if and only if each relation $s \preceq g_1 g_2$ with $s \in S$ implies that $s \preceq g_1$.
A path $g_1|\cdots|g_p$ is S-greedy if and only if each relation $s \preceq g_q \cdots g_r$ with $s \in S$ implies that $s \preceq g_q$ for every $1 \leq q < r \leq p$.

Garside theory

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A path $g_1|\cdots|g_p$ is S-greedy if and only if each relation $s \preceq g_q \cdots g_r$ with $s \in S$ implies that $s \preceq g_q$ for every $1 \leq q < r \leq p$.

Definition (Garside family)

Let $\mathbf{C}$ be a left cancellative small category. A subfamily $\mathbf{G}$ of $\mathbf{C}$ is called a **Garside family** if every element of $\mathbf{C}$ admits at least one $\mathbf{G}$-normal decomposition.

The subspace Ω of $\hat{J}$

Let $\hat{J}$ be the space of characters on J. The subspace Ω of $\hat{J}$ is defined as follows:

Ω consists of characters $\chi : J \longrightarrow \{0, 1\}$ with the property that whenever $e, f_1, \ldots, f_n \in J$ satisfy $e = \bigcup_{i=1}^{n} f_i$ as subsets of $\mathbf{C}$, then $\chi(e) = 1$ implies that $\chi(f_i) = 1$ for some index i.

The topology on Ω is the subspace topology from $\hat{J}$.

Given $s \in I_l$ and $\chi \in \hat{J}$ with requirement that $\chi(s^{-1}s) = 1$, we define **the action of s on χ** as another character $s.\chi : J \longrightarrow \{0, 1\}$ by $(s.\chi)(e) = \chi(s^{-1}es)$.

Groupoid models for the left reduced C*-algebras

Left reduced C*-algebras for the small category $\mathbf{C}$

Let $\mathbf{C}$ be a left cancellative small category and $\mathbb{C}$ denotes the space of complex numbers. Define the $\ell^2(\mathbf{C})$ space to be $\ell^2(\mathbf{C}) = \left\{ f : \mathbf{C} \to \mathbb{C} \,\middle|\, \sum_{c \in \mathbf{C}} |f(c)|^2 < \infty \right\}$ with the "well-known" inner product. The standard orthonormal basis of $\ell^2(\mathbf{C})$ is given by $\{\delta_x\}_{x \in \mathbf{C}}$, where $\delta_x : \mathbf{C} \to \mathbb{C}$, $\delta_x(y) = \begin{cases} 1 \text{ if } y = x, \\ 0 \text{ if } y \neq x. \end{cases}$

For each $c \in \mathbf{C}$, we define a partial isometry λ_c by assigning $\delta_x \mapsto \delta_{cx}$ if $\mathbf{t}(x) = \mathbf{d}(c)$ and $\delta_x \mapsto 0$ if $\mathbf{t}(x) \neq \mathbf{d}(c)$ and extending by linearity on $\ell^2(\mathbf{C})$.

Definition (Left reduced C*-algebra of a left cancellative small category)

The left reduced C*-algebra of $\mathbf{C}$, denoted by $C^*_\lambda(\mathbf{C})$, is defined by the C*-algebra generated by the partial isometries $\{\lambda_c\}_{c \in \mathbf{C}}$.

Groupoid models for the left reduced C*-algebras

The transformation groupoid and its variation

The **transformation groupoid** $I_l \ltimes \Omega$ and its variation $I_l \bar{\ltimes} \Omega$ are defined to be the collection of equivalence classes on the set

$$I_l * \Omega := \{(s, \chi) \in I_l \times \Omega : \chi(s^{-1}s) = 1\}.$$

For $I_l \ltimes \Omega$, the equivalence relation $\sim$ is given by

$(s, \chi) \sim (t, \psi) \iff \chi = \psi$ and there exists an $e \in J$ with $\chi(e) = 1$ and $se = te$.

For $I_l \bar{\ltimes} \Omega$, the equivalence relation $\bar{\sim}$ is given by

$(s, \chi) \bar{\sim} (t, \psi) \iff \chi = \psi$ and there exists an $\varepsilon \in \bar{J}$ with $\chi(\mathrm{Id}_\varepsilon) = 1$ and $s\,\mathrm{Id}_\varepsilon = t\,\mathrm{Id}_\varepsilon$.

Groupoid structure:

- the source map $\mathrm{s}([s, \chi]) = \chi$ and the range map $\mathrm{r}([s, \chi]) = s.\chi$;
- multiplication $[s, t.\chi][t, \chi] = [st, \chi]$;
- inversion $[s, \chi]^{-1} = [s^{-1}, s.\chi]$.

The character from an element

Definition (The Character from an element)

Given $x \in \mathbf{C}$, we define $\chi_x : J \longrightarrow \{0,1\}$ by

$$\chi_x(e) = \begin{cases} 1, & \text{if } x\mathbf{C} \subseteq e, \\ 0, & \text{otherwise.} \end{cases}$$

Observation: $\chi_x = \chi_y$ if and only if $x\mathbf{C} = y\mathbf{C}$.

Lemma

$\{\chi_x : x \in \mathbf{C}\}$ is dense in Ω with respect to the pointwise-convergence topology.

The character from an infinite path

Let $w = s_1|s_2|\cdots$ be an infinite path. We define $w_n := s_1 \cdots s_n$ for the product of first n elements of w.

Definition (Character from an infinite path)

Let S be a subfamily of $\mathbf{C}$ which generates $\mathbf{C}$. For an (infinite) S-path w, we define a map $\chi_w : J \longrightarrow \{0,1\}$ by

$$\chi_w(e) = \begin{cases} 1, & \text{if } w_n \in e \text{ for some } n \in \mathbb{N}_+, \\ 0, & \text{otherwise.} \end{cases}$$

Let S be a subfamily of $\mathbf{C}$ generating $\mathbf{C}$. We see that for every $x \in \mathbf{C}$, χ_x is actually the character from a finite path.

Let $\Omega_\infty = \Omega \setminus \{\chi_x : x \in \mathbf{C}\}$.

Lemma ♣

Let $\mathbf{C}$ be a finitely aligned left cancellative small category which is also countable. Let S be a subfamily of $\mathbf{C}$ generating $\mathbf{C}$.
Then every $\chi \in \Omega_\infty$ is of the form χ_w for some infinite S-path.

Standard assumption

- $\mathbf{C}$ is a finitely aligned, countable, left cancellative small category;
- $\mathbf{G}$ is a Garside family of $\mathbf{C}$ which is $=^*$-transverse, locally bounded, and $\mathbf{G} \cap \mathbf{C}^* = \varnothing$.

Definitions:

Let $\mathbf{C}$ be a small category.

A subfamily S of $\mathbf{C}$ is said to be $=^*$-**transverse** if $a =^* b$ implies that $a = b$ for all $a, b \in S$.

A subfamily S of $\mathbf{C}$ is said to be **locally bounded** if for every $\mathbf{v} \in \mathbf{C}^0$ there is no infinite sequence $s_1, s_2, \ldots$ in $\mathbf{v}S$ with $s_1 \prec s_2 \prec \cdots$.

Given two S-paths $x = s_1|s_2|\cdots$ and $y = t_1|t_2|\cdots$ we mean $x = y$ by requiring $s_i = t_i$ for all indices i. In the case of finite paths, we also require that their lengths are the same.

Theorem ♠

Let $\mathbf{C}$ be a finitely aligned countable left cancellative small category. Let $\mathbf{G}$ be a Garside family of $\mathbf{C}$ which is $=^*$-transverse, locally bounded, and $\mathbf{G} \cap \mathbf{C}^* = \varnothing$. Then every $\chi \in \Omega \setminus \{\chi_{\mathbf{v}} : \mathbf{v} \in \mathbf{C}^0\}$ is of the form χ_p for some $\mathbf{G}$-normal path p. Moreover, such a path is unique, in the sense that for two normal paths p and q, $\chi_p = \chi_q$ if and only if $p = q$.

Let $\mathcal{W}$ be the collection of all $\mathbf{G}$-normal paths, then the above theorem gives a one-to-one correspondence between paths in $\mathcal{W} \sqcup \mathbf{C}^0$ and characters in Ω given by $w \mapsto \chi_w, \mathbf{v} \mapsto \chi_{\mathbf{v}}$.

Admissible pairs, H-invariance, $\max_{\leq}^{\infty}$-closeness

Let $\mathbf{C}$ be a left cancellative small category and $\mathbf{G}$ be a (nontrivial) Garside family.

Also let $\mathbf{I}$ be a subfamily of $\mathbf{G}$ and $\mathbf{D}$ be a subfamily of $\mathbf{C}^0$.

(i) The pair $(\mathbf{I}, \mathbf{D})$ is called **admissible** if for all $t \in \mathbf{I}$, either there is a $t' \in \mathbf{I}$ such that the path $t|t'$ is $\mathbf{G}$-normal or $\mathbf{d}(t) \in \mathbf{D}$.

(ii) $(\mathbf{I}, \mathbf{D})$ is called H-**invariant** if for all $a \in \mathbf{C} \backslash \mathbf{C}^*$ and $x \in \mathbf{I} \cup \mathbf{D}$ with $\mathbf{d}(a) = \mathbf{t}(x)$, $H_{\mathbf{G}}(ax)$ lies in $\mathbf{I}$.

(iii) $(\mathbf{I}, \mathbf{D})$ is called $\max_{\leq}^{\infty}$-**closed** if for every sequence $\{t_i\}_i$ in $\mathbf{I}$, if $\lim_i t_i$ exists in $\mathbf{G}$, then $\lim_i t_i \in \mathbf{I} \cup \mathbf{D}$.

Definitions:

- Let S be a subfamily of $\mathbf{C}$. Given $a \in \mathbf{C}$, an element $s \in S$ is known as an S-**head** of a if s is a greatest left divisor of a is S.

Lemma (Dehornoy et al. 2015)

If $\mathbf{G}$ is a Garside family of $\mathbf{C}$, then every non-invertible element a admits a $\mathbf{G}$-head.

In the case that S is $=^*$-transverse, the S-head is unique if it exists. In this case, the S-head of an element $a \in \mathbf{C}$ is denoted as $H_S(a)$. We may also omit S and write $H(a)$ instead when it is clear in the context.

- For a sequence $\{s^{(i)}\}$ in $\mathbf{G}$ and an element $s \in \mathbf{G} \cup \mathbf{C}^0$, we write $\lim_i s^{(i)} = s$ if s is the greatest element with respect to $\preceq$ among the set

$$\{r \in \mathbf{G} \cup \mathbf{C}^0 : r \preceq s^{(i)} \text{ for all but finitely many } i\}$$

in the sense that $s \preceq s^{(i)}$ for all but finitely many i, and every element r left dividing $s^{(i)}$ is also a left divisor of s.

Theorem ♠ implies also that there is a bijective correspondence between subsets of Ω and subsets of $\mathcal{W} \sqcup \mathbf{C}^0$.

Definitions

- Given $X \subseteq \Omega$, let $\mathcal{V}(X) = \{w \in \mathcal{W}, \mathbf{v} \in \mathbf{C}^0 : \chi_w, \chi_\mathbf{v} \in X\}$. We define

$$\mathbf{I}(X) = \{t \in \mathbf{G} : t = v_{=i} \text{ for some } v \in \mathcal{V}(X) \cap \mathcal{W} \text{ and } i \in \mathbb{N}_+\}$$

  and

$$\mathbf{D}(X) = \mathcal{V}(X) \cap \mathbf{C}^0 = \{\mathbf{v} \in \mathbf{C}^0 : \chi_\mathbf{v} \in X\}.$$

- Let $\mathbf{I}$ be a subfamily of $\mathbf{G}$ and $\mathbf{D}$ be a subfamily of $\mathbf{C}^0$. Define

$$X(\mathbf{I}, \mathbf{D}) = \{\chi_v : v_{=i} \in \mathbf{I}, \forall i \in \mathbb{N}_+\} \cup \{\chi_\mathbf{v} : \mathbf{v} \in \mathbf{D}\}.$$

Main theorem

Theorem

There is an inclusion preserving one-to-one correspondence:

$$\{(I_l \ltimes \Omega)\text{-invariant closed subspaces of } \Omega\} \longrightarrow \left\{\text{admissible, } H\text{-invariant } \max^{\infty}_{\leq}\text{-closed pairs}\right\}$$

$$X \longmapsto (\mathbf{I}(X), \mathbf{D}(X))$$

$$X(\mathbf{I}, \mathbf{D}) \longleftarrow\!\!\shortmid (\mathbf{I}, \mathbf{D})$$

with $\mathbf{I} \subseteq \mathbf{G}$ and $\mathbf{D} \subseteq \mathbf{C}^0$.

A subfamily S of $\mathbf{C}$ is said to be **locally finite** if $\mathbf{v}S$ is finite for all $\mathbf{v} \in \mathbf{C}^0$. If further $\mathbf{G}$ is locally finite, then every pair $(\mathbf{I}, \mathbf{D})$ is automatically $\max^{\infty}_{\lesssim}$-closed.

Theorem

Let $\mathbf{C}$ be a finitely aligned countable left cancellative small category and $\mathbf{G}$ is a Garside family of $\mathbf{C}$ which is $=^*$-transverse, locally bounded and $\mathbf{G} \cap \mathbf{C}^* = \varnothing$. Then we have the following:

- The transformation groupoid $I_l \ltimes \Omega$ is a groupoid model for the left reduced C*-algebra $C^*_\lambda(\mathbf{C})$.
- There is an inclusion preserving one-to-one correspondence between $I_l \ltimes \Omega$-invariant closed subspaces of Ω and admissible, H-invariant $\max^{\infty}_{\lesssim}$-closed pairs $(\mathbf{I}, \mathbf{D})$ with $\mathbf{I} \subseteq \mathbf{G}$ and $\mathbf{D} \subseteq \mathbf{C}^0$.
- If further $\mathbf{G}$ is locally finite, then the $\max^{\infty}_{\lesssim}$-closeness condition can be removed.

Higher-rank graphs

Definition (Graph of rank k)

Let k be an nonnegative integer. A **graph of rank** k (also called a k-graph) is a countable small category $\mathbf{E}$ equipped with a degree functor $\mathrm{d} : \mathbf{E} \longrightarrow \mathbb{N}^k$ satisfying the following unique factorization property: For all $e \in \mathbf{E}$ and $m, n \in \mathbb{N}^k$ with $\mathrm{d}(e) = m + n$, there are unique elements $u \in \mathrm{d}^{-1}(m)$ and $v \in \mathrm{d}^{-1}(n)$ such that $e = vu$.

We often call a graph of rank k a **higher-rank graph** when $k \geq 2$. $\mathbb{N}^k$ has a natural partial order $\leq$.

Basic properties of higher-rank graphs:

(i) $\mathrm{d}^{-1}(0) = \{\mathrm{id}_{\mathbf{v}} : \mathbf{v} \in \mathbf{E}^0\} = \mathbf{E}^0$.

(ii) $\mathbf{E}^* = \mathbf{E}^0$.

(iii) Let $a, b \in \mathbf{E}$. Then $a =^* b$ if and only if $a = b$ (and thus $\leq$ is really a partial order).

(iv) The degree functor d preserves order.

Now we need to answer the following questions:

(Q1) Does a higher-rank graph possess left cancellative property?

(Q2) If so, what can be its Garside family?

(Q3) What do the results obtained previously mean for higher-rank graphs?

Answer to **(Q1)**

A higher-rank graph indeed possesses left cancellative property.

Proposition

Let $\mathbf{E}$ be a graph of rank k. Then $\mathbf{E}$ is both left and right cancellative.

Proof. Let $v, u, w \in \mathbf{E}$ such that $vu = vw$, we set out to verify that $u = w$. Actually we have $\mathrm{d}(vu) = \mathrm{d}(v) + \mathrm{d}(u)$ and $\mathrm{d}(vw) = \mathrm{d}(v) + \mathrm{d}(w)$. Then the identity $\mathrm{d}(v) + \mathrm{d}(u) = \mathrm{d}(v) + \mathrm{d}(w)$ with unique factorization property implies that $u = w$. This means $\mathbf{E}$ is indeed left cancellative. The same argument shows that $\mathbf{E}$ is indeed right cancellative.

Answer to (Q2)

Let $\mathbf{E}$ be a graph of rank k. Let $S_p = \{0,1\}^k \setminus \{(0,\dots,0)\}$ be the set k-tuples whose components are only 0 or 1, without the zero tuple. Then

$$\mathbf{G} := \mathrm{d}^{-1}(S_p)$$

is a Garside family of $\mathbf{E}$.

Moreover, $\mathbf{G}$ has the following properties:

- $\mathbf{G}$ is $=^*$-transverse;

  By basic property (iii), $\mathbf{E}$ itself is already $=^*$-transverse.

- $\mathbf{G}$ is locally bounded;

  For any $\mathbf{v} \in \mathbf{E}^0$, if there were an infinite strictly increasing sequence $\mathrm{id}_{\mathbf{v}}\, s_1 \prec \mathrm{id}_{\mathbf{v}}\, s_2 \prec \cdots$ in $\mathrm{id}_{\mathbf{v}}\, \mathbf{G}$ then $\mathrm{d}(\mathrm{id}_{\mathbf{v}}\, s_1) \prec \mathrm{d}(\mathrm{id}_{\mathbf{v}}\, s_2) \preceq \cdots$ is an infinite strictly increasing sequence in S_p since $\mathrm{d}(\mathrm{id}_{\mathbf{v}}\, s_i) = \mathrm{d}(\mathrm{id}_{\mathbf{v}}) + \mathrm{d}(s_i) = \mathrm{d}(s_i)$ and $\mathrm{d}(\mathrm{id}_{\mathbf{v}}) = 0$. However, there cannot exist any infinitely increasing sequence in S_p because the greatest element in it is $(1,1,\dots,1)$.

- $\mathbf{G} \cap \mathbf{E}^* = \varnothing$.

  By basic properties (i) and (ii), $\mathrm{d}(\mathbf{E}^*) = \{0\}$.

Characterization of admissible pairs, H-invariance and $\max_{\preceq}^{\infty}$-closeness in a higher-rank graph

Lemma (Li 2022)

Let $\mathbf{E}$ be a graph or a higher-rank graph with the Garside family $\mathbf{G}$ defined above. Let $\mathbf{I}$ be a subfamily of $\mathbf{G}$ and $\mathbf{D}$ be a subfamily of $\mathbf{E}^0$.

- The pair $(\mathbf{I}, \mathbf{D})$ is admissible if and only if
  (A) for every $t \in \mathbf{I}$ there exists a $t' \in \mathbf{I}$ with $\mathrm{d}(t') \leq \mathrm{d}(t)$ or $\mathbf{d}(t) \in \mathbf{D}$.
- $(\mathbf{I}, \mathbf{D})$ is H-invariant if and only if
  (I) for every $t \in \mathbf{I} \cup \mathbf{D}$ and every atom a with $\mathbf{d}(a) = \mathbf{t}(t)$ if $\mathrm{d}(a) \nleq \mathrm{d}(t)$ then $at \in \mathbf{I}$, and if $\mathrm{d}(a) \leq \mathrm{d}(t)$ and $t = rs$ with $\mathrm{d}(s) = \mathrm{d}(a)$ then $ar \in \mathbf{I}$.
- $(\mathbf{I}, \mathbf{D})$ is $\max_{\preceq}^{\infty}$-closed if and only if
  (C) for every sequence $\{az_i\}_i$ with a fixed $a \in \mathbf{G} \cup \mathbf{E}^0$ and $\mathrm{d}(z_i) = d \in \mathbb{N}^k$ a constant tuple, if whenever $e \leq d$ is a standard basis element of $\mathbb{N}^k$ and $s_i \preceq z_i$ satisfies $\mathrm{d}(s_i) = e$ we must have $s_i \neq s_j$ for all $i \neq j$, then $a \in \mathbf{I} \cup \mathbf{D}$.

An element g of a left cancellative category $\mathbf{C}$ is called an **atom** if g is not invertible and g is not a product of two non-invertible elements.

Answer to (Q3)

Theorem (Farthing et al. 2005)

The transformation groupoid of a higher-rank graph $\mathbf{E}$ is a groupoid model for the Toeplitz-Cuntz-Kriger algebra $\mathcal{T}C^*(\mathbf{E})$ of $\mathbf{E}$.

Theorem

Let $\mathbf{E}$ be a countable finitely aligned higher-rank graph, with the Garside family $\mathbf{G} = \mathrm{d}^{-1}(S_p)$ and let $I_l \ltimes \Omega$ be the corresponding transformation groupoid. Then $I_l \ltimes \Omega$ is the groupoid model for the Toeplitz-Cuntz-Kriger algebra of $\mathbf{E}$, and there is an inclusion preserving one-to-one correspondence:

$$\{(I_l \ltimes \Omega)\text{-invariant closed subspaces of } \Omega\} \longrightarrow \{\text{pairs satisfying conditions } \textbf{(A)}, \textbf{(I)} \text{ and } \textbf{(C)}\}$$

$$X \longmapsto (\mathbf{I}(X), \mathbf{D}(X))$$

$$X(\mathbf{I}, \mathbf{D}) \longleftarrow (\mathbf{I}, \mathbf{D})$$

with $\mathbf{I} \subseteq \mathbf{G}$ and $\mathbf{D} \subseteq \mathbf{C}^0$.

If further $\mathbf{G}$ is locally finite, then the condition **(C)** can be removed.

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