UNIVERSITY OF COPENHAGEN
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Master thesis

C*-algebras of Left Cancellative Small Categories with Garside Families

A quick tour

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Contents

Abstract

Recent work by Xin Li has shown how to naturally associate C*-algebras to Garside categories and to present these as groupoid algebras for appropriately chosen groupoids, obtaining a unifying theory encompassing many important special cases. This thesis will be a quick tour of Garside theory on left cancellative small categories, as well as groupoids and C*-algebras arising from them, containing the fundamental results, underpinning this endeavor and leading to structural characterizations, with an application to higher-rank graphs as a typical example.

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Preliminaries

1.1 Inverse semigroups and characters

In this subsection we briefly go through necessary topics of inverse semigroups and characters roughly following Cuntz et al. [1].

Definition 1.1 (Semigroup)**.** A semigroup is a set S equipped with a binary operation $S \times S \to S$ called multiplication, satisfying associativity, that is, for all $a,b,c \in S$, $(xy)z = x(yz)$.

In other words, a semigroup is a set with a binary operation that is associative. It does not necessarily have an identity element or inverses.

Definition 1.2 (Inverse semigroup)**.** An inverse semigroup is a semigroup S with the property that for every $x \in S$, there is a unique $y \in S$ with $x = xyx$ and $y = yxy$. We write $y = x^{-1}$ and call y the inverse of x. An inverse semigroup S is called an inverse semigroup with zero if there is a distinguished element $0 \in S$ satisfying $0 \cdot x = x \cdot 0 = 0$ for all $x \in S$.

Definition 1.3 (Idempotent)**.** Let S be an invese semigroup. An element $e \in S$ is called an idempotent if $e^2 := ee = e$.

Proposition 1.4 (Basic properties of an inverse semigroup)**.** *Let S be an inverse semigroup. Then the following hold*

(i) For every x, $(x^{-1})^{-1} = x$;

(ii) For every $x \in S$, $x^{-1}x$ and xx^{-1} are both idempotents;

(iii) If e is an idempotent then $e^{-1} = e$

(iv) Every idempotent $e \in S$ is of the form $e = x^{-1}x$ for some $x \in S$.

Lemma 1.5 (Closure of idempotents under multiplication)**.** *In an inverse semigroup, the product of two idempotents is again an idempotent.*

Proof. Let S be an inverse semigroup and let $e, f \in S$ be two idempotents. We check that $(ef)^{-1} = f(ef)^{-1}e$:

$$(ef)(f(ef)^{-1}e)(ef) = ef^2(ef)^{-1}(e^2f) = ef(ef)^{-1}ef = ef$$

and

$$(f(ef)^{-1}e)(ef)(f(ef)^{-1}e) = f(ef)^{-1}e^2f^2(ef)^{-1}e = f((ef)^{-1}(ef)(ef)^{-1})e = f(ef)^{-1}e.$$

Therefore, by uniqueness of inverses in the definiton of an inverse semigroup,

$$(ef)^{-1} = f(ef)^{-1}e.$$

It follows that

$$((ef)^{-1})^2 = (f(ef)^{-1})^2 = f(ef)^{-1}ef(ef)^{-1}e = f(ef)^{-1}e = (ef)^{-1}.$$

This tells us that the inverse of ef is an idempotent. Thus $ef = (ef)^{-1}$ is also an idempotent by Proposition 1.4 (iii).

□

Lemma 1.6 (Commutativity of idempotents)**.** *In an inverse semigroup, any two idempotents commute.*

Proof. Using the notations from Lemma 1.5, we have

$$(ef)(fe)(ef) = efef = ef$$

and

$$(fe)(ef)(fe) = fefe = fe$$

so $fe = (ef)^{-1} = ef$ as desired. □

Theorem 1.7 (Inverse of a product in an inverse semigroup)**.** *Let S be an inverse semigroup, then $(xy)^{-1} = y^{-1}x^{-1}$ for all $x, y \in S$.*

Proof. We have

$$xy(y^{-1}x^{-1})xy = x(yy^{-1})(x^{-1}x)y = xx^{-1}yy^{-1}y = xy$$

and

$$y^{-1}x^{-1}(xy)y^{-1}x^{-1} = y^{-1}yy^{-1}x^{-1}xx^{-1} = y^{-1}x^{-1},$$

where we used the commutativity of idempotents from Lemma 1.6. □

Before giving the fundamental example of an inverse semigroup, we state the definition of a partial bijection.

Definition 1.8 (Partial bijection)**.** A partial bijection on a set X is a bijection between some subsets of X.

Example 1.9 (Fundamental example: partial bijections)**.** For any set X, let $I(X)$ be the set of all partial bijections on X. Every inverse semigroup can be realized as a partial bijection on a fixed set. Multiplication is given by composition. For partial bijections $s : \mathrm{dom}(s) \to \mathrm{im}(s)$ with another partial bijection $t : \mathrm{dom}(t) \to \mathrm{im}(t)$, we define the domain of their composition $s \circ t$ by

$$\mathrm{dom}(s \circ t) := \mathrm{dom}(t) \cap t^{-1}(\mathrm{dom}(s)) = t^{-1}(\mathrm{dom}(s) \cap \mathrm{im}(t))$$

to make sure that the image of the restriction of t lies in the domain of s. The inverse of a partial bijection $s : \mathrm{dom}(s) \to \mathrm{im}(s)$ is the usual set-theoretical inverse $s^{-1} : \mathrm{im}(s) \to \mathrm{dom}(s)$ Thus $\mathrm{dom}\left(s^{-1}\right) = \mathrm{im}(s)$ and $\mathrm{im}\left(s^{-1}\right) = \mathrm{dom}(s)$. In case there is partial bijection $0 : \varnothing \to \varnothing$, which is nowhere defined, we call it a zero bijection. We see that $s^{-1}s : \mathrm{dom}(s) \to \mathrm{dom}(s)$ and $ss^{-1} : \mathrm{im}(s) \to \mathrm{im}(s)$ are just the identies on $\mathrm{dom}(s)$ and $\mathrm{im}(s)$, denoted by $\mathrm{Id}_{\mathrm{dom}(s)}$ and $\mathrm{Id}_{\mathrm{im}(s)}$, respectively.

Definition 1.10 (Semilattice of idempotents)**.** The semilattice E of idempotent in an inverse semigroup S is given by:

$$E = \left\{x^{-1}x : x \in S\right\} = \left\{xx^{-1} : x \in S\right\} = \left\{e \in S : e = e^2\right\}$$

The order relation "$\leq$" on E is given by

$$e \leq f \iff e = ef$$

Note that the equalities hold because of Proposition 1.4 (iv).

Proposition 1.11. *The order $\leq$ defined above is a partial order.*

Proof. If $e \leq f$ and $f \leq e$, then $e = ef$ and $f = fe$. Since any two idempotents commute by Lemma 1.6, $e = ef = fe = e$. ☐

Example 1.12 (Idempotents of partial bijections)**.** Back to the Fundamental example 1.9, the semilattice of idempotents is given by

$$E(X) := \{\mathrm{dom}(s) : s \in I(X)\} = \{\mathrm{im}(s) : s \in I(X)\}$$

Here $\mathrm{dom}(s)$ is identified with $\mathrm{Id}_{\mathrm{dom}(s)} = s^{-1}s$, and $\mathrm{im}(s)$ is identified with $\mathrm{Id}_{\mathrm{im}(s)} = ss^{-1}$.

Multiplication in this semilattice is given by the intersection of sets, and the partial order is given by the inclusion of sets. The distinguished zero element, if it exists, is given by the empty set.

We will frequently use these identifications from Example 1.9 and Example 1.12 in this thesis.

Proposition 1.13 (Conjugation action)**.** *Let S be an inverse semigroup and let E be its semilattice of idempotents. For every $x \in S$ and $e \in E$, the conjugation $x^{-1}ex$ still lies in E. Moreover, the conjugation action preserves the partial order.*

Proof. We have

$$x^{-1}ex = x^{-1}e^{-1}ex = (ex)^{-1}(ex)$$

which is an idempotent. The first equality comes from Proposition 1.4 (iii) and the second equality is because of Theorem 1.7.

Suppose $e, f \in E$, with $e \leq f$. By definition, we have $e = ef$. Then for any $x \in S$,

$$x^{-1}ex = x^{-1}efx = x^{-1}xx^{-1}efx = (x^{-1}ex)(x^{-1}fx).$$

The second equality is bee ause $x^{-1} = x^{-1}xx^{-1}$ by the axioms of an inverse semigroup, and the third equality is because idempotents in an inverse semigroup commute. Thus $x^{-1}ex \leq x^{-1}fx$. ☐

Definition 1.14 (Character)**.** Let S be an inverse semigroup and E be its semilattice of idempotents. A character on E is a nonzero, multiplicative map $\chi : E \to \{0,1\}$ which sends $0 \in E$ (if it exists), to $0 \in \{0,1\}$.

Note that we always have $\chi(\mathrm{Id}_X) = 1$ for all characters χ because they are required to be nonzero. Let $\hat{E}$ be the set of characters on E, and we endow $\hat{E}$ with pointwise-convergence topology. $\hat{E}$ is called the space of characters. We will see that the family $\{\chi^{-1}(1) : \chi \in \hat{E}\}$ is an $\hat{E}$-valued filter, where for each $\chi \in \hat{E}$, $\chi^{-1}(1) = \{e \in E : \chi(e) = 1\}$. Let's first recall the definition of a filter on a set.

Definition 1.15 (Filter)**.** A filter $\mathscr{F}$ on a set X is a collection of subsets of X that is

(i) downward directed: $A, B \in \mathscr{F}$ implied that there exists a $C \in \mathscr{F}$ such that $C = A \cap B$;

(ii) non-empty: $\mathscr{F} \neq \varnothing$;

(iii) upward closed: $A \in \mathscr{F}$ and $A \subseteq B \subseteq X$ implies $B \in \mathscr{F}$.

The family $\mathscr{F} = \{\chi^{-1}(1) : \chi \in \hat{E}\}$ is a filter because

- $\chi^{-1}(1) \neq \varnothing$ by definition of χ. Thus item (ii) is satisfied;

- For all $e, f \in E$ (viewed as the inclusion of sets) with $e \in f$, $e \in \chi^{-1}(1)$ implies that $f \in \chi^{-1}(1)$. Thus item (iii) is satisfied;
- For all $e, f \in E$ with $e, f \in \chi^{-1}(1)$, we have $ef \in \chi^{-1}(1)$ (viewed as the intersection of sets). Thus item (i) is satisfied.

From the definition of a character, we can see further that there is a one-to-one correspondence between the set of characters in $\hat{E}$ and the filters on E. On the one hand, every $\chi \in \hat{E}$ is uniquely determined by $\chi^{-1}(1)$ simply because each χ has only two values 0 and 1. On the other hand, every filter $\mathscr{F}$ on E determines a unique $\chi \in \hat{E}$ with $\chi^{-1}(1) = \mathscr{F}$, by defining $\chi : E \to \{0,1\}$, with $\chi(e) = 1$ for all $e \in \mathscr{F}$ and zero otherwise.

Definition 1.16 (The boundary)**.** Let E be the semilattice of idempotents of and inverse semigroup. We define the boundary of the space of characters $\hat{E}$ as

$$\hat{E}_{\max} = \{\chi \in \hat{E} : \chi^{-1}(1) \text{ is maximal along all characters on } E\}.$$

Here "$\chi^{-1}(1)$ is maximal along all characters on E" means that if there is another character $\psi \in \hat{E}$ such that $\chi^{-1}(1) \subseteq \psi^{-1}(1)$, then $\chi^{-1}(1) = \psi^{-1}(1)$ (and equivalently $\chi = \psi$ by the remarks given after the definition of filters)

Let $E^* = E \setminus \{0\}$ if $0 \in E$.

Lemma 1.17. *Let E be a semilattice of idempotents of an inverse semigroup with distinguished zero elements* 0*. Suppose that $\chi \in \hat{E}_{\max}$ satisfies $\chi(e) = 0$ for some $e \in E^*$. Then there exists an $f \in \hat{E}$ with $\chi(f) = 1$ and $ef = 0$.*

Proof. If for every $f \in E$ with $\chi(f) = 1$ satisfies $ef = 0$, the we define $\mathscr{F}$ by requiring that for every $f \in E^*$, $\tilde{f} \in \mathscr{F}$ if there exists and $f \in E^*$ with $\chi(f) = 1$ and $ef \leq \tilde{f}$.

Then $\mathscr{F}$ is a filter so that there exists a character $\chi_F \in \hat{E}$ with $\chi_F^{-1}(1) = \mathscr{F}$ by the one-to-one correspondence.

By construction, we have that $\{f \in E^* : \chi(f) = 1\} \subseteq \{f \in E^* : \chi_F(f) = 1\}$. But $\chi_F(e) = 1$ while $\chi(e) = 0$. This contradicts the maximality of $\chi^{-1}(1)$. □

1.2 Groupoids

In this subsection we briefly go through necessary topics in groupoids roughly following Sims [8].

Definition 1.18 (Groupoid)**.** A groupoid is a set $\mathscr{G}$ together with a distinguished subset $\mathscr{G}^{(2)} \subseteq \mathscr{G} \times \mathscr{G}$, a multiplication map $(\alpha, \beta) \mapsto \alpha\beta$ from $\mathscr{G}^{(2)}$ to $\mathscr{G}$ and an inverse map $\gamma \mapsto \gamma^{-1}$ from $\mathscr{G}$ to $\mathscr{G}$ satisfying the following three axioms

(1) $\left(\gamma^{-1}\right)^{-1} = \gamma$ for all $\gamma \in \mathscr{G}$;

(2) if (α, β) and (β, γ) belong to $\mathscr{G}^{(2)}$, then $(\alpha\beta, \gamma)$ and $(\alpha, \beta\gamma)$ belong to $\mathscr{G}^{(2)}$ with $(\alpha\beta)\gamma = \alpha(\beta\gamma)$; and

(3) $\left(\gamma, \gamma^{-1}\right) \in \mathscr{G}^{(2)}$ for all $\gamma \in \mathscr{G}$, and for all $(\gamma, \eta) \in \mathscr{G}^{(2)}$, we have $\gamma^{-1}(\gamma\eta) = \eta$ and $(\gamma\eta)\eta^{-1} = \gamma$.

Elements in $\mathscr{G}^{(2)}$ are called multiplicable pairs. Elements of the form $\gamma^{-1}\gamma$ for some $\gamma \in \mathscr{G}$ are called units. Let $\mathscr{G}^{(0)}$ denote the set of units. $\mathscr{G}^{(0)}$ is called the unit space.

We see from the axioms of a groupoid that not every two elements are multiplicable. Let U and V be two subsets of $\mathscr{G}$, then by axioms we write

$$UV := \{\alpha\beta : \alpha, \beta \in \mathscr{G} \text{ with } (\alpha, \beta) \in \mathscr{G}^{(2)}\}.$$

Definition 1.19 (Source and range)**.** Let $\mathscr{G}$ be a groupoid. For every $\gamma \in \mathscr{G}$ we define its source $\mathrm{s}(\gamma) := \gamma^{-1}\gamma$ and its range $\mathrm{r}(\gamma) := \gamma\gamma^{-1}$. This gives two maps $\mathrm{s}, \mathrm{r} : \mathscr{G} \to \mathscr{G}^{(0)} \subseteq \mathscr{G}$.

Proposition 1.20 (Basic properties of a groupoid)**.** *Let $\mathscr{G}$ be a groupoid.*

(i) For every $\gamma \in \mathscr{G}$, $(\mathrm{r}(\gamma), \gamma), (\gamma, \mathrm{s}(\gamma)) \in \mathscr{G}$ and $\mathrm{r}(\gamma)(\gamma) = \gamma = \gamma\mathrm{s}(\gamma)$;

(ii) For every $\gamma \in \mathscr{G}$, $\mathrm{r}(\gamma^{-1}) = \mathrm{s}(\gamma)$ and $\mathrm{s}(\gamma^{-1}) = \mathrm{r}(\gamma)$;

(iii) Cancellation law: If $(\alpha, \gamma), (\beta, \gamma) \in \mathscr{G}^{(2)}$ such that $\alpha\gamma = \beta\gamma$, then $\alpha = \beta$. Similarly if $(\gamma, \alpha), (\gamma, \beta) \in \mathscr{G}^{(2)}$ such that $\gamma\alpha = \gamma\beta$, then $\alpha = \beta$;

(iv) For all $\gamma \in \mathscr{G}^{(0)}$, we have $\mathrm{r}(\gamma) = \gamma = \mathrm{s}(\gamma)$.

Lemma 1.21 (Uniqueness)**.** *Let $\mathscr{G}$ be a groupoid and let gamma be its element. Then γ^{-1} is the unique element such that $(\gamma, \gamma^{-1}) \in \mathscr{G}^{(2)}$ and $\gamma\gamma^{-1} = \mathrm{r}(\gamma)$, and also the unique element such that $(\gamma^{-1}, \gamma) \in \mathscr{G}^{(2)}$ and $\gamma^{-1}\gamma = \mathrm{s}(\gamma)$.*

Proof. If $(\gamma, \alpha) \in \mathscr{G}^{(2)}$ and $\gamma\alpha = r(\gamma) = \gamma\gamma^{-1}$, then axiom (2) shows that $\left(\gamma^{-1}\gamma, \alpha\right) \in \mathscr{G}^{(2)}$ and $\alpha = \gamma^{-1}\gamma\alpha = \gamma^{-1}r(\gamma) = \gamma^{-1}s\left(\gamma^{-1}\right) = \gamma^{-1}$. A similar argument shows that $\alpha\gamma = s(\gamma)$ only for $\alpha = \gamma^{-1}$ ☐

Lemma 1.22 (Compatiability of the source and range)**.** *Let $\mathscr{G}$ be a groupoid. Then $(\alpha, \beta) \in \mathscr{G}^{(2)}$ if and only if $\mathrm{s}(\alpha) = \mathrm{r}(\beta)$.*

Proof. Suppose that $\mathrm{s}(\alpha) = \mathrm{r}(\beta)$, that is $\alpha^{-1}\alpha = \beta\beta^{-1}$. Then we have $(\alpha, \beta\beta^{-1}) = (\alpha, \alpha^{-1}\alpha) \in \mathscr{G}^{(2)}$. Since $(\beta\beta^{-1}, \beta)$ is also in $\mathscr{G}^{(2)}$, then by axiom (2) $(\alpha, \beta) = (\alpha, \beta\beta^{-1}\beta) \in \mathscr{G}^{(2)}$. Conversely, suppose $(\alpha, \beta) \in \mathscr{G}^{(2)}$, then by axiom (1) and (2), $(\alpha^{-1}, \alpha\beta) \in \mathscr{G}^{(2)}$ with $\alpha^{-1}\alpha\beta = \beta = \mathrm{r}(\beta)$. Then Proposition 1.20 (iii) cancellation law shows that $\mathrm{s}(\alpha) = \alpha^{-1}\alpha = \mathrm{r}(\beta)$. ☐

Lemma 1.23. *Let $\mathscr{G}$ be a groupoid. If $(\alpha, \beta) \in \mathscr{G}^{(2)}$ then so is $(\beta^{-1}, \alpha^{-1})$.*

Proof. Suppose that $(\alpha, \beta) \in \mathscr{G}^{(2)}$. Then

$$(\beta^{-1}, \mathrm{s}(\beta^{-1})) = (\beta^{-1}, \mathrm{r}(\beta)) = (\beta^{-1}, \mathrm{s}(\alpha)) = (\beta, \mathrm{r}(\alpha^{-1})) \in \mathscr{G}^{(2)},$$

where the first and third equalities are from Proposition 1.20 (ii), and the second equality id from Lemma 1.23. Also, $(\mathrm{r}(\alpha^{-1}), \alpha^{-1}) \in \mathscr{G}^{(2)}$ by Lemma 1.23.

Therefore, $(\beta^{-1}, \alpha^{-1}) = (\beta, \alpha^{-1}\alpha\alpha^{-1}) = (\beta^{-1}, \mathrm{r}(\alpha^{-1})\alpha^{-1}) \in \mathscr{G}^{(2)}$. ☐

Theorem 1.24. *Let $\mathscr{G}$ be a groupoid. Then $(\alpha\beta)^{-1} = \beta^{-1}\alpha^{-1}$ for all $(\alpha, \beta) \in \mathscr{G}^{(2)}$.*

Proof. $(\beta^{-1}, \alpha^{-1}) \in \mathscr{G}^{(2)}$ is seen in Lemma 1.23. We have $(\alpha\beta)(\beta^{-1}\alpha^{-1}) = \alpha\alpha^{-1}$. Multiplying $(\alpha\beta)^{-1}$ on the left of both sides and by axiom (3), we have

$$\beta^{-1}\alpha^{-1} = (\alpha\beta)^{-1}(\alpha\beta)\beta^{-1}\alpha^{-1} = (\alpha\beta)^{-1}\alpha\alpha^{-1} = (\alpha\beta)^{-1}.$$

☐

Proposition 1.25 (Characterization of the unit space)**.** *Let $\mathscr{G}$ be a groupoid. Then the unit space $\mathscr{G}^{(0)}$ has an explicit form*

$$\mathscr{G}^{(0)} = \{\gamma \in \mathscr{G} : (\gamma, \gamma) \in \mathscr{G}^{(2)} \text{ with } \gamma^2 = \gamma\}.$$

Also, $\gamma = \gamma^{-1}\gamma = \gamma\gamma^{-1}$ for every $\gamma \in \mathscr{G}^{(0)}$.

Proof. For every $\gamma \in \mathscr{G}$, $\gamma = \alpha^{-1}\alpha$ for some $\alpha \in \mathscr{G}$. Then $(\gamma,\gamma) \in \mathscr{G}^{(2)}$ because $(\alpha^{-1},\alpha),(\alpha,\alpha^{-1}) \in \mathscr{G}^{(2)}$ give $(\alpha^{-1}\alpha,\alpha^{-1}) \in \mathscr{G}^{(2)}$, together with $(\alpha^{-1},\alpha) \in \mathscr{G}^{(2)}$ again yielding $(\alpha^{-1}\alpha,\alpha^{-1}\alpha) = (\gamma,\gamma) \in \mathscr{G}^{(2)}$, by repeatedly using axioms (1) and (2). Conversely, suppose that $(\gamma,\gamma) \in \mathscr{G}^{(2)}$, with $\gamma^2 = \gamma$. Then $\gamma^2 = \gamma = \gamma(\gamma^{-1}\gamma)$. Using Proposition 1.20 (iii) cancellation law, $\gamma = \gamma^{-1}\gamma \in \mathscr{G}^{(0)}$. □

Corollary 1.26. *If $\gamma,\eta \in \mathscr{G}^{(0)}$, then $(\gamma,\eta) \in \mathscr{G}^{(2)}$ (if and) only if $\gamma = \eta$.*

Proof. If $(\gamma,\eta) \in \mathscr{G}^{(2)}$ then by Lemma 1.22 and Proposition 1.25, $\gamma = \mathrm{s}(\gamma) = \mathrm{r}(\eta) = \eta$. □

Definition 1.27 (Groupoid homomorphism)**.** Let $\mathscr{G}$ and $\mathscr{H}$ be two groupoids. A map $\varphi : \mathscr{G} \to \mathscr{H}$ is a groupoid homomorphism if for all $(\alpha,\beta) \in \mathscr{G}^{(2)}$, $(\varphi(\alpha),\varphi(\beta)) \in \mathscr{H}^{(2)}$ and $\varphi(\alpha\beta) = \varphi(\alpha)\varphi(\beta)$.

Proposition 1.28. *Let $\mathscr{G}$ and $\mathscr{H}$ be groupoids and $\varphi : \mathscr{G} \to \mathscr{H}$ be a groupoid homomorphism. Then for every $\gamma \in \mathscr{G}$,*

(i) $\varphi(\mathscr{G}^{(0)}) \subseteq \mathscr{H}^{(0)}$;

(ii) $\varphi(\gamma^{-1}) = \varphi(\gamma)^{-1}$;

(iii) $\varphi(\mathrm{s}(\gamma)) = \mathrm{s}(\varphi(\gamma))$ *and* $\varphi(\mathrm{r}(\gamma)) = \mathrm{r}(\varphi(\gamma))$.

Proof. (i) We use the characterization in Proposition 1.25. Let $\gamma \in \mathscr{G}^{(0)}$. Then $(\gamma,\gamma) \in \mathscr{G}^{(2)}$ and $\gamma^2 = \gamma$. Then $(\varphi(\gamma),\varphi(\gamma)) \in \mathscr{H}^{(2)}$ and $\varphi(\gamma) = \varphi(\gamma^2) = \varphi(\gamma)\varphi(\gamma) = \varphi(\gamma)^2$, so that $\varphi(\gamma) \in \mathscr{H}^{(0)}$.

(ii) By axiom (3) of a groupoid, $(\gamma^{-1},\gamma) \in \mathscr{G}^{(2)}$ and hence $(\varphi(\gamma^{-1}),\varphi(\gamma)) \in \mathscr{H}^{(2)}$ with $\varphi(\gamma^{-1}\gamma) = \varphi(\gamma^{-1})\varphi(\gamma)$. Multiplying $\varphi(\gamma)^{-1}$ on the right of each term of both sides and using (i),

$$\varphi(\gamma)^{-1} = \varphi(\gamma^{-1}\gamma)\varphi(\gamma)^{-1} = \varphi(\gamma^{-1})\varphi(\gamma)\varphi(\gamma)^{-1} = \varphi(\gamma^{-1}).$$

(iii) This can be directly checked using (ii).

□

Let $\mathscr{G}$ be a groupoid. For a given $x \in \mathscr{G}^{(0)}$ we write $\mathscr{G}_x := \{\gamma \in \mathscr{G} : \mathrm{s}(\gamma) = x\}$, $\mathscr{G}^x := \{\gamma \in \mathscr{G} : \mathrm{r}(\gamma) = x\}$ and $\mathscr{G}_x^y := \mathscr{G}_x \cap \mathscr{G}^y$.

Definition 1.29 (Isotropy subgroupoid)**.** Let $\mathscr{G}$ be a groupoid. The isotropy subgroupoid of $\mathscr{G}$ is defined as the subset

$$\mathrm{Iso}(\mathscr{G}) := \bigcup_{x \in \mathscr{G}^{(0)}} \mathscr{G}_x^x = \{\gamma \in \mathscr{G} : \mathrm{r}(\gamma) = \mathrm{s}(\gamma)\}$$

endowed with the same multiplication and inverse as for $\mathscr{G}$.

It is straightforward to see that the isotropy subgroupoid is indeed a subgroupoid.

Definition 1.30 (Topological groupoid)**.** A topological groupoid is a groupoid $\mathscr{G}$ endowed with a locally compact topology under which $\mathscr{G}^0 \subseteq \mathscr{G}$ is Hausdorff in the relative topology. Moreover, we require that the source s, the range r and the inverse operation $(\cdot)^{-1}$ are continuous, and the multiplication $(\alpha,\beta) \mapsto \alpha\beta$ with respect to the relative topology on $\mathscr{G}^{(2)}$ as a subset of $\mathscr{G} \times \mathscr{G}$ is also continuous.

Definition 1.31 (Étale groupoid)**.** A topological groupoid $\mathscr{G}$ is étale if the range map $\mathrm{r} : \mathscr{G} \to \mathscr{G}$ is a local homeomorphism.

Lemma 1.32. *If $\mathscr{G}$ is an étale groupoid, then $\mathscr{G}^{(0)}$ is open in $\mathscr{G}$.*

Proof. For every $\gamma \in \mathscr{G}$ we can choose an open subset U_γ such that $\mathrm{r} : U_\gamma \to \mathrm{r}(U_\gamma)$ is a homeomorphism onto an open set. Then $\mathscr{G}^{(0)} = \bigcup_{\gamma \in \mathscr{G}} r(U_\gamma)$ is open. □

Definition 1.33 (Bisection)**.** A subset B of an étale groupoid $\mathscr{G}$ is called a bisection if there is an open subset U containing B such that $\mathrm{r} : U \to \mathrm{r}(U)$ and $\mathrm{s} : U \to \mathrm{s}(U)$ are both homeomorphisms onto their images.

We will mainly focus on second-countable Hausdorff étale groupoids because they possess some nice properties.

Lemma 1.34. *Every second-countable Hausdorff étale groupoid $\mathscr{G}$ has a countable base of open bisections.*

Proof. Recall from topology that a second countable topological space is separable, so we can take countable dense subset $\{\gamma_n\}_{n=1}^{\infty}$ of $\mathscr{G}$. For each γ_n we can choose countable neighborhood bases $\{U_{n,i}\}_i$ and $\{V_{n,i}\}_i$ at γ_n such that the range map is a homeomorphism from each $U_{n,i}$ onto an open set and the source map is also a homeomorphism from each $V_{n,i}$ onto an open set. Then we obtain the family $\{U_{n,i} \cap V_{n,i} : n, i \in \mathbb{N}\}$ which is a countable base of open bisections. ☐

From Lemma 1.34 we immediately have the following conclusion.

Corollary 1.35. *Let $\mathscr{G}$ be an étale groupoid. Then $\mathscr{G}_x$ and $\mathscr{G}^x$ are both discrete sets.*

Proof. It suffices to show that each singleton $\{\gamma\}$ with $\gamma \in \mathscr{G}_x$ is open in $\mathscr{G}_x$.

For each $\gamma \in \mathscr{G}_x$ we can choose an open bisection U_γ containing γ. Then $U_\gamma \cap \mathscr{G}_x = \{\gamma\}$ is a singleton because $\mathrm{s}|_{U_\gamma} : U_\gamma \to \mathrm{s}(U_\gamma) \subseteq \mathscr{G}$ is a homeomorphism with $\mathrm{s}(U_\gamma \cap \mathscr{G}_x) = x$ by definition. Therefore $\{\gamma\}$ is open in $\mathscr{G}_x$. ☐

1.3 Left cancellative small categories

Definition 1.36 (Category)**.** A category $\mathbf{C}$ consists of the following data:

(i) A collection of objects $\mathrm{Ob}(\mathbf{C})$;

(ii) For every pair of objects $A, B \in \mathrm{Ob}(\mathbf{C})$, a collection of "arrows" called morphisms $\mathrm{Hom}_{\mathbf{C}}(A, B)$;

(iii) For every triple of objects $A, B, C \in \mathrm{Ob}(\mathbf{C})$, a binary operation $\circ : \mathrm{Hom}_{\mathbf{C}}(B,C) \times \mathrm{Hom}_{\mathbf{C}}(A,B) \to \mathrm{Hom}_{\mathbf{C}}(A,C)$, called composition, which satisfies the following axioms:

  (a) Associativity: For all $f \in \mathrm{Hom}_{\mathbf{C}}(C,D), g \in \mathrm{Hom}_{\mathbf{C}}(B,C), h \in \mathrm{Hom}_{\mathbf{C}}(A,B)$, we have $f \circ (g \circ h) = (f \circ g) \circ h$;

  (b) Identity: For every object $A \in \mathrm{Ob}(\mathbf{C})$, there exists a morphism $\mathrm{id}_A \in \mathrm{Hom}_{\mathbf{C}}(A,A)$, called the identity morphism, such that for all $f \in \mathrm{Hom}_{\mathbf{C}}(A,B)$, we have $f \circ \mathrm{id}_A = f$ and $\mathrm{id}_B \circ f = f$.

Definition 1.37 (Isomorphism)**.** In a category $\mathbf{C}$, an isomorphism is a morphism $f : A \to B$ between two objects A and B that has an inverse, i.e., a morphism $g : B \to A$ such that $f \circ g = \mathrm{id}_B$ and $g \circ f = \mathrm{id}_A$, where id_A and id_B are the identity morphisms on A and B, respectively. We also say that f is invertible.

Definition 1.38 (Small category)**.** A small category $\mathbf{C}$ is a category where the collection of objects $\mathrm{Ob}(\mathbf{C})$ and the collection of morphisms $\mathrm{Hom}_{\mathbf{C}}(A,B)$ between any two objects $A, B \in \mathrm{Ob}(\mathbf{C})$ are sets.

A very typical example of a small category is a groupoid we have just seen before.

Example 1.39 (Monoid and groupoid as small categories)**.** From the categorical perspective,

- A monoid is a small category with only one object.

- A groupoid is a small category in which every morphism is an isomorphism.

For the sake of convenience, in a small category $\mathbf{C}$ we often abuse notations by writing $\mathbf{C} = \mathrm{Mor}(\mathbf{C})$ to refer to the set of morphisms. The set of isomorphisms is denoted by $\mathbf{C}^*$. The set of objects is denoted by $\mathbf{C}^0$. By identifying each $\mathbf{v} \in \mathbf{C}^0$ as the identity morphism $\mathrm{id}_\mathbf{v}$, we regard $\mathbf{C}^0$ as a subset of $\mathbf{C}^*(\subseteq \mathbf{C})$.

Let $\mathbf{d} : \mathbf{C} \to \mathbf{C}^0$ and $\mathbf{t} : \mathbf{C} \to \mathbf{C}^0$ be the maps indicating the source and the target of a morphism respectively. By the definition of a category, we require that for every $c, d \in \mathbf{C}$, the composition cd if and only if $\mathbf{t}(d) = \mathbf{d}(c)$, that is, $\mathbf{d}(d) \xrightarrow{d} \mathbf{t}(d) = \mathbf{d}(c) \xrightarrow{c} \mathbf{t}(c)$.

Given $c \in \mathbf{C}$ and $S \subseteq \mathbf{C}$, we define $cS = \{cs : s \in S, \mathbf{t}(s) = \mathbf{d}(c)\}$. We call the subset of the form $c\mathbf{C}$ a principal ideal of $\mathbf{C}$. For an object $\mathbf{v} \in \mathbf{C}^0$, we can simply write $\mathbf{v}\mathbf{C}$ for the principle ideal $\mathrm{id}_\mathbf{v}\,\mathbf{C}$.

In the remaining part of the thesis, we will mainly focus on left cancellative small categories.

Definition 1.40 (Left cancellative small category)**.** A small category $\mathbf{C}$ is left cancellative if for all $c, x, y \in \mathbf{C}$ with $\mathbf{t}(x) = \mathbf{t}(y) = \mathbf{d}(c)$,

$$cx = cy \Longrightarrow x = y.$$

That is to say, we can cancel the same elements from the left in an equality.

Definition 1.41 (Left divisor, Right multiple)**.** Let $\mathbf{C}$ be a left cancellative small category. For a given $b \in \mathbf{C}$, we say an element $a \in \mathbf{C}$ is a left divisor of b if $b = ac$ for some $c \in \mathbf{C}$ (with $\mathbf{t}(c) = \mathbf{d}(a)$). We also say that b is a right multiple of a, and we write this by $a \preceq b$. If $a \preceq b$ but $a \neq b$, we say that a is a strict (or proper) left divisor of b, written as $a \prec b$.

For each $b \in \mathbf{C}$, the set of left divisors of b is written as $\mathrm{Div}(b)$. Some easy but useful properties come as follows.

Proposition 1.42. *Let* $\mathbf{C}$ *be a small category. Then*

(i) $a \preceq b \iff b \in a\mathbf{C} \iff b\mathbf{C} \subseteq a\mathbf{C}$*;*

(ii) Transitivity: $a \preceq b, b \preceq c \Rightarrow a \preceq c$*.*

The proof of Proposition 1.42 is just a direct verification. We see that the left divisibility relation $\preceq$ behaves somewhat like a partial order.

Definition 1.43 (Closed under left divisors)**.** A subfamily S of a small category $\mathbf{C}$ is said to be closed under left divisors if for all $s \in S$, every left divisor of s still lies in S.

Definition 1.44 (Greatest left divisor in a subfamily)**.** Let S be a subfamily of a small category $\mathbf{C}$. Given $a \in \mathbf{C}$, an element $s \in S$ is a divisor of a in S if $s \preceq a$ and is the greatest in S in the sense that every $r \in S$ with $r \preceq a$ satisfies $r \preceq s$.

Definition 1.45 (Closed under right comultiples)**.** Let $\mathbf{C}$ be a small category. A subset S of $\mathbf{C}$ is a closed under right comultiples if for all $r, s \in S$ and $a \in \mathbf{C}$ with $r \preceq a$ and $s \preceq a$, there exists a $t \in S$ with $r \preceq t, s \preceq t$ and $t \preceq a$.

Definition 1.46 (Left Noetherian)**.** A small category $\mathbf{C}$ is called left Noetherian if there exists no infinite sequence $\cdots \prec a_3 \prec a_2 \prec a_1$ in $\mathbf{C}$.

A small category $\mathbf{C}$ is left Noetherian means that for every $a \in \mathbf{C}$, every strictly decreasing sequence in $\mathrm{Div}(a)$ with respect to left divisibility is finite.

Similarly, we can define the concepts of right cancellative small categories, right divisors, and right Noetherianity. We mentioned that a small category is said to be Noetherian if it is both left and right Noetherian.

Definition 1.47 (Atom)**.** An element g of a left cancellative category $\mathbf{C}$ is called an atom if g is not invertible and g is not a product of two non-invertible elements.

Definition 1.48 ($=^*$-equivalence)**.** Let $a,b \in \mathbf{C}$ be two elements in a left cancellative small category. We say that $a =^* b$ if there exists an element $c \in \mathbf{C}^*$ such that $a = bc$.

Remark 1.49. It is easy to see that the binary relation $=^*$ on $\mathbf{C}$ is indeed an equivalence relation. A special warning is that one may thinks that any two invertible elements are $=^*$-equivalent, for $b = aa^{-1}b$ for all $a,b \in \mathbf{C}^*$. In fact, this is not true. This is because $\mathbf{t}(b)$ may not be the same as $\mathbf{d}(a^{-1})$ as required in the definition of composition in a category. In particular, we note that $\mathbf{vC} \cap \mathbf{wC} = \varnothing$ for distinct $\mathbf{v},\mathbf{w} \in \mathbf{C}^0$, but $\mathbf{C}^0 \subseteq \mathbf{C}^*$. Therefore, $\mathbf{C}^*$ is actually not an equivalence class itself.

Proposition 1.50. *Let* $\mathbf{C}$ *be a left cancellative small category. Then we have*

$$a =^* b \iff a \in b\mathbf{C}^* \iff a\mathbf{C} = b\mathbf{C} \iff b \in a\mathbf{C}^*,$$

and hence $a \preceq b, b \preceq a \iff a =^* b$.

Proof. The first equivalence is straightforward. Also, if the second equivalence holds, then by "symmetry" so does the third one. Thus we focus on showing the second equivalence.

Suppose $a \in b\mathbf{C}^*$. Then $a = bc_0$ for some $c_0 \in \mathbf{C}^*$. For all $c \in \mathbf{C}$ such that $\mathbf{t}(c) = \mathbf{d}(a) = \mathbf{d}(c_0)$, we have $ac = bc_0c = b(c_0c) \in b\mathbf{C}$ because c_0c is still an element of $\mathbf{C}$. Thus $a\mathbf{C} \subseteq b\mathbf{C}$. For $bc \in b\mathbf{C}$ with $\mathbf{t}(c) = \mathbf{d}(b)$, we have $bc = bc_0c_0^{-1}c = a(c_0^{-1}c) \in a\mathbf{C}$ with $\mathbf{t}(c) = \mathbf{d}(c_0^{-1}) = \mathbf{t}(c_0)$. Thus $b\mathbf{C} \subseteq a\mathbf{C}$. Therefore $b\mathbf{C} = a\mathbf{C}$.

Conversely, suppose $a\mathbf{C} = b\mathbf{C}$. Then $a = bc$ for some $c \in \mathbf{C}$ and $b = ac'$ for some $c' \in \mathbf{C}$. Thus $b = b\,\mathrm{id}_{\mathbf{d}(b)} = ac' = bcc'$ and $a = a\,\mathrm{id}_{\mathbf{d}(a)} = bc = ac'c$. By left cancellative property of $\mathbf{C}$, we have $cc' = \mathrm{id}_{\mathbf{d}(b)}$ and $c'c = \mathrm{id}_{\mathbf{d}(a)}$. By definition of isomorphisms, both c and c' lie in $\mathbf{C}^*$, so that $a = bc \in b\mathbf{C}^*$. □

From Proposition 1.50 we see in particular that the greatest left divisor of each element in $\mathbf{C}$ is unique up to $=^*$-equivalence.

We know in general that the product of two invertible elements is again invertible. In fact, the converse also holds in a left cancellative small category.

Proposition 1.51. *Let* $\mathbf{C}$ *be a left cancellative small category. The product* ab *of two elements* $a,b \in \mathbf{C}$ *with* $\mathbf{t}(b) = \mathbf{d}(a)$ *is invertible (i.e., in* $\mathbf{C}^*$*) if and only if both* a *and* b *are invertible.*

Proof. If $a,b \in \mathbf{C}^*$ with $\mathbf{t}(b) = \mathbf{d}(a)$ then so is ab. The crucial part is to see that ab is invertible implying that both a and b are invertible.

Let $x \in \mathbf{C}$ be such that $abx = \mathrm{id}_{\mathbf{d}(x)}$ and $xab = \mathrm{id}_{\mathbf{d}(b)}$. Let $y = bxa$ the $yb = bxab = b\,\mathrm{id}_{\mathbf{d}(b)} = b$. By left cancellative property of $\mathbf{C}$ we have that $(bx)a = y = \mathrm{id}_{\mathbf{t}(b)} = \mathrm{id}_{\mathbf{d}(a)}$. Thus a is invertible, and b is also invertible because $b = a^{-1}(ab)$ is a product of two elements. □

Definition 1.52 (Commom right multiple)**.** Let a,b be two elements in a small category $\mathbf{C}$. An element $c \in \mathbf{C}$ is a common right multiple of a and b if there is an element $a' \in \mathbf{C}$ for a and $b' \in \mathbf{C}$ for b such that $c = aa' = bb'$. Equivalently, $c \in a\mathbf{C} \cap b\mathbf{C}$.

Definition 1.53 (Minimal common right multiple)**.** A common right multiple of a and b in a small category $\mathbf{C}$ is called a minimal common right multiple if in addition, there is no proper left divisor d of c that is still a common right multiple of a and b.

Given two elements a,b in a small category $\mathbf{C}$, there may be many or no common right multiples of them. In this thesis, we write $\mathrm{mcm}(a,b)$ for the set of minimal common right multiples of a and b. In the case of finitely aligned left cancellative small categories, the minimal common right multiples of any two given elements exist are finite up to $=^*$-equivalence. This means there is no common right multiple for distinct objects.

Definition 1.54 (Finitely aligned small category)**.** A small category $\mathbf{C}$ is finitely aligned if for all $a,b \in \mathbf{C}$ there exists a finite subset $F \subseteq \mathbf{C}$ such that $a\mathbf{C} \cap b\mathbf{C} = \bigcup_{c \in F} c\mathbf{C}$.

Observe that inductively, a small category $\mathbf{C}$ is finitely aligned if and only if for finitely many $a_1, a_2, \ldots, a_n \in \mathbf{C}$, there is a finite subset F of $\mathbf{C}$ such that $\bigcap_{i=1}^{n} a_i\mathbf{C} = \bigcup_{c \in F} c\mathbf{C}$.

Lemma 1.55. *Let $\mathbf{C}$ be a left cancellative small category and $a,b \in \mathbf{C}$. Then $\mathbf{C}$ is finitely aligned if and only if $\mathrm{mcm}(a,b)$ is non-empty and contains only finitely many $=^*$-classes.*

Proof. By definition, there exists a finite subset $F \subseteq \mathbf{C}$ such that $a\mathbf{C} \cap b\mathbf{C} = \bigcup_{c \in F} c\mathbf{C}$. We can arrange F so that each $=^*$-equivalence contains only one element by choosing a representative for each equivalence class in $a\mathbf{C} \cap b\mathbf{C} = \bigcup_{c \in F} c\mathbf{C}$ and delete the other $=^*$-equivalent elements without affecting the union. Therefore we may assume that any two elements are not $=^*$-equivalent in F.

We claim that $F \subseteq \mathrm{mcm}(a,b)$. Let $c \in F$. Suppose that $d \in a\mathbf{C} \cap b\mathbf{C}$ with $c \in d\mathbf{C}$. Then there is a $c' \in F$ such that $d \in c'\mathbf{C}$ by finite alignment of $\mathbf{C}$. Then $c \in c'\mathbf{C}$, so $c = c'$. Hence $d \in c\mathbf{C}$. There fore $d =^* c$ and $c \in \mathrm{mcm}(a,b)$. We further claim that for all $f \in \mathrm{mcm}(a,b)$ there is a $c \in F$ such that $f =^* c$. To see this, let $f \in \mathrm{mcm}(a,b)$. Then $f \in a\mathbf{C} \cap b\mathbf{C}$, so there is a $c \in f$ such that $f \in c\mathbf{C}$. Since f is minimal, we have that $c =^* f$. Therefore, $\mathrm{mcm}(a,b)$ contains only finitely many $=^*$-equivalence class.

Conversely suppose $\mathrm{mcm}(a,b)$ is nonempty and contains only finitely many $=^*$-classes. By keeping only one element in each $=^*$-equivalence class, we may assume that $\mathrm{mcm}(a,b)$ is a finite set. Note that each element $d \in a\mathbf{C} \cap b\mathbf{C}$ is a common right multiple of a and b, hence must be a multiple of some minimal common right multiple of a and b. That is to say, $d \in c\mathbf{C}$ for some $c \in \mathrm{mcm}(a,b)$. Thus $a\mathbf{C} \cap b\mathbf{C} = \bigcup_{c \in \mathrm{mcm}(a,b)} c\mathbf{C}$ is the union of finitely many principal ideals. $\square$

♪ 2

Inverse semigroups from left cancellative small categories

Let $\mathbf{C}$ be a left cancellative small category. Every $c \in \mathbf{C}$ induces a partial bijection $\sigma_c : \mathbf{d}(c)\mathbf{C} \longrightarrow c\mathbf{C}$ given by $x \mapsto cx$. We note that the left cancellativity assumption is essential because the injectivity of σ_c comes from it. Using these partial bijections we can generate an inverse semigroup, as in the fundamental example 1.9 given before.

Remark 2.1. Given $c \in \mathbf{C}$, very often we may denote the induced partial bijection σ_c by c again, but here we will keep the notation σ_c instead of c to avoid confusion between the inverse function σ_c^{-1} (which must exist by construction) of σ_c and the inverse of invertible elements c^{-1} (which does not necessarily exist) of c in $\mathbf{C}$.

Definition 2.2 (Left inverse hull of induced partial bijections)**.** Let $\mathbf{C}$ be a left cancellative small category. The inverse hull of $\mathbf{C}$ is defined to be the inverse semigroup generated by the set of all the partial bijections $\{\sigma_c : c \in \mathbf{C}\}$. That is, the smallest inverse semigroup containing all the partial bijections which is closed under composition and inverses. We denote such an inverse semigroup by I_l.

Definition 2.3 (A variation of the inverse semigroup I_l)**.**

- Define $\bar{J}$ to be the collection of subsets of $\mathbf{C}$ of the form $e \setminus \bigcup_{i=1}^{n} f_i$ for some $e, f_1, \ldots, f_n \in J$ with $f_1, \ldots, f_n \leq e$.
- Define $\bar{I}_l$ to be the set of all partial bijections on $\mathbf{C}$ of the form $s\,\mathrm{Id}_\varepsilon$ with $s \in \bar{I}_l$ and $\varepsilon \in \bar{J}$ such that $\varepsilon \leq s^{-1}s$.

Note that J is naturally a subset of $\bar{J}$. I_l is again an inverse semigroup, whose semilattice of idempotents is given by $\bar{J}$.

The following proposition is a characterization of elements of I_l.

Proposition 2.4 (Characterization of I_l)**.** *Every element of I_l is of the so-called zigzag form*

$$\sigma_{d_n}^{-1}\sigma_{c_n}\cdots\sigma_{d_1}^{-1}\sigma_{c_1}$$

for some $d_i, c_i \in \mathbf{C}$ with $\mathbf{t}(c_i) = \mathbf{t}(d_i)$ and $\mathbf{d}(d_i) = \mathbf{d}(c_{i+1}), i = 1, 2, \ldots, n,\ n \in \mathbb{N}_+$.

Proof. Every element of I_l, by definition, is of the form $\sigma_{a_m}\sigma_{a_{m-1}}\cdots\sigma_{a_1}$, where $m \in \mathbb{N}$ and $a_1, \ldots, a_m \in \mathbf{C}$ with compatiable sources and targets. The desired form is obtained by inserting some $\mathrm{id}_{\mathbf{v}} = \mathrm{id}_{\mathbf{v}}^{-1}$ for appropriate $\mathbf{v} \in \mathbf{C}^0$ in the product if necessary. ☐

As described in Section 1.1, we denote semilattice of idempotents of I_l by J, and the space of characters of J by $\hat{J}$, endowed with pointwise-convergence topology. A basis of compact open sets of $\hat{J}$ is given by the sets of the form

$$\hat{J}(e;\mathfrak{f}) = \{\chi \in \hat{J} : \chi(e) = 1 \text{ and } \chi(f) = 0 \text{ for all } f \in \mathfrak{f}\}$$

where $e \in J$ and $\mathfrak{f}$ is a finite subset of J. We also write $\hat{J}(e) = \{\chi \in \hat{J} : \chi(e) = 1\}$ for a given $e \in J$.

Definition 2.5 (The subspace Ω of $\hat{J}$)**.** We define the subspace Ω of $\hat{J}$ consisting of characters $\chi : J \to \{0,1\}$ with the property that whenever $e, f_1, \ldots, f_n \in J$ satisfy $e = \bigcup_{i=1}^n f_i$ as subsets of $\mathbf{C}$, then $\chi(e) = 1$ imples that $\chi(f_i) = 1$ for some index i. The topology on Ω is the subspace topology from $\hat{J}$.

A basis of compact open topology of Ω is given by $\Omega(e;\mathfrak{f}) = \{\chi \in \Omega : \chi(e) = 1 \text{ and } \chi(f) = 0 \text{ for all } f \in \mathfrak{f}\}$, where $e \in J$ and $\mathfrak{f}$ is a finite subset of J.

Example 2.6 (Character from an element)**.** Let $\mathbf{C}$ be a left cancellative small category. Given $x \in \mathbf{C}$, we define $\chi_x : J \to \{0,1\}$ by

$$\chi_x(e) = \begin{cases} 1, & \text{if } x\mathbf{C} \subseteq e, \\ 0, & \text{otherwise.} \end{cases}$$

A basic property of such kind of characters is that two characters χ_x and χ_y are the same if and only if x and y differ from an invertible element.

Proposition 2.7. *Let x, y be two elements in a left cancellative small category $\mathbf{C}$. Then $\chi_x = \chi_y$ if and only if $x\mathbf{C} = y\mathbf{C}$ and if and only if $x =^* y$.*

Proof. If $x = y \in \mathbf{C}$ then trivially we have $\chi_x = \chi_y$. Conversely suppose that $\chi_x = \chi_y$ then $x\mathbf{C} \subseteq e$ if and only if $y\mathbf{C} \subseteq e$. Take $e = x\mathbf{C}$, and then $y\mathbf{C} \subseteq x\mathbf{C}$ because $x\mathbf{C} \subseteq x\mathbf{C}$ is always true. Similarly take $e = y\mathbf{C}$, and then $x\mathbf{C} \subseteq y\mathbf{C}$. Therefore $x\mathbf{C} = y\mathbf{C}$. $\square$

It is straightforward to see that $\chi_x \in \Omega$ for all $x \in \mathbf{C}$. Such a type of characters will play an important role in the remaining text. We state here two related lemmas without proof.

Lemma 2.8 (Li [5], Lemma 2.12)**.** $\{\chi_x : x \in \mathbf{C}\}$ *is a dense subset of Ω with respect to the pointwise-convergence topology.*

Lemma 2.9 (Li [5], Lemma 2.13)**.** $\Omega = \hat{J}$ *if and only if whenever $e, f_1, \ldots, f_n \in J$ satisfy $e = \bigcup_{i=1}^n f_i$ as subsets of $\mathbf{C}$, then $e = f_i$ for some i.*

Definition 2.10. Given $s \in I_l$ and $\chi \in \hat{J}$ with requirement that $\chi(s^{-1}s) = 1$, we define another character $s.\chi : J \to \{0,1\}$ by $(s.\chi)(e) = \chi(s^{-1}es)$.

The requirement in the above definition is necessary since this ensures that the character is indeed a nonzero map.

By Proposition 1.13, the characters $s.\chi$ for all $s \in I_l$ are well defined. This gives an action of I_l on $\hat{J}$. In fact we have a homeomorphism $\hat{J}(s^{-1}s) \to \hat{J}(ss^{-1}), \chi \mapsto s.\chi$.

Lemma 2.11 (Invariance of Ω)**.** *The subspace Ω from Definition 2.5 is I_l-invariant, meaning that $\chi \in \Omega$ implies that $s.\chi$ still lies in Ω for all $s \in I_l$ with $\chi(s^{-1}s) = 1$.*

Proof. Suppose $\chi \in \Omega$ and $s \in I_l$. Let $e, f_1, \ldots, f_n \in J$ such that $e = \bigcup_{i=1}^n f_i$ such that $s.\chi(e) = 1$. Note that $s^{-1}es = s^{-1}\left(\bigcup_{i=1}^n f_i\right)s = \bigcup_{i=1}^n s^{-1}f_is$. Then $\chi(s^{-1}es) = \chi(\bigcup_{i=1}^n s^{-1}f_is) = 1$. Since $\chi \in \Omega$, there is an i such that $s.\chi(f_i) = \chi(s^{-1}f_is) = 1$. Thus $s.\chi \in \Omega$. $\square$

Because of Lemma 2.11, the characters $s.\chi$ also give an action of I_l on Ω and similarly we have a homeomorphism $\Omega(s^{-1}s) \to \Omega(ss^{-1})$. The subspace Ω is invariant under the action of I_l. This property is important since this enables us to construct the transformation groupoid.

The following lemma will be frequently used later.

Lemma 2.12. *Let $s \in I_l$ be a partial bijection on a small category $\mathbf{C}$. If x is an element of $\mathbf{C}$ such that $x\mathbf{C} \subseteq \mathrm{dom}(s)$ then $s(x\mathbf{C}) = s(x)\mathbf{C}$.*

Proof. By Proposition 2.4, we write s in the zigzag form $s = \sigma_{d_n}^{-1}\sigma_{c_n}\cdots\sigma_{d_1}^{-1}\sigma_{c_1}$ for some $d_i, c_i \in \mathbf{C}$ with $\mathbf{t}(c_i) = \mathbf{t}(d_i)$ and $\mathbf{d}(d_i) = \mathbf{d}(c_{i+1}), i = 1, 2, \ldots, n$, $n \in \mathbb{N}_+$. Since $x\mathbf{C} \subseteq \mathrm{dom}(s)$, $x\mathbf{C} \subseteq \bigcup_{i=1}^{n}(\mathrm{dom}(\sigma_{c_i}) \cap \mathrm{dom}(\sigma_{d_i}^{-1})) = \bigcup_{i=1}^{n}(\mathbf{d}(c_i)\mathbf{C} \cap d_i\mathbf{C})$. Hence it suffices to see that for each $a, b \in \mathbf{C}$ with $x\mathbf{C} \subseteq a\mathbf{C} \cap b\mathbf{C}$, we always have that $\sigma_a(x\mathbf{C}) = \sigma_a(x)\mathbf{C}$ and $\sigma_b^{-1}(x\mathbf{C}) = \sigma_b^{-1}(x)\mathbf{C}$.

By definition, let $xc \in x\mathbf{C}$, $\sigma_a(xc) = axc = (ax)c \in \sigma_a(x)\mathbf{C}$, so $\sigma_a(x\mathbf{C}) \subseteq \sigma_a(x)\mathbf{C}$. This equality also gives the reverse inclusion. Hence $\sigma_a(x\mathbf{C}) = \sigma_a(x)\mathbf{C}$. Since $x \in x\mathbf{C} \subseteq b\mathbf{C}$, we write $x = bc'$ for another $c \in \mathbf{C}$ we have $\sigma_b^{-1}(xc) = \sigma_b^{-1}(bc'c) = c'c = \sigma_b^{-1}(x)c \in \sigma_b^{-1}(x)\mathbf{C}$. Similarly, this equality also gives the reverse inclusion. Hence $\sigma_b^{-1}(x\mathbf{C}) = \sigma_b^{-1}(x)\mathbf{C}$. □

Remark 2.13. In Li [5], Lemma 2.12 is also frequently used and regarded as an obvious result without being stated and proved, but it would be better to make it explicit.

Lemma 2.14. *Let $x \in \mathbf{C}$ and $s \in I_l$ be such that $\chi_x(s^{-1}s) = 1$. Then $s.\chi_x = \chi_{s(x)}$ for every x satisfying $x\mathbf{C} \subseteq \mathrm{dom}(s)$.*

Proof. By definition, $s.\chi_x(e) = 1$ if and only if $x\mathbf{C} \subseteq s^{-1}es$, and $\chi_{s(x)} = 1$ if and only if $s(x)\mathbf{C} \subseteq e$. We show that $x\mathbf{C} \subseteq \mathrm{dom}(es)$ if and only if $s(x)\mathbf{C} \subseteq \mathrm{dom}(e)$ when $x\mathbf{C} \subseteq \mathrm{dom}(s)$.

Since $x\mathbf{C} \subseteq \mathrm{dom}(es) = \mathrm{dom}(s) \cap s^{-1}(\mathrm{dom}(e))$, we have by Lemma 2.12 that

$$s(x)\mathbf{C} = s(x\mathbf{C}) \subseteq \mathrm{im}(s) \cap \mathrm{dom}(e) \subseteq \mathrm{dom}(e)$$

.

Conversely, if $s(x)\mathbf{C} \subseteq \mathrm{dom}(e)$, then again by Lemma 2.12,

$$x\mathbf{C} = s^{-1}s(x\mathbf{C}) = s^{-1}s(x)\mathbf{C} \subseteq s^{-1}(\mathrm{dom}(e)) = s^{-1}(\mathrm{dom}(e) \cap \mathrm{im}(s)) = \mathrm{dom}(es).$$

□

♩ 3

Groupoid models for the left reduced C*-algebra

3.1 The left reduced C*-algebra of a groupoid

Now we briefly go through the concepts of the left reduced C*-algebra of a groupoid, roughly following Sims [8] and Li [5], respectively.

Let $C_c(\mathscr{G})$ be the space of complex-valued continuous functions $f : \mathscr{G} \to \mathbb{C}$ of compact support on the topological groupoid $\mathscr{G}$, where a support of f is defined to be $\mathrm{supp}(f) := \overline{\{\gamma \in \mathscr{G} : f(\gamma) \neq 0\}}$.

Proposition 3.1 (*-algebra structure on $C_c(\mathscr{G})$)**.** *Let $\mathscr{G}$ be a second-countable locally compact Hausdorff étale groupoid. Then we have the following*

- *For every $\gamma \in \mathscr{G}$, the set $\{\alpha \in \mathscr{G}^{\mathrm{r}(\gamma)} : f(\alpha)g(\alpha^{-1}\gamma) \neq 0\}$ is finite.*
- *$C_c(\mathscr{G})$ becomes a *-algebra when equipped with multiplication $(f * g)(\gamma) = \sum_{\alpha \in \mathscr{G}^{\mathrm{r}(\gamma)}} f(\alpha)g(\alpha^{-1}\gamma)$ and the involution $f^*(\gamma) = f(\gamma^{-1})$.*

We note that the multiplication in Proposition 3.1 is well defined because $\mathrm{supp}(f * g) \subseteq \mathrm{supp}(f)\,\mathrm{supp}(g)$, and it is easy to verify that $\mathrm{supp}(f)\,\mathrm{supp}(g)$ is also compact.

Definition 3.2 (Convolution)**.** Let $\mathscr{G}$ be a second-countable locally compact Hausdorff étale groupoid. For $f, g \in C_c(\mathscr{G})$, the multiplication

$$(f * g)(\gamma) = \sum_{\alpha \in \mathscr{G}^{\mathrm{r}(\gamma)}} f(\alpha)g(\alpha^{-1}\gamma)$$

defined in Proposition 3.1 is called the convolution of f and g.

In a second-countable locally compact Hausdorff étale groupoid $\mathscr{G}$, there is a *-representation

$$\pi_x : C_c(\mathscr{G}) \longrightarrow \mathscr{B}(\ell^2(\mathscr{G}_x))$$

for each $x \in \mathscr{G}^{(0)}$ defined by $\pi_x(f)(\delta_\gamma) = f * \delta_\gamma$. π_x is known as the regular representation of $C_c(\mathscr{G})$ associated to $x \in \mathscr{G}^{(0)}$. Then we can state the definition of the left reduced C*-algebra.

Definition 3.3 (Left reduced C*-algebra of a groupoid)**.** Let $\mathscr{G}$ be a second-countable locally compact Hausdorff étale groupoid. The left reduced C*-algebra $C_r^*(\mathscr{G})$ of $\mathscr{G}$ is the completion of the subspace

$$\left(\bigoplus_{x \in \mathscr{G}^{(0)}} \pi_x\right)(C_c(\mathscr{G})) \subseteq \bigoplus_{x \in \mathscr{G}^{(0)}} \mathscr{B}(\ell^2(\mathscr{G}_x))$$

with respect to the norm given in the direct sum of Hilbert spaces, where $\left(\bigoplus_{x\in\mathscr{G}^{(0)}}\pi_x\right)$ is a *-representation given by

$$\begin{aligned}\left(\bigoplus_{x\in\mathscr{G}^{(0)}}\pi_x\right): C_c(\mathscr{G}) &\longrightarrow \bigoplus_{x\in\mathscr{G}^{(0)}}\mathscr{B}(\ell^2(\mathscr{G}_x)),\\ f &\longmapsto (\pi_x(f))_{x\in\mathscr{G}^{(0)}}.\end{aligned}$$

We should note that this *-representation is actually injective. Roughly speaking, this is because $f*\delta_\gamma = 0$ for every $\gamma\in\mathscr{G}$ implies that $f=0$ on $\mathscr{G}$.

3.2 The left reduced C*-algebra of a left cancellative small category

Let $\mathbf{C}$ be a left cancellative small category and $\mathbb{C}$ denotes the space of complex numbers. Define the $\ell^2(\mathbf{C})$ space to be

$$\ell^2(\mathbf{C}) = \left\{ f:\mathbf{C}\to\mathbb{C} \;\middle|\; \sum_{c\in\mathbf{C}}|f(c)|^2<\infty \right\},$$

where

$$\sum_{c\in\mathbf{C}}|f(c)|^2 := \sup_{F\subseteq\mathbf{C}\text{ finite}}\sum_{c\in F}|f(c)|^2.$$

$\ell^2(\mathbf{C})$ is a Hilbert space when we equip it with the well-defined inner product

$$\begin{aligned}\langle\cdot,\cdot\rangle:\mathbf{C}&\longrightarrow\mathbb{C},\\ \langle f,g\rangle &= \sum_{c\in\mathbf{C}} f(c)\overline{g(c)}.\end{aligned}$$

The standard orthonormal basis of $\ell^2(\mathbf{C})$ is given by $\{\delta_x\}_{x\in\mathbf{C}}$, where $\delta_x:\mathbf{C}\to\mathbb{C}$, $y\mapsto 1$ if $y=x$ and $y\mapsto 0$ if $y\neq x$.

Given $x\in\mathbf{C}$, for each $c\in\mathbf{C}$, the assignment $\delta_x\mapsto\delta_{cx}$ if $\mathbf{t}(x)=\mathbf{d}(c)$ and $\delta_x\mapsto 0$ if $\mathbf{t}(x)\neq\mathbf{d}(c)$ extends by linearity to a bounded linear operator on $\ell^2(\mathbf{C})$, which is denoted by λ_c, i.e., $\lambda_c\in\mathscr{B}(\ell^2(\mathbf{C}))$. Left cancellative property is essential here to ensure boundedness and also this implies that this is actually a partial isometry.

Definition 3.4 (Left reduced C*-algebra of a left cancellative small category)**.** Let $\mathbf{C}$ be a left cancellative small category. The left reduced C*-algebra of $\mathbf{C}$, denoted by $C^*_\lambda(\mathbf{C})$, is defined by the C*-algebra generated by the partial isometries $\{\lambda_c\}_{c\in\mathbf{C}}$.

Formally, we have that $C^*_\lambda(\mathbf{C}) := C^*(\{\lambda_c : c\in\mathbf{C}\})\subseteq\mathscr{B}(\ell^2(\mathbf{C}))$.

In the remaining part of this section, we provide two candidates of groupoid models for the left reduced C*-algebra.

3.3 Two groupoid models for the left reduced C*-algebra

Define the set

$$I_l*\Omega := \{(s,\chi)\in I_l\times\Omega : \chi(s^{-1}s)=1\}.$$

Definition 3.5 (Transformation groupoid)**.** Define an equivalence relation $\sim$ on $I_l*\Omega$ by

$$(s,\chi)\sim(t,\psi) \iff \chi=\psi \text{ and there exists an } e\in J \text{ with } \chi(e)=1 \text{ and } se=te.$$

The equivalence classes with respect to $\sim$ are denoted by $[\cdot]$.

The transformation groupoid $I_l\ltimes\Omega$ is the set of equivalence classes $I_l*\Omega/\sim$ equipped with

- the source map $\mathrm{s}([s,\chi]) = \chi$ and the range map $\mathrm{r}([s,\chi]) = s.\chi$;
- multiplication $[s,t.\chi][t,\chi] = [st,\chi]$;
- inversion $[s,\chi]^{-1} = [s^{-1}, s.\chi]$.

For a subset U of Ω, write $[s,U] := \{[s,\chi] : \chi \in U\}$. The topology on $I_l \ltimes \Omega$ is given by requiring $[s,\Omega(s^{-1}s)]$ to be open sets for all $s \in I_l$ and the source map restricted to $[s,\Omega(s^{-1}s)]$ is a homeomorphism onto $\Omega(s^{-1}s)$ by $[s,\chi] \mapsto \chi$.

Definition 3.6 (A variation of the transformation groupoid)**.** Define an equivalence relation $\bar{\sim}$ on $I_l * \Omega$ by

$$(s,\chi)\bar{\sim}(t,\psi) \iff \chi = \psi \text{ and there exists an } \varepsilon \in \bar{J} \text{ with } \chi(\mathrm{Id}_\varepsilon) = 1 \text{ and } s\,\mathrm{Id}_\varepsilon = t\,\mathrm{Id}_\varepsilon\,.$$

The equivalence classes with respect to $\sim$ are denoted by $[\cdot]^{\bar{\sim}}$.

The groupoid $I_l \bar{\ltimes} \Omega$ is the set of equivalence classes $I_l * \Omega/\bar{\sim}$ with source map, range map, multiplication and inversion defined in the same way as for $I_l \ltimes \Omega$.

For a subset U of Ω, write $[s,U]^{\bar{\sim}} := \{[s,\chi]^{\bar{\sim}} : \chi \in U\}$. The topology on $I_l \bar{\ltimes} \Omega$ is given by requiring $[s,\Omega(s^{-1}s)]^{\bar{\sim}}$ to be open sets for all $s \in I_l$ and the source map restricted to $[s,\Omega(s^{-1}s)]^{\bar{\sim}}$ is a homeomorphism onto $\Omega(s^{-1}s)$ by $[s,\chi]^{\bar{\sim}} \mapsto \chi$.

We now see what the unit space of $I_l \ltimes \Omega$ is. As expected, the unit space $(I_l \ltimes \Omega)^{(0)}$ should be homeomorphic to Ω.

Proposition 3.7. *For any $s,t \in I_l$, $(s^{-1}s,\chi) \sim (t^{-1}t,\chi)$ in the set $I_l * \Omega$. In other words, when fixing an $\chi \in \Omega$, for all idempotents e of I_l with $\chi(e) = 1$, (e,χ) are in the same equivalence class.*

Proof. We have by assumption that $\chi(s^{-1}s) = \chi(t^{-1}t) = 1$. Let $e = s^{-1}st^{-1}t$. Then

$$\chi(e) = \chi(s^{-1}st^{-1}t) = \chi(s^{-1}s)\chi(t^{-1}t) = 1,$$

Also,

$$s^{-1}se = s^{-1}ss^{-1}st^{-1}t = s^{-1}(ss^{-1}s)t^{-1}t = s^{-1}st^{-1}t = e$$

and

$$t^{-1}te = et^{-1}t = s^{-1}st^{-1}tt^{-1}t = s^{-1}st^{-1}t = e,$$

where we used the property that idempotents commute in an inverse semigroup (Lemma 1.6). Hence indeed, $(s^{-1}s,\chi) \sim (t^{-1}t,\chi)$. □

From the above proposition we know that the unit space $(I_l \ltimes \Omega)^{(0)}$ is given by

$$(I_l \ltimes \Omega)^{(0)} = \{[e,\chi] : \chi \in \Omega,\ (\text{any})\ e \in J \text{ with } \chi(e) = 1\}\,.$$

We can check by definition that for every $[s,\chi] \in I_l \ltimes \Omega$, the range $\mathrm{r}([s,\chi])$ and the source $\mathrm{s}([s,\chi])$ both lie in $(I_l \ltimes \Omega)^{(0)}$. For example,

$$\begin{aligned} \mathrm{r}([s,\chi]) &= [s,\chi][s,\chi]^{-1} = [s,\chi][s^{-1},s.\chi] \\ &= [s,(s^{-1}).(s.\chi)][s^{-1},s.\chi] \\ &= [ss^{-1},\chi] = [e,\chi] \end{aligned}$$

with $e = ss^{-1}$ which is an idempotent. Similarly we can see $\mathrm{s}([s,\chi]) = [e,\chi]$ for some $e \in J$. Moreover, $(I_l \ltimes \Omega)^{(0)} \cong \Omega$ through the map $[e,\chi] \mapsto \chi$, so we can identify $(I_l \ltimes \Omega)^{(0)}$ with Ω without concern.

By construction, there is a canonical projection $I_l \ltimes \Omega \twoheadrightarrow I_l \bar{\ltimes} \Omega$. We now explore under which kinds of further assumptions these two groupoids are isomorphic. But let's quickly go through some easy results related to these two groupoids without proof.

Proposition 3.8 (Li [5], Lemma 3.1)**.** *(i) The canonical projection $I_l \ltimes \Omega \twoheadrightarrow I_l \bar{\ltimes} \Omega$ is an open quotient map.*

(ii) The canonical projection $I_l \ltimes \Omega \twoheadrightarrow I_l \bar{\ltimes} \Omega$ takes bisections to bisections.

(iii) The identity map on Ω induces a one-to-one correspondence between $(I_l \ltimes \Omega)$-invariant subsets to $I_l \bar{\ltimes} \Omega$-invariant subsets.

We will see that under the assumption that either the ambient category $\mathbf{C}$ is finitely aligned or the transformation (topological) groupoid is Hausdorff, the canonical projection is a groupoid isomorphism.

Before giving the next lemma, we mention the notion of the union of a family of functions. Let $\{f_i : i \in I\}$ be a family of functions such that for all $i, j \in I$, f_i and f_j agree on the intersection of their domains. Then we define the union of this family of functions as a new function $\bigcup_{i\in I} f_i$ by mapping x to $f_i(x)$ if $x \in \mathrm{dom}(f_i)$.

Lemma 3.9. *Let $\mathbf{C}$ be a finitely aligned left cancellative small category. Then the following statements hold.*

(i) Every $e \in J$ is a union of finitely many principal ideals. That is, $e = \bigcup_{x\in F} x\mathbf{C}$ for some finite subset $F \subseteq \mathbf{C}$.

(ii) Every $\chi \in \Omega$ is determined by the family of principal ideals where the value of χ is 1, that is, $\mathscr{F}_p^\chi := \{x\mathbf{C} : x \in \mathbf{C}$ with $\chi(x\mathbf{C}) = 1\}$, in the sense that for every $e \in J$, $\chi(e) = 1$ if and only if there is an $x\mathbf{C} \in \mathscr{F}_p$ such that $x\mathbf{C} \subseteq e$. Moreover, a basis of compact open sets for Ω is given by sets of the form $\Omega(x\mathbf{C}; y_1\mathbf{C}, \ldots, y_n\mathbf{C})$.

(iii) Every $s \in I_l$ is a finite union of partial bijections of the form $\sigma_c\sigma_d^{-1}$ with $c, d \in \mathbf{C}$ and $\mathbf{d}(c) = \mathbf{d}(d)$.

(iv) We have
$$I_l \ltimes \Omega = \left\{[\sigma_c\sigma_d^{-1}, \chi] : c, d \in \mathbf{C}, \mathbf{d}(c) = \mathbf{d}(d), (\sigma_c\sigma_d^{-1}, \chi) \in I_l * \Omega\right\}.$$

Proof. (i) For each $e \in J \subseteq I_l$, e is identified as $\mathrm{dom}(e)$ and we write e in the zigzag form $e = \sigma_{d_n}^{-1}\sigma_{c_n}\cdots\sigma_{d_1}^{-1}\sigma_{c_1}$ with $c_i, d_i \in \mathbf{C}$, $i = 1, \ldots, n$. For every $c, d \in \mathbf{C}$, the domain of $\sigma_d^{-1}\sigma_c$ is $\sigma_c(d\mathbf{C} \cap c\mathbf{C})$. Using finite alignment assumption, there is a finite subset $F \subseteq \mathbf{C}$ such that $d\mathbf{C} \cap c\mathbf{C} = \bigcup_{f\in F} f\mathbf{C}$. Thus,

$$\mathrm{dom}(\sigma_d^{-1}\sigma_c) = \sigma_c^{-1}(d\mathbf{C} \cap c\mathbf{C}) = \sigma_c^{-1}\left(\bigcup_{f\in F} f\mathbf{C}\right) = \bigcup_{f\in F} \sigma_c^{-1}(f\mathbf{C}) = \bigcup_{f\in F} \sigma_c^{-1}(f)\mathbf{C},$$

where the last equality is from Lemma 2.12. Repeated application of this reduces the domain of every zigzag map to the required form.

(ii) Since $\chi \in \Omega$ and $e = \bigcup_{i=1}^n$ for some $\{x_i\}_{i=1}^n \subseteq \mathbf{C}$ by (i), then $\chi(e) = 1$ implies that there is an x_i for some index i such that $\chi(x_i\mathbf{C}) = 1$. Thus $x_i\mathbf{C} \in \mathscr{F}_p^\chi$ with $x\mathbf{C} \subseteq e$. Conversely, if there exists an $x\mathbf{C} \in \mathscr{F}_p^\chi$ with $x\mathbf{C} \subseteq e$, then $\chi(x\mathbf{C}) = 1$. Indeed, $x\mathbf{C} \cap e = x\mathbf{C}$ and by multiplicative property of χ, $\chi(x\mathbf{C})\chi(e) = \chi(x\mathbf{C} \cap e) = \chi(x\mathbf{C}) = 1$. Hence $\chi(e) = 1$.

Now we set out to prove the latter statement. On the one hand, $\Omega(x\mathbf{C}; y_1\mathbf{C}, \ldots, y_n\mathbf{C})$ is indeed of the form $\Omega(e; \mathfrak{f})$ by simply taking $e = x\mathbf{C}$ and $\mathfrak{f} = \{y_1\mathbf{C}, \ldots, y_n\mathbf{C}\}$. On the other hand, let $\Omega(e; \mathfrak{f})$ be a basis element of Ω with $e \in J$ and $\mathfrak{f}$ a finite subset of J. By (i), $\chi(e) = 1$ if and only if there is an $x \in \mathbf{C}$ with $\chi(x\mathbf{C}) = 1$. For $\mathfrak{f} = \{f_1, \ldots, f_m\}$, $f_i = \bigcup_{k=1}^{n_i} y_{i,k}\mathbf{C}$ for each $i = 1, \ldots, m$. Thus $\chi(f_i) = 0$ for each $i = 1, \ldots, m$ implies that $\chi(y_{i,k}\mathbf{C}) = \chi(f_i \cap y_{i,k}\mathbf{C}) = \chi(f_i)\chi(y_{i,k}\mathbf{C}) = 0$ for each $k = 1, \ldots, n_i$. Thus we have that $\Omega(e; \mathfrak{f}) = \Omega(x\mathbf{C}; y_{1,1}\mathbf{C}, \ldots, y_{m,n_m}\mathbf{C})$.

(iii) From the proof of (i) we see that

$$\sigma_d^{-1}\sigma_c = \bigcup_{f\in F}\left(\sigma_{\sigma_d^{-1}(f)}\right)\left(\sigma^{-1}_{\sigma_c^{-1}(f)}\right),\ F\subseteq \mathbf{C} \text{ finite,}$$

with both $\sigma_d^{-1}(f)$ and $\sigma_c^{-1}(f)$ lying in $\mathbf{C}$. $\mathbf{d}(\sigma_d^{-1}(f)) = \mathbf{d}(\sigma_c^{-1}(f))$ is because we need $\mathbf{d}(\sigma_d^{-1}(f)) \cap \mathbf{d}(\sigma_c^{-1}(f)) \neq \varnothing$. This is equivalent to say that $\mathbf{d}(\sigma_d^{-1}(f)) = \mathbf{d}(\sigma_c^{-1}(f))$.

Repeated application of this reduces a zigzag map to the required form and every $s \in I_l$ is a finite union of the form $\sigma_d^{-1}\sigma_c$ by Proposition 2.4.

(iv) The characterization of $I_l \ltimes \Omega$ and $I_l \bar{\ltimes} \Omega$ follows immediately from (iii) and the definition of the transformation groupoid.

□

Theorem 3.10. *The canonical projection $I_l \ltimes \Omega \twoheadrightarrow I_l \bar{\ltimes} \Omega$ is a groupoid isomorphism if either of the following conditions holds:*

(1) $\mathbf{C}$ is finitely aligned.

(2) $I_l \ltimes \Omega$ is Hausdorff.

Proof. Since the canonical projection is a surjective map, it remains to see that this is also injective. Let (s,χ) and (t,χ) be two elements in $I_l * \Omega$ such that $(s,\chi)\tilde{\sim}(t,\chi)$. By definition there is an $\varepsilon \in \bar{J}$ with $\chi(\mathrm{Id}_\varepsilon) = 1$ and $s\,\mathrm{Id}_\varepsilon = t\,\mathrm{Id}_\varepsilon$.

Suppose firstly that $\mathbf{C}$ is finitely aligned. Then by definition of an element in $\bar{J}$ and Lemma 3.9, we may assume $\varepsilon = x\mathbf{C} \setminus (\bigcup_{i=1}^n y_i\mathbf{C})$ for some $x, y_1, \ldots, y_n \in \mathbf{C}$. Since $s\,\mathrm{Id}_\varepsilon = t\,\mathrm{Id}_\varepsilon$, then $x\mathbf{C} \subseteq \mathrm{dom}(s)$ and in particular,$s(x) = t(x)$, so that when taking $e = x\mathbf{C}$, $se = s\,\mathrm{Id}_{x\mathbf{C}} = s(x)\mathbf{C} = t(x)\mathbf{C} = t\,\mathrm{Id}_{x\mathbf{C}} = te$, by Lemma 2.12. Moreover, $\chi(\varepsilon) = 1$ implies that $\chi(e) = 1$ because $\varepsilon \leq e$ and $1 = \chi(\varepsilon) = \chi(\varepsilon e) = \chi(\varepsilon)\chi(e)$.

Suppose secondly that $I_l \ltimes \Omega$ is Hausdorff. By the density of $\{\chi_x : x \in \mathbf{C}\}$ (Lemma 2.8) in Ω we can find a net $\{x_i\}_{i\in I}$ in $\mathbf{C}$ with $\lim_i \chi_{x_i} = \chi$ pointwise. As $\chi(\varepsilon) = 1$, then $\chi_{x_i}(\varepsilon) = 1$ for all but finitely many $i \in I$ because each character has at most two values 0 and 1. Hence we may assume that $\chi_{x_i}(\varepsilon) = 1$ for all i, i.e., $x_i\mathbf{C} \subseteq \varepsilon$. Take $e_i = x_i\mathbf{C}$, then $s\,\mathrm{Id}_\varepsilon = t\,\mathrm{Id}_\varepsilon$ implies in particular that $se_i = te_i$ for each i. Thus $(s,\chi_{x_i}) \sim (t,\chi_{x_i})$ so that $[s,\chi_{x_i}] = [t,\chi_{x_i}]$ for each i. Now, in $I_l \ltimes \Omega$ with the homeomorphism $[s,\Omega(s^{-1}s)] \cong \Omega(s^{-1}s)$ given in the definition, $\lim_i[s,\chi_{x_i}] = [s,\chi]$ and $\lim_i[t,\chi_{x_i}] = [t,\chi]$, then $[s,\chi] = [t,\chi]$ because in a Hausdorff space, limit points are unique. □

Theorem 3.11. *Let $\mathbf{C}$ be a left cancellative small category. The groupoid $I_l \bar{\ltimes} \Omega$ is a groupoid model for $C^*_\lambda(\mathbf{C})$, meaning that $C^*_\lambda(\mathbf{C}) \cong C^*_r(I_l \bar{\ltimes} \Omega)$.*

With Theorem 3.10 and 3.11, we have an immediate consequence:

Corollary 3.12. *The transformation groupoid $I_l \ltimes \Omega$ is a groupoid model for $C^*_\lambda(\mathbf{C})$ if either $\mathbf{C}$ is finitely aligned or $I_l \ltimes \Omega$ is a Hausdorff space.*

3.4 Topological free and effective properties

As a side dish, we briefly mention the topological free and effective properties of the transformation groupoid $I_l \ltimes \Omega$ and its variation $I_l \bar{\ltimes} \Omega$.

Definition 3.13 (Topological free groupoid, Effective groupoid).
- An étale groupoid $\mathscr{G}$ is topological free if for every open bisection U not containing units of $\mathscr{G}$, the set $\{x \in \mathscr{G}^{(0)} : \mathscr{G}_x^x \cap U \neq \varnothing\}$ has empty interior.

- An étale groupoid $\mathscr{G}$ effective if the interior of the isotropy subgroupoid coincide with the unit space, that is to say, $\mathrm{Iso}(\mathscr{G})^\circ = \mathscr{G}^{(0)}$.

Note that from the above definitions we can see that an étale groupoid $\mathscr{G}$ is topological free if and only if for every open bisection U not containing units of $\mathscr{G}$, the subspace $\{x \in \mathrm{s}(U) : \mathscr{G}_x^x \cap U = \varnothing\}$ is dense in $\mathrm{s}(U)$. Also, effectiveness implies topological freeness.

We present some results of topological freeness and effectiveness of the transformation groupoid and its variation.

Theorem 3.14 (Characterization of topological freeness of $I_l \ltimes \Omega$)**.** *Define* $\mathbf{C}^{*,0} := \{u \in \mathbf{C}^* : \mathbf{t}(u) = \mathbf{d}(u)\}$ *and* $\mathbf{C}^{*,0} \ltimes \Omega := \{[u,\chi] \in I_l \ltimes \Omega : u \in \mathbf{C}^{*,0}\}$*. Then the following are equivalent:*

(i) $I_l \ltimes \Omega$ *is topologically free;*

(ii) $\mathbf{C}^{*,0} \ltimes \Omega$ *is topologically free;*

(iii) For every $\mathbf{v} \in \mathbf{C}^0$*, every* $u \in \mathrm{id}_{\mathbf{v}}\, \mathbf{C}^*\, \mathrm{id}_{\mathbf{v}}$ *and every finite subset* $\{f_1,\dots,f_n\} \subseteq J$ *with* $f_i \subsetneq \mathbf{vC}$ *for all* $i = 1,\dots,n$,

$uz =^* z$ for all $z \in \mathbf{vC} \setminus \bigcup_{i=1}^n f_i$ implies that there is an $x \in \mathbf{vC} \setminus \bigcup_{i=1}^n f_i$ with $ux = x$.

Proof. (i) implying (ii): By definition, $\mathbf{C}^{*,0} \ltimes \Omega$ is an open subgroupoid of $I_l \ltimes \Omega$. An is an open bisection $U \subseteq \mathbf{C}^{*,0} \ltimes \Omega$ with $U \subseteq (\mathbf{C}^{*,0} \ltimes \Omega) \setminus \Omega$ is also an open bisection of $I_l \ltimes \Omega$ contained in $(I_l \ltimes \Omega) \setminus \Omega$. Here we identify the unit space of these two groupoids with Ω as described after Proposition 3.7. If $(I \ltimes \Omega)_x^x \cap U = \varnothing$, then so the intersection of $(\mathbf{C}^{*,0} \ltimes \Omega)_x^x$ with U is empty. Thus the topological free property of $I_l \ltimes \Omega$ implies the the topological free property of $\mathbf{C}^{*,0} \ltimes \Omega$.

(ii) implying (i): Suppose on the converse that $I_l \ltimes \Omega$ is not topologically free. This implies that $\mathrm{Iso}(I_l \ltimes \Omega)$ has a nonempty interior. We can take an $s \in I_l$ and an open subset U of $\Omega(s^{-1}s)$ such that $[s,U] \subseteq (I_l \ltimes \Omega) \setminus \Omega$. By Lemma 2.8, there is an $x \in \mathbf{C}$ with $\chi_x \in U$, and thus $[s,\chi_x] \neq \chi_x$. However by construction, $s.\chi_x = \chi_x$, so Lemma 2.14 implies that $s(x) = xu$ for some $u \in \mathbf{C}^{*,0} \setminus \mathbf{C}^0$. Take $V = \Omega(x\mathbf{C}) \cap U \neq \varnothing$. Now a direct computation shows that

$$[x,\Omega(\mathbf{d}(x)\mathbf{C})]^{-1}[s,V][x,\Omega(\mathbf{d}(x)\mathbf{C})] = [u,\sigma_x^{-1}.V],$$

and $\sigma_x^{-1}.V \neq \varnothing$ because V is. The conjugation $[u,\sigma_x^{-1}.V] = [x,\Omega(\mathbf{d}(x)\mathbf{C})]^{-1}[s,V][x,\Omega(\mathbf{d}(x)\mathbf{C})] \subseteq \mathrm{Iso}(\mathbf{C}^{*,0} \ltimes \Omega) \setminus \Omega$ because $[s,V]$ is contained in $\mathrm{Iso}(I_l \ltimes \Omega) \setminus \Omega$. Therefore, $\mathbf{C}^{*,0} \ltimes \Omega$ is not topologically free.

(ii) implying (iii): Assume $uz =^* z$ for all $z \in \mathbf{vC} \setminus \bigcup_{i=1}^n f_i$. The subset $\Omega(\mathbf{vC}; f_1,\dots,f_n)$ is an compact open basis element of Ω and $[u,\Omega(\mathbf{vC}; f_1,\dots,f_n)] \subseteq \mathrm{Iso}(\mathbf{C}^{*,0} \ltimes \Omega)$. Since $\mathbf{C}^{*,0} \ltimes \Omega$ is topological free by assumption, there exists $\chi \in \Omega(\mathbf{vC}; f_1,\dots,f_n)$ with $[u,\chi] = \chi$, i.e., there is a $e \in J$ with $\chi(e) = 1$ and $ue = e$. Now $\chi(\mathbf{vC} \setminus \bigcup_{i=1}^n f_i) = 1$ implies that $e \not\subseteq \bigcup_{i=1}^n f_i$. Hence choose an x from $e \setminus \bigcup_{i=1}^n f_i$ and we have $ux = x$.

We claim that (iii) is equivalent to the following statement:

For every $\mathbf{v} \in \mathbf{C}^0$, every $u \in \mathrm{id}_{\mathbf{v}}\, \mathbf{C}^*\, \mathrm{id}_{\mathbf{v}}$ and every finite subset $e, f_1,\dots,f_n \in J$ with $e, f_i \subsetneq \mathbf{vC}$ for all $i = 1,\dots,n$ and $\bigcup_{i=1}^n f_i \subsetneq e$,

$uz =^* z$ for all $z \in e \setminus \bigcup_{i=1}^n f_i$ implies that there is an $x \in e \setminus \bigcup_{i=1}^n f_i$ with $ux = x$.

We just sketch the idea of the verifying process. It is straightforward to see that the above statement implies (iii). For the reverse direction, take a $y \in e \setminus \bigcup_{i=1}^n f_i$, and $\mathbf{v} \in \mathbf{d}(y)$. Then by assumption $uy =^* y$ and we have $uy = y\tilde{u}$ for some $v \in \mathbf{vC}^*\mathbf{v}$. Take $f_i' = y\mathbf{C} \cap f_i$. Then $\bigcup_{i=1}^n f_i' \subsetneq \mathbf{vC}$ implies that $\bigcup_{i=1}^n \sigma_y^{-1} f_i' \subsetneq \mathbf{vC}$. For every $\tilde{x} \in \mathbf{vC} \setminus \bigcup_{i=1}^n \sigma_y^{-1} f_i'$, $y\tilde{u}\tilde{x} = uy\tilde{x} \in y\tilde{x}\mathbf{C}^*$ so that $\tilde{u}\tilde{x} \in \tilde{x}\mathbf{C}^*$. Hence (iii) implies that there is an $x \in \mathbf{vC} \setminus \bigcup_{i=1}^n \sigma_y^{-1} f_i$ with $\tilde{u}x = x$. Thus $yx = e \setminus \bigcup_{i=1}^n f_i$ and $u(yx) = y\tilde{u}x = yx$.

So we can assume the new condition above to prove (ii). Let $u \in \mathbf{C}^{*,0}$. By assumption we can take a compact open basis element $\Omega(e; f_1,\dots,f_n)$ such that $[u,\Omega(e; f_1,\dots,f_n)] \subseteq \mathrm{Iso}(\mathbf{C}^{*,0} \ltimes \Omega)$. Then we have that $uz \in z\mathbf{C}^*$ hence $uz =^* z$ for all $e \setminus \bigcup_{i=1}^n f_i$. Hence the new statement given above implies that there exists an $x \in e \setminus \bigcup_{i=1}^n f_i$ with $ux = x$. Then $\chi_x \in \Omega(e; f_1, \ldots, f_n)$ because $x \in e \setminus \bigcup_{i=1}^n f_i$. Furthermore, $ux = x$ implies that $[u, \chi_x] = \chi_x \in \Omega$. Hence $\mathbf{C}^{*,0} \ltimes \Omega$ is topologically free. $\square$

The idea to obtain the following analogous statement for $I_l \bar{\ltimes} \Omega$ is similar. We record it without proof.

Theorem 3.15 (Characterization of topological freeness of $I_l \bar{\ltimes} \Omega$, Li [5], Theorem 4.5)**.** *Let $\mathbf{C}^{*,0}$ be defined as in the previous theorem. Define $\mathbf{C}^{*,0} \bar{\ltimes} \Omega := \{[u,\chi] \in I_l \bar{\ltimes} \Omega : u \in \mathbf{C}^{*,0}\}$. Then the following are equivalent:*

(i) *$I_l \bar{\ltimes} \Omega$ is effective;*

(ii) *$\mathbf{C}^{*,0} \bar{\ltimes} \Omega$ is effective;*

(iii) *For every $\mathbf{v} \in \mathbf{C}^0$, every $u \in \mathrm{id}_{\mathbf{v}}\, \mathbf{C}^*\, \mathrm{id}_{\mathbf{v}}$ and every finite subset $\{f_1, \ldots, f_n\} \subseteq J$ with $f_i \subsetneq \mathbf{v}\mathbf{C}$ for all $i = 1, \ldots, n$,*
$uz =^* z$ for all $z \in \mathbf{v}\mathbf{C} \setminus \bigcup_{i=1}^n f_i$ implies that $ux = x$ for all $x \in \mathbf{v}\mathbf{C} \setminus \bigcup_{i=1}^n f_i$.

We can see that Theorem 3.15 (iii) implies Theorem 3.14 (iii), so we immediately have the following implication.

Corollary 3.16. *The effectiveness of $I_l \bar{\ltimes} \Omega$ implies the topological freeness of $I_l \ltimes \Omega$.*

♪ 4

Garside theory

In this section, we recall necessary parts of the Garside theory for our discussion following Dehornoy et al. [2]. However, we note that the directions of arrows are opposite to those in [2].

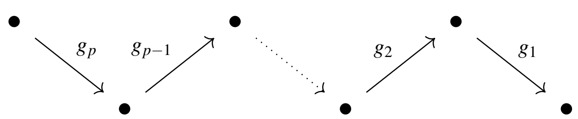
4.1 Paths

Definition 4.1 (**C**-path)**.** Let **C** be a small category.

- A finite **C**-path is a finite sequence $(g_1,\ldots,g_p)$ such that $\mathbf{t}(g_{k+1}) = \mathbf{d}(g_k)$ for all $k = 1,2,\ldots,p-1$.
- An infinite **C**-path is an infinite sequence $(g_1,g_2,\ldots)$ such that $\mathbf{t}(g_{k+1}) = \mathbf{d}(g_k)$ for all $k \in \mathbb{N}_+$.
- In finite cases, we say $(g_1,\ldots g_p)$ is of length p. In infinite cases, we say that the length of the **C**-path is ∞.
- For every object $x \in \mathbf{C}^0$, we introduce an empty path $()_x$ or written as ε_x whose source and target are x and whose length is required to be zero.

If the small category is clear in the context, we often just call a **C**-path simply a path.

The following is a visualization of a finite path.



Definition 4.2 (Concatenation)**.** If $(f_1,\ldots,f_p)$ and $(g_1,\ldots,g_q)$ are two **C**-paths and $\mathbf{t}(f_1) = \mathbf{d}(g_q)$, we define the concatenation $(g_1,\ldots,g_q)|(f_1,\ldots,f_p)$ of these two paths as a new path

$$(g_1,\ldots,g_q)|(f_1,\ldots,f_p) := (g_1,\ldots,g_q,f_1,\ldots,f_p).$$

Moreover, when $x = \mathbf{d}(g_q)$ and $y = \mathbf{t}(g_1)$, we put $(g_1,\ldots,g_q)|()_x = (g_1,\ldots,g_q) = ()_y|(g_1,\ldots,g_q)$. Also for each $x \in \mathbf{C}$ we put $()_x|()_x = ()_x$.

Remark 4.3. We see from the definition that adjacent elements in a path can be composed. To avoid confusion, we keep writing $g_1|\cdots|g_p$ to refer to a finite path $(g_1,\ldots,g_p)$, or $g_1|g_2|\cdots$ to refer to an infinite path and write $g_1\cdots g_p$ to refer to the product of these elements in the path $(g_1,\ldots,g_p)$. In some literature, for example, Li [5] (Definition 6.4), the author does not tell the difference between a product and a (finite) path.

4.2 Greediness

Definition 4.4 (S-greedy)**.** Let S be a subfamily of a left cancellative small category $\mathbf{C}$. A length-two path $g_1|g_2$ is said to be S-greedy if each relation $s \preceq fg_1g_2$ with $s \in S$ and $f \in \mathbf{C}$ implies that $s \preceq fg_1$.

A path $g_1|\cdots|g_p$ is said to be S-greedy if $g_i|g_{i+1}$ is S-greedy for each $i = 1, 2, \ldots, p-1$.

By definition, a path of length zero or one is always S-greedy.

Two simple observations are useful.

Proposition 4.5. *Let S be a subfamily of a left cancellative small category $\mathbf{C}$.*

(i) If two length-two paths $g_1|g_2$ and $g_2|g_3$ are both S-greedy, then so is the concatenated path $g_1|g_2|g_3$.

(ii) If $g_1|\cdots|g_p$ is S-greedy then for every $1 \leq q < r \leq p$, $g_q|\cdots|g_r$ is S-greedy. This in turn implies that $g_q\cdots g_{r-1}|g_r$ is S-greedy.

Proof. The first statement is straightforward and let's prove the second statement. By definition, the length-two paths $g_q|g_{q+1}$,..., $g_{r-1}|g_r$ are all S-greedy, so $g_q|\cdots|g_r$ is S-greedy by (i). The latter half of the second statement is because $s \preceq fg_q\cdots g_r = (fg_q\cdots g_{r-2})g_{r-1}g_r$ implies that $s \preceq fg_q\cdots g_{r-1}$. □

The general statement is given as follows.

Theorem 4.6 (Grouping entries)**.** *Let $\mathbf{C}$ be a left cancellative small category and $S \subseteq \mathbf{C}$ be a subfamily. If a finite path $g_1|\cdots|g_p$ is S-greedy, then so is every path obtained from $g_1|\cdots|g_p$ by replacing adjacent entries with their product.*

Proof. We first claim that if $g_1|\cdots|g_p$ is S-greedy and $s \in S$ satisfies $s \preceq fg_1\cdots g_p$, then for each q with $1 \leq q \leq p$ we have $s \preceq fg_1\cdots g_q$. This is an argument for decreasing induction. For $q = p$, this is just the initial assumption. For $q < p$, the induction hypothesis givs that $s \preceq fg_1\cdots g_q \preceq fg_1\cdots g_qg_{q+1} = (fg_1\cdots g_{q-1})g_qg_{q+1}$. Since $g_1|\cdots|g_p$ is S-greedy, $g_q|g_{q+1}$ is greedy by definition. This implies that $s \preceq (fg_1\cdots g_{q-1})g_q = fg_1\cdots g_{q-1}g_q$ and the induction goes on to finish proving the claim.

Suppose that $g_1|\cdots|g_p$ is S-greedy and $1 \leq q < r \leq p$. Applying the above claim to the S-greedy path $g_{q-1}|\cdots|g_r$ shows that every element $s \in S$ with $s \preceq fg_{q-1}(g_q\cdots g_r)$ left divides fg_{q-1} which implies that $g_{q-1}|g_q\cdots g_r$ is S-greedy. Applying the above claim to the S-greedy path $g_q|\cdots|g_{r+1}$ shows that every element $s \in S$ with $s \preceq f(g_q\cdots g_r)g_{r+1}$ left divides $f(g_q\cdots g_r)$, which implies that $g_q\cdots g_r|g_{r+1}$ is S-greedy. Hence $g_1|\cdots|g_{q-1}|g_q\cdots g_r|g_{r+1}|\cdots|g_p$ is S-greedy. □

The following corollary from the above results can be used as another definition of the greediness of a path. It means that to determine whether a path is greedy, we do not need to use the auxiliary arbitrary element $f \in \mathbf{C}$ in Definition 4.4.

Corollary 4.7 (An equivalent definition for greediness)**.** *Let $\mathbf{C}$ be a left cancellative small category and $S \subseteq \mathbf{C}$ be a subfamily. A length-two path $g_1|g_2$ is S-greedy if and only if each relation $s \preceq g_1g_2$ with $s \in S$ implies that $s \preceq g_1$.*

Moreover, a path $g_1|\cdots|g_p$ is S-greedy if and only if each relation $s \preceq g_q\cdots g_r$ with $s \in S$ implies that $s \preceq g_q$ for every $1 \leq q < r \leq p$.

Proof. Suppose $g_1|g_2$ is S-greedy then we take $f = \mathrm{id}_{\mathbf{t}(g_1)}$ in Definition 4.4 and obtain the "only if" statement. Suppose conversely that each relation $s \preceq g_1g_2$ with $s \in S$ implies that $s \preceq g_1$. Then we replace g_1 by fg_1 and obtain the "if" statement.

In general, if the path $g_1|\cdots|g_p$ is S-greedy then so is $g_q|\cdots|g_r$ by Proposition 4.5 (ii) and hence $g_q|g_{q+1}\cdots g_r$ by Theorem 4.6. Thus each relation $s \preceq g_q\cdots g_r$ with $s \in S$ implies that $s \preceq g_q$ for every

$1 \le q < r \le p$ by the first statement. On the other hand, suppose each relation $s \preceq g_q \cdots g_r$ with $s \in S$ implies that $s \preceq g_q$. Then we take $q = 1$ and $r = p$ and replace g_1 by fg_1, and we obtain the greediness of $g_1|\cdots|g_p$. $\square$

Proposition 4.8 (S^m-greedy)**.** *Let S be a subfamily of a left cancellative small category $\mathbf{C}$ and $g_1|\cdots|g_p$ is S-greedy. Then for every m with $1 \le m \le p$ the (length two) path $g_1 \cdots g_m|g_{m+1} \cdots g_p$ is S^m-greedy. That is, each relation $s \preceq fg_1 \cdots g_p$ with $s \in S^m$ implies $s \preceq fg_1 \cdots g_m$.*

Proof. For $s \in S^m$, write $s = s_1 \cdots s_m$ with each $s_i \in S$, $i = 1, 2, \dots, m$. Suppose now we have a relation $s \preceq fg_1 \cdots g_p$ with $s = s_1 \cdots s_m \in S^m$. We take a $t \in \mathbf{C}$ such that $s_1 \cdots s_m t = fg_1 \cdots g_p$. Then first we have $s_1 \preceq s \preceq fg_1 \cdots g_p$, and by the assumption that $g_1|\cdots|g_p$ is S-greedy, we have $s_1 \preceq fg_1$, so that $s_1 h_1 = fg_1$ for some $h_1 \in \mathbf{C}$. For the sake of convenience, we let $h_0 = f$. By left cancellativity of $\mathbf{C}$, $s_2 \cdots s_m t = h_1 g_2 \cdots g_p$. Now we have the following sequence of statements from greediness:

$$\begin{aligned} s_2 \cdots s_m t = h_1 g_2 \cdots g_p &\Rightarrow s_2 \preceq h_1 g_2 \text{ say, } s_2 h_2 = h_1 g_2, \\ s_3 \cdots s_m t = h_2 g_3 \cdots g_p &\Rightarrow s_3 \preceq h_2 g_3 \text{ say, } s_3 h_3 = h_2 g_3, \\ &\vdots \\ s_m t = h_{m-1} g_m \cdots g_p &\Rightarrow s_m \preceq h_{m-1} h_m \text{ say, } s_m h_m = h_{m-1} g_m. \end{aligned}$$

Thus, $s_{m-1} s_m = s_{m-1} h_m g_m = h_{m-2} g_{m-1} g_m$, and inductively,

$$s h_m = s_1 \cdots s_m h_m = fg_1 \cdots g_m.$$

Therefore, $s \preceq fg_1 \cdots g_m$ finishing the proof. $\square$

Lemma 4.9. *Let $\mathbf{C}$ be a left cancellative small category and let S be a subfamily of $\mathbf{C}$. Then a finite $\mathbf{C}$-path is S-greedy of and only if it is $S^\sharp$-greedy, if and only if $\mathbf{C}^* S^\sharp$-greedy.*

Proof. Observe that if a $\mathbf{C}$-path is S-greedy then it is S'-greedy for every $S' \subseteq S$. Since $S \subseteq S^\sharp \subseteq \mathbf{C}^* S^\sharp$, being $\mathbf{C}^* S^\sharp$-greedy implies being $S^\sharp$-greedy and in turn implies being S-greedy.

Conversely, suppose that the path $g_1|\cdots|g_p$ is S-greedy, and let $s \in \mathbf{C}^* S^\sharp$, say $\varepsilon_1 s' \varepsilon_2$ with $\varepsilon_1, \varepsilon_2 \in \mathbf{C}^*$ and $s' \in S$. Given $s \preceq fg_i g_{i+1}$, then we have $s' = (\varepsilon_1^{-1} f) g_i g_{i+1}$. As $s \in S$ and $g_i|g_{i+1}$ is S-greedy, this implies that $s' \preceq (\varepsilon_1^{-1} f) g_i$, say $\varepsilon_1^{-1} f g_i = s' f'$ for some $f' \in \mathbf{C}$. We deduce that $s(\varepsilon_2^{-1} f') = \varepsilon_1 s' \varepsilon_2^{-1} f' = \varepsilon_1 \varepsilon_1' f g_i$, whence $s \preceq fg_i$ so $g_i|g_{i+1}$ is $\mathbf{C}^* S^\sharp$-greedy. $\square$

4.3 Normal decomposition

Definition 4.10 (The family $S^\sharp$, closeness under $=^*$)**.** Let S be a subfamily of a left cancellative small category $\mathbf{C}$. We define the family $S^\sharp$ as

$$S^\sharp := S\mathbf{C}^* \cup \mathbf{C}^*.$$

We say that S is closed under $=^*$ or $=^*$-closed if for every $g' \in \mathbf{C}^*$, $g' =^* g$ for some $g \in S$ implies that $g' \in S$.

We call $S^\sharp$ the closure of the family S with respect to the right-multiplication by invertible elements.

Some remarks come as follows:

- We have that $(S^\sharp)^\sharp = S^\sharp$. This is one of the reasons why $S^\sharp$ is called the closure of S.
- $\mathbf{C}^* = (\mathbf{id_C})^\sharp$, where $\mathbf{id_C} := \{\mathrm{id}_\mathbf{v} : \mathbf{v} \in \mathbf{C}^0\} = \mathbf{C}^0$.
- If $\mathbf{C}$ has no nontrivial invertible elements (i.e.,$\mathbf{C}^* = \mathbf{id_C}$) then the relation $=^*$ is an equality, so every subfamily S of $\mathbf{C}$ is $=^*$-closed and $S^\sharp$ is precisely $S \cup \mathbf{id_C}$.

Lemma 4.11. *Let $\mathbf{C}$ be a left cancellative small category.*

(i) For every subfamily S of $\mathbf{C}$, the closure $S^\sharp$ is the smallest subfamily with respect to the inclusion of sets that is $=^*$-closed and includes $S \cup \mathbf{C}^0$.

(ii) The intersection of arbitrarily many $=^*$-closed subfamily of $\mathbf{C}$ is $=^*$-closed.

Lemma 4.11 (i) is another reason why $S^\sharp$ is called the closure of S.

Proof. Observe that S is $=^*$-closed if and only if $S\mathbf{C}^* \subseteq S$, hence if and only if $S\mathbf{C}^* = S$ because $S \subseteq S\mathbf{C}^*$ trivially.

$$\begin{aligned} S^\sharp\mathbf{C} &= (S\mathbf{C}^* \cup \mathbf{C}^*)\mathbf{C}^* \\ &= S\mathbf{C}^*\mathbf{C}^* \cup \mathbf{C}^*\mathbf{C}^* \\ &= S\mathbf{C}^* \cup \mathbf{C}^* = S^\sharp, \end{aligned}$$

where we have obviously $\mathbf{C}^*\mathbf{C}^* = \mathbf{C}^*$.

It is straightforward that $S^\sharp$ includes $S \cup \mathbf{C}^0$. If there is another family S' that is $=^*$-closed and includes $S \cup \mathbf{C}^0$ then S' contains $(S \cup \mathbf{C}^0)\mathbf{C}^* = S\mathbf{C}^* \cup \mathbf{C}^* = S^\sharp$.

For the second item, $\{S_\alpha\}_{\alpha \in A}$ be a subfamilies of $\mathbf{C}$ which are $=^*$-closed. Then $S_\alpha\mathbf{C}^* = S_\alpha$ for all $\alpha \in A$. Taking the intersection on both sides yields the desired result.

□

Definition 4.12 (S-normal, strict normal)**.** Let $\mathbf{C}$ be a left cancellative small category and S is a subfamily of $\mathbf{C}$. A finite or infinite $\mathbf{C}$-path is S-normal if it is S-greedy and every entry lies in $S^\sharp$.

An S-normal path is strict if no entry is invertible and when it is finite, all entries except possibly the last one lie in S.

Definition 4.13 (S-normal decomposition)**.** We say that a path $s_1|\cdots|s_p$ is an S-normal decomposition (or a strict S-normal decomposition) for an element g if $s_1|\cdots|s_p$ is an S-normal path (or a strict S-normal path) and $g = s_1 \cdots s_p$.

Proposition 4.14. *Let $\mathbf{C}$ be a left cancellative small category and S is a subfamily of $\mathbf{C}$. A $\mathbf{C}$-path is S-normal if and only if it is $S^\sharp$-normal.*

Proof. Let $s_1|\cdots|s_p$ be an S-normal path. By Lemma 4.9 $s_1|\cdots|s_p$ is greedy and each $s_i, i = 1,2,\ldots,p$, is an element of $S^\sharp$, which is the same as $(S^\sharp)^\sharp$ by the remark given below Definition 4.10. Hence $s_1|\cdots|s_p$ is $S^\sharp$-normal.

Conversely, assume that $s_1|\cdots|s_p$ is $S^\sharp$-normal. Then it is $S^\sharp$-greedy and hence S-greedy, again by Lemma 4.9, with each $s_i, (i = 1,2,\ldots,p)$ lying in $(S^\sharp)^\sharp = S^\sharp$. Hence $s_1|\cdots|s_p$ is S-normal.

□

Definition 4.15 (Deformation by invertible elements)**.** Let $\mathbf{C}$ be a left cancellative small category. A $\mathbf{C}$-path $s_1|\cdots|f_p$ is said to be a deformation by invertible elements or $\mathbf{C}^*$-deformation of another $\mathbf{C}$-path $t_1|\cdots|t_q$ if there exist $\varepsilon_0,\ldots,\varepsilon_m, m = \max(p,q)$, such that ε_0 and ε_m are identity elements and $\varepsilon_{i-1}t_i = s_i\varepsilon_i$ holds for $1 \le i \le m$, where for $p \ne q$ the shorter path is expanded by identity elements.

The following commutative diagram visualizes Definition 4.15 in the case $q \ge p$.

$$\begin{tikzcd}
x \arrow[r, "g_q"] \arrow[d, equal, "\varepsilon_q=\mathrm{id}_x"'] & \bullet \arrow[r, dotted] \arrow[d, shift right] \arrow[d, shift left, leftarrow] & \bullet \arrow[r, "g_{p+1}"] \arrow[d, shift right] \arrow[d, shift left, leftarrow] & \bullet \arrow[r, "g_p"] \arrow[d, shift right, "\varepsilon_p"'] \arrow[d, shift left, leftarrow, "\varepsilon_p^{-1}"] & \bullet \arrow[r, dotted] \arrow[d, shift right] \arrow[d, shift left, leftarrow] & \bullet \arrow[r, "g_2"] \arrow[d, shift right] \arrow[d, shift left, leftarrow] & \bullet \arrow[r, "g_1"] \arrow[d, shift right, "\varepsilon_1"'] \arrow[d, shift left, leftarrow, "\varepsilon_1^{-1}"] & y \arrow[d, equal, "\varepsilon_0=\mathrm{id}_y"] \\
x \arrow[r, equal, "\mathrm{id}_{-}"'] & \bullet \arrow[r, dotted] & \bullet \arrow[r, equal, "\mathrm{id}_{-}"'] & \bullet \arrow[r, "f_p"'] & \bullet \arrow[r, dotted] & \bullet \arrow[r, "f_2"'] & \bullet \arrow[r, "f_1"'] & y
\end{tikzcd}$$

The following theorem with proof omitted, gives the uniqueness of a normal decomposition up to deformation by invertible elements.

Theorem 4.16 (Normal uniqueness, Dehornoy et al. [2], Chapter III, Proposition 1.25)**.** *Let S be a subfamily of a left cancellative small category* $\mathbf{C}$*. Then any two S-normal decompositions of an element of* $\mathbf{C}$ *are a deformation by invertible elements of one another.*

Definition 4.17 ($\sim$-transverse)**.** Suppose $\sim$ is an equivalence relation on a set S. A subset S' of S is said to be $\sim$-transverse if distinct element of S' are not $\sim$-equivalent.

Definition 4.18 ($\sim$-selector)**.** Assumptions as above. A subset S'' is said to be a $\sim$-selector in S' if $S'' \subseteq S$ and contains exactly one element in each $\sim$-equivalence class intersecting S'.

By definition, we can see that a $\sim$-selector is $\sim$-transverse. If we accept the Axiom of Choice, selectors always exist. Also, if $\mathbf{C}$ contains no nontrivial invertible element, then every subfamily of $\mathbf{C}$ is $=^*$-transverse.

Corollary 4.19. *Let* $\mathbf{C}$ *be a left cancellative small category and S is a subfamily of* $\mathbf{C}$ *that is $=^*$-transverse. Then every non-invertible element of* $\mathbf{C}$ *admits at most one strict S-normal decomposition.*

Proof. Assume that $s_1|\cdots|s_p$ and $t_1|\cdots|t_q$ are strict S-normal decompositions of some element g. We first show $p = q$. By assumption, none of s_i and t_i is invertible and Theorem 4.16 gives the equality of p and q. Then an induction on $i < p$ with Theorem 4.16 gives $s_i =^* t_i$ so $s_i = t_i$ because S is $=^*$-transverse and each $s_i, t_i \in S$. Finally $s_1 \cdots s_{p-1}s_p = t_1 \cdots t_{p-1}t_p$ whence $s_p = t_p$ by left cancellating $s_1 \cdots s_{p-1}$. □

Corollary 4.20. *Let S be a subfamily of a left cancellative small category* $\mathbf{C}$*, then for every element g of* $\mathbf{C}$ *the number of non-invertible elements in an S-normal decomposition of g (if any) does not depend on the choice of decomposition.*

Proof. Let $s_1|\cdots|s_p$ and $t_1|\cdots|t_q$ be two S-normal decomposition of a given element g. Theorem 4.16 implies that one is a deformation by invertible elements of the other. This in turn implies that s_i is invertible if and only if t_i is. □

From Corollary 4.20, the number of non-invertible elements in a normal decomposition of $g \in \mathbf{C}$ depends only on g itself. Hence we can record this number as its net length.

Definition 4.21 (Net S-length)**.** Let S be a subfamily of a left cancellative small category $\mathbf{C}$ and g is an element of $\mathbf{C}$ that admits at least one S-normal decomposition. The (unique) number of non-invertible elements in (all) S-normal decompositions of g is called the net S-length of g, denoted by $\|g\|_S$.

Remark 4.22. In Dehornoy et al. [2], the term "net S-length" is simply called S-length. Here use the term "net" to emphasize that we ignore all the invertible elements in a path and count the number of non-invertible elements, and avoid confusion with the length given in Definition 4.1.

Definition 4.23 (S-head)**.** Let S-be a subfamily of a left cancellative small category $\mathbf{C}$. Given $a \in \mathbf{C}$, an element $s \in S$ is an S-head of a if s is a greatest left divisor of a is S.

In the case that S is $=^*$-transverse, the S-head is unique if it exists. In this case, the S-head of an element $a \in \mathbf{C}$ is denoted as $H_S(a)$. We may also omit S and write $H(a)$ instead when it is clear in the context.

4.4 Garside families

Definition 4.24 (Garside family)**.** Let $\mathbf{C}$ be a left cancellative small category. A subfamily $\mathbf{G}$ of $\mathbf{C}$ is called a Garside family if every element of $\mathbf{C}$ admits at least one $\mathbf{G}$-normal decomposition.

A trivial example of a Garside family is the small category $\mathbf{C}$ itself. Thus we are focusing on non-trivial Garside families of $\mathbf{C}$.

Proposition 4.25 (Invariance under closure)**.** *Let $\mathbf{C}$ be a left cancellative small category. The subfamily $\mathbf{G}$ is a Garside family of $\mathbf{C}$ if and only if $\mathbf{G}^\sharp$ is.*

Proof. Suppose that $\mathbf{G}$ is a Garside family of $\mathbf{C}$. Then every element of $\mathbf{C}$ admits a $\mathbf{G}$-normal decomposition. By Proposition 4.14, this is also an $\mathbf{G}^\sharp$-normal decomposition. Hence $S^\sharp$ is a Garside family. Conversely, suppose that $\mathbf{G}^\sharp$ is a Garside family of $\mathbf{C}$. Then every element of $\mathbf{C}$ admits a $\mathbf{G}^\sharp$-normal decomposition and again by Proposition 4.14 this is $\mathbf{G}$-normal. Hence $\mathbf{G}$ is a Garside family. □

Corollary 4.26. *Let $\mathbf{C}$ be a left cancellative small category.*

(i) If $\mathbf{G}$ and $\mathbf{G}'$ are subfamilies of $\mathbf{C}$ and satisfy $\mathbf{G}^\sharp = (\mathbf{G}')^\sharp$, then $\mathbf{G}$ is a Garside family if and only if $\mathbf{G}'$ is.

(ii) If $\mathbf{G}'$ is a $=^*$-selector in $\mathbf{G}$, then $\mathbf{G}$ is a Garside family if and only if $\mathbf{G}' \setminus \mathbf{id}_\mathbf{C}$ is.

Proof. (i) directly follows from Proposition 4.25. For (ii) observe that $\mathbf{G}'$ being an $=^*$-selector in $\mathbf{G}$ implies that $\mathbf{G}^\sharp = (\mathbf{G}')^\sharp = (\mathbf{G}' \setminus \mathbf{id}_\mathbf{C})^\sharp$ and the desired result follows by applying (i). □

We record a lemma from about the existence of a $\mathbf{G}$-head when $\mathbf{G}$ is a Garside family without proof, which will be used later.

Lemma 4.27 (Dehornoy et al. [2] Chapter IV, Proposition 1.24)**.** *If $\mathbf{G}$ is a Garside family of a left cancellative small category $\mathbf{C}$, then every non-invertible element a admits a $\mathbf{G}$-head.*

♪ 5

Main results

Let $\mathbf{C}$ be a left cancellative small category and S be a subfamily of $\mathbf{C}$ which generates $\mathbf{C}$. Let $w = s_1|s_2|\cdots$ be an infinite $\mathbf{C}$-path with every $s_i, (i \in \mathbb{N}_+)$ in S. In the case that every s_i lies in S, we also say that w is an infinite S-path, and we should note that an S-normal path is an $S^\sharp$-path. We write

$w_{\leq n} := s_1|\cdots|s_n$ for the finite path formed by the first n elements of the infinite path w,

$w_n := s_1\cdots s_n$ for the product of elements of $w_{\leq n}$,

$w_{=n} := s_n$ for the n-th element of w, and

$w_{>n} := s_{n+1}|s_{n+2}|\cdots$ for the path obtained by deleting first n-elements from w.

Also we define $\Omega_\infty := \Omega \setminus \{\chi_x : x \in \mathbf{C}\}$.

Definition 5.1 (Character from an infinite path)**.** Let S be a subfamily of a left cancellative small category $\mathbf{C}$ which generates $\mathbf{C}$. Given a S-path w, we define a map $\chi_w : J \to \{0,1\}$ by

$$\chi_w(e) = \begin{cases} 1, & \text{if } w_n \in e \text{ for some } n \in \mathbb{N}_+, \\ 0, & \text{otherwise.} \end{cases}$$

Proposition 5.2. *$\chi_w \in \Omega$ for every infinite S-path w defined above.*

This is straightforward from the corresponding definitions.

Lemma 5.3. *Let $\mathbf{C}$ be a finitely aligned left cancellative small category which is also countable. Let S be a subfamily of $\mathbf{C}$ generating $\mathbf{C}$. Then every $\chi \in \Omega_\infty$ is of the form χ_w for some infinite S-path.*

Proof. Let $\chi \in \Omega_\infty = \Omega \setminus \{\chi_x : x \in \mathbf{C}\}$ and recall that $\mathscr{F}_p^\chi := \{x\mathbf{C} : x \in \mathbf{C} \text{ with } \chi(x\mathbf{C}) = 1\}$. Since $\mathbf{C}$ is assumed to be countable, we can write $\mathscr{F}_p^\chi = \{x_1\mathbf{C}, x_2\mathbf{C}, \ldots\}$. Now let $w^{(1)} = x_1$ so that $\chi(w^{(1)}\mathbf{C}) = 1$. We now carry out an induction showing that for each $n \in \mathbb{N}_+$, there is an element $w^{(n)}$ which is a common right multiple of $x_1, \ldots, x_n$ and satisfies $\chi(w^{(n)}\mathbf{C}) = 1$. Suppose that there is a $w^{(n-1)}$ such that $\chi(w^{(n-1)}\mathbf{C}) = 1$. Since $\chi(w^{(n-1)}\mathbf{C} \cap x_n\mathbf{C}) = \chi(w^{(n-1)}\mathbf{C})\chi(x_n\mathbf{C}) = 1$, then by Lemma 1.55, there exists a minimal common right multiple of $w^{(n-1)}$ and x_n, say, $w^{(n)}$, such that $\chi(w^{(n)}\mathbf{C}) = 1$. Otherwise $\chi(w^{(n-1)}\mathbf{C} \cap x_n\mathbf{C})$ would be 0. Now since $\chi \notin \{\chi_x : x \in \mathbf{C}\}$ we may assume that $w^{(n)}$ is not invertible for all $n \in \mathbb{N}_+$. Thus we can write $w^{(n)} = s_1^{(n)} \cdots s_{l_n}^{(n)}$ as an S-path for each $n \in \mathbb{N}_+$, where l_n is an natural number depending on $w^{(n)}$. Now we define a new S-path

$$w := s_1^{(1)}|\cdots|s_{l_n}^{(1)}|s_1^{(2)}|\cdots|s_{l_2}^{(2)}|s_1^{(3)}|\cdots|s_{l_3}^{(3)}|\cdots$$

and we claim that χ is indeed χ_w. We make use of Lemma 3.9 (ii) to show this. We see that $\chi_w(x\mathbf{C}) = 1$ if and only if $w_n \in x\mathbf{C}$ for some $n \in \mathbb{N}_+$ if and only if $w_n\mathbf{C} \subseteq x\mathbf{C}$ (for this n by Proposition 1.42 (i)). This holds again if and only if $\chi(x\mathbf{C}) = 1$ using Lemma 3.9 because $\chi(w^{(n)}\mathbf{C}) = 1$ and $w^{(n)}\mathbf{C} \subseteq x_n\mathbf{C}$ give $\chi(x_n\mathbf{C}) = 1$ for all $x_n\mathbf{C} \in \mathscr{F}_p$.

□

Inspired by Lemma 2.14, we would like to see how $s.\chi_w$ behaves given $s \in I_l$ and an (infinite) S-path w. For a given $s \in I_l$, $s.\chi_w$ is defined if and only if $\chi_w(s^{-1}s) = 1$, if and only if there is an $m \in \mathbb{N}_+$ such that $w_m \in \mathrm{dom}(s)$.

Now let $s.\chi_w$ be defined. For an $e \in J$, $s.\chi_w(e) = \chi_w(s^{-1}es) = 1$ if and only if there is an $n \in \mathbb{N}_+$ such that $w_n \in \mathrm{dom}(es) = s^{-1}(\mathrm{dom}(e) \cap \mathrm{im}(s))$. This is equivalent to say that $\chi_{s(w_n)}(e) = s.\chi_{w_n}(e) = 1$ for some n. Thus, $s.\chi_w = \chi_{s(w_n)} = s.\chi_{w_n} = s.\chi_{w_n|w_{>n}}$ for this n. In particular, if an element $c \in \mathbf{C}$ satisfies that $\mathbf{d}(c) = \mathbf{t}(w_1) = \mathbf{t}(w_n)$, then for every e such that $\sigma_c.\chi_w(e) = 1$, we have $1 = \sigma_c.\chi_w(e) = \chi_{\sigma_c(w_n)}(e) = \chi_{cw_n}(e)$, where $n \in \mathbb{N}_+$ (and hence w_n) depends on e. Thus $1 = \chi_{cw_n} = \chi_{cw_n|w_{>n}}(e) = \chi_{c|w_{\leq n}|w_{>n}}(e) = \chi_{c|w}(e)$ for all $e \in J$ such that $\sigma_c.\chi_w(e) = 1$. This proves that $\sigma_c.\chi_w = \chi_{c|w}$ for every nonempty normal path w and every $c \in \mathbf{C}$ satisfying $\mathbf{d}(c) = \mathbf{t}(w_1)$.

Lemma 5.4. *Let $\mathbf{C}$ be a finitely aligned left cancellative small category and let S be a subfamily of $\mathbf{C}$ which generates $\mathbf{C}$. Given $c, d \in \mathbf{C}$ with $\mathbf{d}(d) = \mathbf{d}(c)$ and an infinite $\mathbf{G}$-path w, then $\sigma_c\sigma_d^{-1}.\chi_w$ is defined if and only if there is an $n \in \mathbb{N}_+$ such that $d \preceq w_n$. In this case, if we have $dx_n = w_n$, then $\sigma_c\sigma_d^{-1}.\chi_w = \chi_{cx_n|w_{>n}}$.*

Proof. $\sigma_c\sigma_d^{-1}.\chi_w$ is defined if and only if $\chi_w((\sigma_c\sigma_d^{-1})^{-1}(\sigma_c\sigma_d^{-1})) = 1$, if and only if there is an $n \in \mathbb{N}_+$ such that $w_n \in \mathrm{dom}(\sigma_c\sigma_d^{-1}) = \sigma_c^{-1}(\mathbf{d}(c)\mathbf{C} \cap \mathbf{d}(d)\mathbf{C})$. However, in order to guarantee that $\sigma_c\sigma_d^{-1}.\chi_w$ is a nonzero character, we need to require that $\mathbf{d}(c)\mathbf{C} \cap \mathbf{d}(d)\mathbf{C} \neq \varnothing$. This holds if and only if $\mathbf{d}(c) = \mathbf{d}(d)$, and hence $\mathrm{dom}(\sigma_c\sigma_d^{-1}) = d\mathbf{C}$. That is to say, $\sigma_c\sigma_d^{-1}.\chi_w$ is defined if and only if there is an $n \in \mathbb{N}_+$ such that $w_n \in d\mathbf{C}$, or equivalently, $d \preceq w_n$.

In this case, if we have $dx_n = w_n$, where x_n depends on w_n, then

$$\sigma_c\sigma_d^{-1}.\chi_w = \sigma_c\sigma_d^{-1}.\chi_{w_n} = \sigma_c\sigma_d^{-1}.\chi_{dx_n} = \sigma_c.\chi_{x_n} = \chi_{cx_n} = \chi_{cx_n|w_{>n}}.$$

□

Remark 5.5. Here we have corrected the typos arising from Li [5] Lemma 6.15 (iii) and its proof.

Definition 5.6 (Local finiteness, Local boundedness)**.** Let $\mathbf{C}$ be a small category. A subfamily S of $\mathbf{C}$ is said to be

- locally finite if $\mathbf{v}S$ is finite for all $\mathbf{v} \in \mathbf{C}^0$;
- locally bounded if for every $\mathbf{v} \in \mathbf{C}^0$ there is no infinite sequence $s_1, s_2, \ldots$ in $\mathbf{v}S$ with $s_1 \prec s_2 \prec \cdots$.

A direct observation from the second item in Definition 5.6 is that if $S \subseteq \mathbf{C}$ is locally bounded, then there is also no strictly increasing sequence in S itself.

Given two S-paths $x = s_1|s_2|\cdots$ and $y = t_1|t_2|\cdots$ we mean $x = y$ by requiring $s_i = t_i$ for all indices i. In the case of finite paths, we also require that their lengths are the same. Then we have the following lemma:

Lemma 5.7. *Let $\mathbf{C}$ be a finitely aligned countable left cancellative small category. Let $\mathbf{G}$ be a Garside family of $\mathbf{C}$ which is $=^*$-transverse, locally bounded, and $\mathbf{G} \cap \mathbf{C}^* = \varnothing$. Then every $\chi \in \Omega \setminus \{\chi_{\mathbf{v}} : \mathbf{v} \in \mathbf{C}^0\}$ is of the form χ_p for some $\mathbf{G}$-normal path p. Moreover, for two normal paths p and q, $\chi_p = \chi_q$ if and only if $p = q$.*

Proof. Since $\Omega_\infty = \Omega \setminus \{\chi_x : x \in \mathbf{C}\}$, we write

$$\Omega \setminus \{\chi_{\mathbf{v}} : \mathbf{v} \in \mathbf{C}^0\} = \Omega_\infty \cup \{\chi_x : x \in \mathbf{C} \setminus \mathbf{C}^0\}.$$

This is a disjoint union. Since $\mathbf{G}$ is a Garside subfamily of $\mathbf{C}$, every x is the product of elements in some finite $\mathbf{G}$-normal path, say, w_x. Hence every χ_x with $x \in \mathbf{C}$ is of the form χ_{w_x}. It remains to see the case that $\chi \in \Omega_\infty$.

By Lemma 5.3, every $\chi \in \Omega_\infty$ is of the form χ_w for some infinite path $w = r_1|r_2|\cdots$. We proceed to construct an (infinite) $\mathbf{G}$-normal path $p = s_1|s_2|\cdots$ so that $\chi = \chi_w = \chi_p$. By Lemma 4.27 and the assumption that $\mathbf{G}$ is $=^*$-transverse, we can take $s_1^{(n)}$ to be the $\mathbf{G}$-head of w_n for every $n \in \mathbb{N}_+$. As $s_1^{(n)} \preceq w_n \preceq w_{n+1}$ by construction, we must have $s_1^{(n)} \preceq s_1^{(n+1)}$. This gives an increasing sequence $\{s_1^{(n)}\}_{n=1}^\infty$ with respect to $\preceq$. Since $\mathbf{G}$ is locally bounded, then from the remark after Definition 5.6 this increasing sequence eventually terminates at some index $N \in \mathbb{N}_+$ and remains constant. We may assume that N is minimal. Let $s_1 := s_1^{(N)}$ and call it $H(w)$, the head of the infinite path w.

By construction, $s_1 = s_1^{(N)} \preceq w_N$ so we can write $w_N = s_1 d_1$ for some $d_1 \in \mathbf{C}$. Define temporary notations $s_1^{-1} w_N := d_1$ (which is well defined because $\mathbf{C}$ is left cancellative) and $s_1^{-1} w$ as the $\mathbf{C}$-path $d_1 | w_{>N}$ and let $s_2 := H(s_1^{-1} w)$. This is well defined because when looking at the construction of s_2, each $\mathbf{G}$-head $s_2^{(n)}$ of the product $d_1 r_{N+1} \cdots r_n$ exists (by Lemma 4.27), is increasing with respect to n, and terminates at s_2.

Similarly we can inductively define for each $m \in \mathbb{N}_+$ the path $s_{m-1}^{-1} s_{m-2}^{-1} \cdots s_1^{-1} w$, and take the head $s_m = H(s_{m-1}^{-1} s_{m-2}^{-1} \cdots s_1^{-1} w)$. Let p be the path obtained by concatenating each $s_n, n \in \mathbb{N}_+$, $p = s_1|s_2|\cdots$. By construction, p a is a well-defined $\mathbf{G}$-normal path. We further claim that $\chi = \chi_w$ is indeed this χ_p. First, we show that for every $n \in \mathbb{N}_+$, there is an integer $N(n)$ (depending on n) large enough such that $p_n \preceq w_{N(n)}$. We do this by induction. This is true by construction for $n = 1$. Suppose $p_n \preceq w_{N(n)}$. Then $s_{n+1} = H(p_n^{-1} w) = H(p_n^{-1} w_{N(n)} | w_{>N(n)})$ meaning that there is an integer $N(n+1)$ (depending on $n+1$) large enough such that $s_{n+1} \preceq p_n^{-1} w_{N(n+1)}$ as desired. Note that the index $N(n)$ is no less than n according to this process so that $w_n \preceq w_{N(n)}$. Next, we prove the claim with the help of Lemma 3.9 (ii).

Given $x \in \mathbf{C}$ with $\chi_p(x\mathbf{C}) = 1$, then by definition, $p_n \in x\mathbf{C}$ for some n, or equivalently, $x \preceq p_n$, and hence $x \preceq p_n \preceq w_{N(n)}$, which implies that $w_{N(n)} \in x\mathbf{C}$ and hence $\chi_w(x\mathbf{C}) = 1$. Conversely, given $x \in \mathbf{C}$ with $\chi_w(x\mathbf{C}) = 1$, we show that $\chi_p(x\mathbf{C}) = 1$. By definition $x \preceq w_n$ for some n. From the above induction argument, we have that $p_n \preceq w_{N(n)}$ for each $n \in \mathbb{N}_+$ with an $N(n) \geq n$ depending on n. Since $\mathbf{G}$ is a Garside family of $\mathbf{C}$ which is also $=^*$-transverse, then the $\mathbf{G}$-normal decomposition of $w_{N(n)}$ exists and unique by Corollary 4.20, so the $\mathbf{G}$-normal decomposition of $w_{N(n)}$ starts with the path $p_{\leq n}$, and we can write $w_n \preceq w_{N(n)} \preceq p_n g_{n_1} \cdots g_{n_m}$ for some $\mathbf{G}^\sharp$-path (actually $\mathbf{G}$-normal) $g_{n_1}|\cdots|g_{n_m}$. Now Proposition 4.8 applied here gives that $w_n \preceq p_n$. Therefore, $x \preceq w_n \preceq p_n$ and $p_n \subseteq x\mathbf{C}$, which indeed implies that $\chi_p(x\mathbf{C}) = 1$.

The uniqueness is easier. Let $p = s_1|s_2|\cdots$ and $q = t_1|t_2|\cdots$ be two $\mathbf{G}$-normal paths such that $\chi_p = \chi_q$. Then $\chi_q(s_1\mathbf{C}) = \chi_p(s_1\mathbf{C}) = 1$ and hence $q_n \in s_1\mathbf{C}$ for some n, or equivalently, $s_1 \preceq q_n$. Since t_1 is the greatest left divisor of q_n by the above construction, we have $s_1 \preceq t_1$. By interchanging the role of p and q we also have that $t_1 \preceq s_1$ and hence $s_1 = t_1$ by transversality of $\mathbf{G}$. Now applying this argument to the path $s^{-1}p = s_2|s_3|\cdots = p_{>1}$ and $t^{-1}q = t_2|t_3|\cdots = q_{>1}$. Using Lemma 2.14, we obtain that $\chi_{s_1^{-1}p} = \sigma_{s_1}^{-1}.\chi_p = \sigma_{t_1}^{-1}.\chi_q = \chi_{t_1^{-1}q}$ so that $s_2 = t_2$ and then $s_i = t_i$ for each $i \in \mathbb{N}_+$ follows inductively. □

Let $\mathbf{C}$ be a finitely aligned left cancellative small category and $\mathbf{G}$ be a Garside family of $\mathbf{C}$ which is $=^*$-transverse and $\mathbf{G} \cap \mathbf{C}^* = \varnothing$. Denote by $\mathscr{W}$ the collection of all $\mathbf{G}$-normal path. If $\mathbf{C}$ is countable and $\mathbf{G}$ is locally bounded, Lemma 5.7 implies that there is a one to one correspondence $\mathscr{W} \sqcup \mathbf{C}^0 \to \Omega$ by $w \mapsto \chi_w, \mathbf{v} \mapsto \chi_{\mathbf{v}}$.

Remark 5.8. The argument of Lemma 5.7 given in Li [5] did not specify for example what the element $s_1^{-1} w_n$ and the path $s_1^{-1} w$ are, which makes it hard to understand. From the context, we can see that these are formal notations, and we should bear in mind that $s_1, s_2, \dots$ are not invertible because of the setting $\mathbf{G} \cap \mathbf{C}^* = \varnothing$.

Lemma 5.9. *Let the* $\mathbf{G}$ *be a Garside family of* $\mathbf{C}$ *as defined above, and let* $\mathscr{W}$ *be the collection of* $\mathbf{G}$*-normal paths.*

- *Given a sequence* $\{w^{(i)}\}$ *in* $\mathscr{W}$ *and an element* $w \in \mathscr{W}$, $\lim_i \chi_{w^{(i)}} = \chi_w$ *if and only if for all* $n \in \mathbb{N}_+$, w_n *is the greatest element with repect to* $\preceq$ *among the set* $\{v \in \mathbf{C} : \|v\|_{\mathbf{G}} \leq n, v \preceq w_n^{(i)}$ *for all but finitely many* $i\}$, *in the sense that* w_n *is in the set and every (other) element* v *in that set is a left divisor of* w_n.
- *Suppose further that* $\mathbf{G}$ *is locally finite. Then the singleton* $\{\chi_x\}$ *is open for every* $x \in \mathbf{C}$, *and thus* Ω_∞ *is closed. Given a sequence* $\{w^{(i)}\}$ *in* $\mathscr{W}$ *and an element* $w \in \mathscr{W}$, $\lim_i \chi_{w^{(i)}} = \chi_w$ *if and only if for all* $n \in \mathbb{N}_+$, $w_n = w_n^{(i)}$ *for all but finitely many* i.

Remark 5.10. In Li [5] the term "greatest element" is called "maximal element" instead, but from the context, it is better referred to as the greatest element.

Proof. We will do the proof with the help of Lemma 3.9 (ii). Let v be an element of $\mathbf{C}$ with $\|v\|_{\mathbf{G}} \leq n$. Then $\lim_i \chi_{w^{(i)}}(v\mathbf{C}) = 1$ if and only if $w_n^{(i)} \in v\mathbf{C}$ for all but finitely many i, if and only if $v \preceq w_n^{(i)}$ for all but finitely many i. This is because the first $\|v\|_{\mathbf{G}}$-many (which is no more than n) elements in the normal decomposition of $w_n^{(i)}$ is the same as that of v by uniqueness of normal decompositions from transversality of $\mathbf{G}$ (Corollary 4.19). On the other hand, for the same reason, $\chi_w(v\mathbf{C}) = 1$ if and only if $w_n \in v\mathbf{C}$, if and only if $v \preceq w_n$. Therefore $\lim_i \chi_{w^{(i)}} = \chi_w$ if and only if $v \preceq w_n^{(i)}$ implying $v \preceq w_n$ for $v \in \mathbf{C}$ with $\|v\|_{\mathbf{G}} \leq n$.

Suppose further that $\mathbf{G}$ is locally finite. Then $\{\chi_x\} = \Omega(x\mathbf{C}; \{y\mathbf{C} : y \in \mathbf{d}(x)\mathbf{G}\})$ is an element of a basis of the topology for Ω, which is open, and hence $\Omega_\infty = \Omega \setminus \bigcup_{x \in \mathbf{C}} \{\chi_x\}$ is closed. By the first item we show that w_n is the greatest with respect to $\preceq$ among the set $\{v \in \mathbf{C} : \|v\|_{\mathbf{G}} \leq n, v \preceq w_n^{(i)}$ for all but finitely many $i\}$ if and only if $w_n = w_n^{(i)}$ for all but finitely many i. By deleting those finite elements we may assume that $w_n \preceq w_n^{(i)}$ for all i. If we do not have $w_n = w_n^{(i)}$ for all but finitely many i, then by passing to a subsequence, we may further assume that $w_n \prec w_n^{(i)}$ for all i. Now $\mathbf{G}^n$ is locally finite because $\mathbf{G}$ is. Then by passing to a subsequence a second time, we may assume that the sequence $w_n^{(i)}$ is a constant, say v. It follows that $w_n \prec v$ and therefore $\chi_{w^{(i)}}$ does not converge to χ_w. □

Let $\mathbf{G}$ be a Garside family. For a sequence $\{s^{(i)}\}$ in $\mathbf{G}$ and an element $s \in \mathbf{G} \cup \mathbf{C}^0$, we write $\lim_i s^{(i)} = s$ if s is the greatest element with respect to $\preceq$ among the set $\{r \in \mathbf{G} \cup \mathbf{C}^0 : r \preceq s^{(i)}$ for all but finitely many $i\}$ in the sense that $s \preceq s^{(i)}$ for all but finitely many i, and every element r left dividing $s^{(i)}$ is also a left divisor of s.

Proposition 5.11. *Let* $\mathscr{V}$ *be a subset of* $\mathscr{W} \sqcup \mathbf{C}^0$.

(1) For a $\mathbf{G}$*-normal path* $w = s_1|s_2|\cdots \in \mathscr{W}$, $\chi_w \in \overline{(I_l \ltimes \Omega).\{\chi_v : v \in \mathscr{V}\}}$ *if and only if for all indices* j, *there is a sequence* $\{v^{(i)}\}$ *in* $\mathscr{V}$ *such that for all* i,

- *there exists an* $a_i \in \mathbf{C}$ *and* $m_i \in \mathbb{N}_+$ *with* $\|v^{(i)}\|_{\mathbf{G}} < m_i$ *if* $v^{(i)} \in \mathscr{V} \setminus \mathbf{C}$*; or*
- *there exists an* $a_i \in \mathbf{C}\mathbf{d}(v^{(i)})$ *if* $v^{(i)} \in \mathbf{C}$*;*

such that if we set $s_j^{(i)} := H(a_i v_{=m_i}^{(i)})$ *in the first case or* $s_j^{(i)} := H(a_i)$ *in the second case, then* $\lim_i s_j^{(i)} = s_j$.

(2) For $\mathbf{w} \in \mathbf{C}^0$, $\chi_{\mathbf{w}} \in \overline{(I_l \ltimes \Omega).\{\chi_v : v \in \mathscr{V}\}}$ *if and only if* $\mathbf{w} \in \mathscr{V}$ *or there exists a sequence* $\{v^{(i)}\}$ *in* $\mathscr{V}$ *such that for all* i,

- *there exists an $a_i \in \mathbf{C}$ and $m_i \in \mathbb{N}_+$ with $\|v^{(i)}\|_{\mathbf{G}} < m_i$ if $v^{(i)} \in \mathscr{V} \setminus \mathbf{C}$; or*
- *there exists an $a_i \in \mathbf{Cd}(v^{(i)})$ if $v^{(i)} \in \mathbf{C}$;*

such that if we set $s^{(i)} := H(a_i v^{(i)}_{=m_i})$ in the first case or $s^{(i)} := H(a_i)$ in the second case, then $\lim_i s^{(i)} = \mathbf{w}$.

Proof. We just prove the first statement and the argument for the second one is similar.

The "only if" part: For every i, write $v^{(i)} = r_1^{(i)}|r_2^{(i)}|\cdots$ (finite or infinite normal path). Define $\tilde{v}^{(i)}$ be a unique normal path such that $\chi_{\tilde{v}^{(i)}} := \sigma_{a_i}\sigma^{-1}_{r_1^{(i)}\cdots r_{m_i-1}^{(i)}}.\chi_{v^{(i)}}$ in the first case and $\chi_{\tilde{v}^{(i)}} := \sigma_{a_i}\sigma^{-1}_{v^{(i)}}.\chi_{\tilde{v}^{(i)}}$ in the second case. The existence and uniqueness of $\tilde{v}^{(i)}$ is guaranteed by Lemma 5.7. Then $\chi_{\tilde{v}^{(i)}} \in (I_l \ltimes \Omega).\{\chi_v : v \in \mathscr{V}\}$ and the normal decomposition of $\tilde{v}^{(i)}$ starts with $s_j^{(i)}$ by the assumption that $s_j^{(i)} := H(a_i v^{(i)}_{=m_i})$ in the first case or $s_j^{(i)} := H(a_i)$ in the second case, as well as transversality of $\mathbf{G}$. Since $\Omega(\mathbf{t}(s_j))$ is compact, by passing to a subsequence, we may assume that $\lim_i \chi_{\tilde{v}^{(i)}} = \chi_x$ for some unique normal path x, which lies in $\overline{(I_l \ltimes \Omega).\{\chi_v : v \in \mathscr{V}\}}$. This means that $\chi_{\tilde{v}^{(i)}} = \chi_x$ for all but finitely many i in the sense of and hence $\tilde{v}^{(i)} = x$ for all but finitely many i by Lemma 5.7. Since $\lim_i s_j^{(i)} = s_j$, then $s_j \preceq s_j^{(i)}$ and the normal decomposition of $\tilde{v}^{(i)}$ and hence x start with s_j for all i. Take $w^{(j)}$ to be a normal path such that $\chi_{w^{(j)}} := \sigma_{s_1\cdots s_{j-1}}.\chi_x = \chi_{s_1|\cdots|s_{j-1}|x}$. The normal decomposition of $w^{(j)}$ starts with $s_1|\cdots|s_j$. Since $w = s_1|s_2|\cdots$ also starts with $s_1|\cdots|s_j$, we conclude that $\lim_j \chi_{w^{(j)}} = \chi_w$.

The "if" part: We can just discuss the case where $j = 1$. For general indices j the arguments are the same. Assume that for each i we can find $c_i, d_i \in \mathbf{C}$ with $\mathbf{d}(d_i) = \mathbf{d}(c_i)$ and $v^{(i)} \in \mathscr{V}$ such that $\lim_i \sigma_{d_i}^{-1}\sigma_{c_i}.\chi_{v^{(i)}} = \chi_w$. Lemma 5.4 implies that $\sigma_{d_i}^{-1}\sigma_{c_i}.\chi_{v^{(i)}} = \chi_{a_i|v^{(i)}_{>N_i}}$ for some $a_i \in \mathbf{C}$ in the first case or $\sigma_{d_i}^{-1}\sigma_{c_i}.\chi_{v^{(i)}} = \chi_{a_i}$ in the second case, where N_i is an integer satisfying the conditions of Lemma 5.4. The normal decomposition of $a_i|v^{(i)}_{>N_i}$ starts with $s_1^{(i)} = H(a_i|v^{(i)}_{>N_i})$ in the first case and the normal decomposition of a_i starts with $s_1^{(i)} = H(a_i)$ in the second case. Then $\lim_i \sigma_{d_i}^{-1}\sigma_{c_i}.\chi_{v^{(i)}} = \chi_w$ implies that $\lim_i s_1^{(i)} = s_1$.

$\square$

Definition 5.12 (Admissible pair, H-invariance, $\max^\infty_\preceq$-closeness)**.** Let $\mathbf{C}$ be a left cancellative small category and $\mathbf{G}$ be a Garside family. Also let $\mathbf{I}$ be a subfamily of $\mathbf{G}$ and $\mathbf{D}$ be a subfamily of $\mathbf{C}^0$.

(i) The pair $(\mathbf{I}, \mathbf{D})$ is called admissible if for all $t \in \mathbf{I}$, either there is a $t' \in \mathbf{I}$ such that $t|t'$ is $\mathbf{G}$-normal or $\mathbf{d}(t) \in \mathbf{D}$.

(ii) $(\mathbf{I}, \mathbf{D})$ is called H-invariant if for all $a \in \mathbf{C}\backslash\mathbf{C}^*$ and $x \in \mathbf{I} \cup \mathbf{D}$ with $\mathbf{d}(a) = \mathbf{t}(x)$, $H(ax)$ lies in $\mathbf{I}$.

(iii) $(\mathbf{I}, \mathbf{D})$ is called $\max^\infty_\preceq$-closed if for every sequence $\{t_i\}_i$ in $\mathbf{I}$, if $\lim_i t_i$ exists in $\mathbf{G}$, then $\lim_i t_i \in \mathbf{I} \cup \mathbf{D}$.

Lemma 5.7 implies that there is a bijective correspondence between subsets of Ω and subsets of $\mathscr{W} \sqcup \mathbf{C}^0$: Given $X \subseteq \Omega$, the corresponding subset of $\mathscr{W} \sqcup \mathbf{C}^0$ is $\mathscr{V}(X) := \{w \in \mathscr{W}, \mathbf{v} \in \mathbf{C}^0 : \chi_w, \chi_{\mathbf{v}} \in X\}$. Given $\mathscr{V} \subseteq \mathscr{W} \sqcup \mathbf{C}^0$, the corresponding subset of Ω is $X(\mathscr{V}) := \{\chi_w, \chi_{\mathbf{v}} \in \Omega : w \in \mathscr{V} \cap \mathscr{W}, \mathbf{v} \in \mathscr{V} \cap \mathbf{C}^0\}$.

Definition 5.13 (Subfamilies from subsets of Ω)**.** Given $X \subseteq \Omega$, let $\mathscr{V}(X) = \{w \in \mathscr{W}, \mathbf{v} \in \mathbf{C}^0 : \chi_w, \chi_{\mathbf{v}} \in X\}$. We define

$$\mathbf{I}(X) := \{t \in \mathbf{G} : t = v_{=i} \text{ for some } v \in \mathscr{V}(X) \cap \mathscr{W} \text{ and } i \in \mathbb{N}_+\}$$

and

$$\mathbf{D}(X) := \mathscr{V}(X) \cap \mathbf{C}^0 = \{\mathbf{v} \in \mathbf{C}^0 : \chi_{\mathbf{v}} \in X\}.$$

The following lemma, which is given with proof omitted, will be used in discussing invariance and closeness properties.

Lemma 5.14. *Let $\mathbf{G}$ be a Garside family of a left cancellative small category $\mathbf{C}$. Also suppose that $\mathbf{G}$ is $=^*$-transverse and $\mathbf{G} \cap \mathbf{C}^* = \varnothing$.*

If $g_1|\cdots|g_n$ be a finite $\mathbf{G}$-path, then $H(g_1\cdots g_n) = H(g_1\cdots H(g_{n-2}H(g_{n-1}g_n))\cdots)$.

If $s_1|s_2|\cdots$ be an infinite $\mathbf{G}$-path which is also normal and let $g_1|\cdots|g_n$ be a finite $\mathbf{G}$-path with $\mathbf{t}(s_1) = \mathbf{d}(g_n)$, then the normal decomposition of $g_1|\cdots|g_n|s_1|s_2|\cdots$ starts with $H(g_1\cdots g_n r_1)$.

Lemma 5.15. *We have the following two necessary and sufficient statements:*

(1) The pair $(\mathbf{I},\mathbf{D})$ is admissible if and only if there is a subset $X \subseteq \Omega$ such that $\mathbf{I} = \mathbf{I}(X)$ and $\mathbf{D} = \mathbf{D}(X)$.

(2) $(\mathbf{I}(X),\mathbf{D}(X))$ is H-invariant and $\max^{\infty}_{\precsim}$-closed if and only if X is $(I_l \ltimes \Omega)$-invariant and closed.

Proof. (1) Suppose that the pair $(\mathbf{I},\mathbf{D})$ is admissible. We take a set X whose elements will be determined later. Let $t \in \mathbf{I}$. If there is a $t' \in \mathbf{I}$ such that $t|t'$ is a $\mathbf{G}$-normal path, then we require X to be the set such that $\mathscr{V}(X) \cap \mathscr{W}$ contains (finite or infinite) $\mathbf{G}$-normal paths in which two of the adjacent elements are t and t', i.e., each element of $\mathscr{V}(X) \cap \mathscr{W}$ is a (finite or infinite) path v such that there exists an $i \leq \|v\|_{\mathbf{G}}$ with $v_{=i} = t$ and $v_{=i+1} = t'$. If $\mathbf{d}(t) \in \mathbf{D}$, then we require X to be the set such that $\mathscr{V}(X) \cap \mathbf{C}^0 = \mathbf{D}$. Then we obtain $\mathscr{V}(X) = (\mathscr{V}(X) \cap \mathscr{W}) \sqcup (\mathscr{V}(X) \cap \mathbf{C}^0)$ and hence we obtain the desired X from the correspondence given before Definition 5.13. Conversely, let $X \subseteq \Omega$, we show that $(\mathbf{I}(X),\mathbf{D}(X))$ is admissible. Let $t \in \mathbf{I}(X)$. We can take a (finite or infinite) path $v \in \mathscr{V}(X) \cap \mathscr{W}$ such that $t = v_{=i}$. If $\|v\|_{\mathbf{G}} \geq i+1$ we take $t' = v_{=i+1}$ Then $t|t'$ is a normal path. If $\|v\|_{\mathbf{G}} = i$, then $\mathbf{d}(t) \in \mathbf{D}(X)$.

(2) Suppose that $(\mathbf{I}(X),\mathbf{D}(X))$ is H-invariant and $\max^{\infty}_{\precsim}$-closed. Then the "if" conditions in Proposition 5.11 are satisfied. It implies directly that for every nonempty $\mathbf{G}$-normal path $w \in \mathscr{W}$ and every object $\mathbf{w} \in \mathbf{C}^0$, $\chi_w, \chi_{\mathbf{w}} \in \overline{(I_l \ltimes \Omega).\{\chi_v : v \in \mathscr{V}(X)\}}$ so that $X \subseteq \overline{(I_l \ltimes \Omega).\{\chi_v : v \in \mathscr{V}(X)\}}$. Moreover, the $\max^{\infty}_{\precsim}$-closeness condition implies that all the limit points of $(I_l \ltimes \Omega).\{\chi_v : v \in \mathscr{V}(X)\}$ lie in X, and equivalently, its closure is contained in X. Therefore $X = \overline{(I_l \ltimes \Omega).\{\chi_v : v \in \mathscr{V}(X)\}}$ and is $(I_l \ltimes \Omega)$-invariant and closed. Conversely suppose that X is $(I_l \ltimes \Omega)$-invariant and closed. Let $\mathscr{V}(X)$ be the corresponding subfamily of $\mathscr{W} \sqcup \mathbf{C}^0$. Then because of $(I_l \ltimes \Omega)$-invariance of X, the pair $(\mathbf{I}(X),\mathbf{D}(X))$ is H-invariant by Lemma 5.14. By closeness of X, the pair $(\mathbf{I}(X),\mathbf{D}(X))$ is $\max^{\infty}_{\precsim}$-closed because of the compactness of $\Omega(\mathbf{vC})$ for every $\mathbf{v} \in \mathbf{C}^0$ implying that none of the limit points will go outside and Proposition 5.11 concludes the proof. □

Let $\mathbf{I}$ be a subfamily of $\mathbf{G}$ and $\mathbf{D}$ be a subfamily of $\mathbf{C}^0$. We can always find a smallest H-invariant and $\max^{\infty}_{\precsim}$-closed pair $(\overline{\mathbf{I}},\overline{\mathbf{D}})$ with respect to inclusion of sets, such that $\mathbf{I} \subseteq \overline{\mathbf{I}}$ and $\mathbf{D} \subseteq \overline{\mathbf{D}}$. This is done by adjoining all the elements of the form $H(ax)$ where $a \in \mathbf{C} \setminus \mathbf{C}^0$ and $x \in \mathbf{I} \cup \mathbf{D}$ with $\mathbf{d}(a) = \mathbf{t}(x)$ and all the limit points $\lim_i t_i$ of sequences $\{t_i\}$ in $\mathbf{I}$. We can also find a biggest admissible pair $(\check{\mathbf{I}},\check{\mathbf{D}})$ with respect to the inclusion of sets, such that $\check{\mathbf{I}} \subseteq \mathbf{I}$ and $\check{\mathbf{D}} \subseteq \mathbf{D}$. This is done by deleting elements t for which there exists no $t' \in \mathbf{I}$ such that tt' is $\mathbf{G}$-normal and for which $\mathbf{d}(t) \in \mathbf{D}$. We write $(\mathbf{I},\mathbf{D}) \subseteq (\overline{\mathbf{I}},\overline{\mathbf{D}})$ to mean that $\mathbf{I} \subseteq \overline{\mathbf{I}}$ and $\mathbf{D} \subseteq \overline{\mathbf{D}}$. Also we write $(\mathbf{I},\mathbf{D}) = (\mathbf{I}',\mathbf{D}')$ to mean that $\mathbf{I} = \mathbf{I}'$ and $\mathbf{D} = \mathbf{D}'$. Thus we have the following corollary.

Corollary 5.16.

- *That the pair $(\mathbf{I},\mathbf{D})$ is admissible implies that $(\overline{\mathbf{I}},\overline{\mathbf{D}})$ is also admissible.*
- *That the pair $(\mathbf{I},\mathbf{D})$ is H-invariant and $\max^{\infty}_{\precsim}$-closed implies that $(\check{\mathbf{I}},\check{\mathbf{D}})$ is also H-invariant and $\max^{\infty}_{\precsim}$-closed.*

Proof. We make use of Lemma 5.15.

For the first item, there exists a subset $X \subseteq \Omega$ such that $\mathbf{I} = \mathbf{I}(X)$ and $\mathbf{D} = \mathbf{D}(X)$. Then by Proposition 5.11 with $\mathscr{V} = \mathscr{V}(X)$ we obtain that $(\overline{\mathbf{I}},\overline{\mathbf{D}})$ is actually the pair $(\mathbf{I}(\overline{(I_l \ltimes \Omega).\{\chi_v : v \in \mathscr{V}(X)\}}),\mathbf{D}(\overline{(I_l \ltimes \Omega).\{\chi_v : v \in \mathscr{V}(X)\}}))$ and hence admissible again by Lemma 5.15.

For the second item, since the pair $(\check{\mathbf{I}},\check{\mathbf{D}})$ is admissible by construction, then $(\bar{\check{\mathbf{I}}},\bar{\check{\mathbf{D}}})$ is admissible by the first item. Since $(\check{\mathbf{I}},\check{\mathbf{D}})$ is H-invariant and $\max^{\infty}_{\precsim}$-closed, we must have $\bar{\check{\mathbf{I}}} \subseteq \mathbf{I}$ and $\bar{\check{\mathbf{D}}} \subseteq \mathbf{D}$. By maximality of $(\check{\mathbf{I}},\check{\mathbf{D}})$ we also have $(\bar{\check{\mathbf{I}}},\bar{\check{\mathbf{D}}}) \subseteq (\check{\mathbf{I}},\check{\mathbf{D}})$. Hence they are equal, so $(\check{\mathbf{I}},\check{\mathbf{D}})$ is H-invariant and $\max^{\infty}_{\precsim}$-closed. □

Definition 5.17. Let $\mathbf{I}$ be a subfamily of $\mathbf{G}$ and $\mathbf{D}$ be a subfamily of $\mathbf{C}^0$. Define

$$X(\mathbf{I},\mathbf{D}) := \{\chi_v : v_{=i} \in \mathbf{I}, \forall i \in \mathbb{N}_+\} \cup \{\chi_{\mathbf{v}} : \mathbf{v} \in \mathbf{D}\}.$$

Theorem 5.18 (Main theorem)**.** *There is an inclusion preserving one-to-one correspondence:*

$$\{(I_l \ltimes \Omega)\text{-invariant closed subspaces of } \Omega\} \longrightarrow \{\text{admissible, } H\text{-invariant } \max^{\infty}_{\precsim}\text{-closed pairs}\}$$
$$X \longmapsto (\mathbf{I}(X),\mathbf{D}(X))$$
$$X(\mathbf{I},\mathbf{D}) \longleftarrow (\mathbf{I},\mathbf{D})$$

with $\mathbf{I} \subseteq \mathbf{G}$ *and* $\mathbf{D} \subseteq \mathbf{C}^0$.

Proof. Given an $(I_l \ltimes \Omega)$-invariant closed $X \subseteq \Omega$, then Lemma 5.15 implies that the pair $(\mathbf{I}(X),\mathbf{D}(X))$ is admissible, H-invariant and $\max^{\infty}_{\precsim}$-closed.

Given an admissible, H-invariant and $\max^{\infty}_{\precsim}$-closed pair $(\mathbf{I},\mathbf{D})$, then Lemma 5.15 implies that there is a subset $Y \subseteq \Omega$ such that $(\mathbf{I},\mathbf{D}) = (\mathbf{I}(Y),\mathbf{D}(Y))$. Since the pair is futher H-invariant and $\max^{\infty}_{\precsim}$-closed, then Y is $(I_l \ltimes \Omega)$-invariant and closed. It remains to check that $X(\mathbf{I}(Y),\mathbf{D}(Y))$ is indeed Y. By definition we clearly have $Y \subseteq X(\mathbf{I}(Y),\mathbf{D}(Y))$. Let χ_v be an element in $X(\mathbf{I}(Y),\mathbf{D}(Y))$. If $v = \mathbf{v} \in \mathbf{C}^0$, then $\chi_{\mathbf{v}} \in Y$ is straightforward. If $v = w \in \mathscr{W}$, then for all $i \in \mathbb{N}_+$ with $i \leq \|v\|_{\mathbf{G}}$, there is a $w' \in \mathscr{W}$ such that $\chi_{w'} \in Y$ and $w_{=i} = w'_{=i}$. Then Proposition 5.11 with the assumption that Y is $(I_l \ltimes \Omega)$-invariant and closed implies that $\chi_w \in Y$, giving the other direction of inclusion.

The above argument also gives that $X(\mathbf{I}(X),\mathbf{D}(X)) = X$. The identity that $(\mathbf{I}(X(\mathbf{I},\mathbf{D})),\mathbf{D}(X(\mathbf{I},\mathbf{D}))) = (\mathbf{I},\mathbf{D})$ is easily seen, as for some $I_l \ltimes \Omega$-invariant and closed subspace Y of Ω,

$$(\mathbf{I},\mathbf{D}) = (\mathbf{I}(Y),\mathbf{D}(Y)) \mapsto Y \mapsto (\mathbf{I}(Y),\mathbf{D}(Y)) = (\mathbf{I},\mathbf{D})$$

by using the above argument and Lemma 5.15.

Finally, the inclusion preserving property is clear to verify. □

Corollary 5.19. *Suppose further that the Garside family* $\mathbf{G}$ *is locally finite. Then the maps given in Theorem 5.18 establish an inclusion preserving one-to-one correspondence between* $(I_l \ltimes \Omega)$*-invariant, closed subspaces of* Ω *and admissible* H*-invariant pairs contained in* $(\mathbf{G},\mathbf{C}^0)$.

The difference in statements between Corollary 5.19 and Theorem 5.18 is that we do not need to care about $\max^{\infty}_{\precsim}$-closeness. This is because under the assumption that $\mathbf{G}$ is locally finite, every pair $(\mathbf{I},\mathbf{D})$ is automatically $\max^{\infty}_{\precsim}$-closed.

Combining Theorems 3.11, 5.18, and Corollary 5.19, we finally come to the following conclusion.

Theorem 5.20 (Li [5], Theorem B)**.** *Let* $\mathbf{C}$ *be a finitely aligned countable left cancellative small category and* $\mathbf{G}$ *is a Garside family of* $\mathbf{C}$ *which is* $=^*$*-transverse, locally bounded and* $\mathbf{G} \cap \mathbf{C}^* = \varnothing$*. Then we have the following:*

- *The transformation groupoid* $I_l \ltimes \Omega$ *is a groupoid model for the left reduced C*-algebra* $C^*_\lambda(\mathbf{C})$.
- *There is an inclusion preserving one-to-one correspondence between* $I_l \ltimes \Omega$*-invariant closed subspaces of* Ω *and admissible,* H*-invariant* $\max^{\infty}_{\precsim}$*-closed pairs* $(\mathbf{I},\mathbf{D})$ *with* $\mathbf{I} \subseteq \mathbf{G}$ *and* $\mathbf{D} \subseteq \mathbf{C}^0$.
- *If further* $\mathbf{G}$ *is locally finite, then the* $\max^{\infty}_{\precsim}$*-closeness condition can be removed.*

♪ 6

Example: Higher-rank graphs

In this section, we discuss an application of the Garside theory to higher-rank graphs. Let's recall the definition and some properties of directed graphs and higher-rank graphs, roughly following Raeburn et al. [7].

Definition 6.1 (Directed graph, Path, Concatenation)**.**

- A directed graph $E = (E^0, E^1, r, s)$ consists of a countable family of vertices E^0 and a countable family of edges E^1 together with the range map $r : E^1 \to E^0$ and the source map $s : E^1 \to E^0$.
- A (finite) path in E is a (finite) sequence $g = g_1 | \cdots | g_n$ with each $g_i \in E^1$ and $r(g_{i+1}) = s(g_i)$ for all $i = 1, 2, \ldots, n-1$. We define the source and range of the path g by $s(g) := s(g_n)$ and $r(g) := r(g_1)$.
- Let $f = f_1 | \cdots | f_m$ and $g = g_1 | \cdots | g_n$ be two finite paths with $s(g_n) = r(f_1)$, then the concatenation of $f_1 | \cdots | f_m$ by $g_1 | \cdots | g_n$ is given by the finite path $g_1 | \cdots | g_n | f_1 | \cdots | f_m$.

Let $E^\bullet$ be the set of all finite paths in E. If a finite path contains n elements in E^1, we say that the path is of length n. This defines a length function $l : E^\bullet \to \mathbb{N}$, and the length function is additive for the concatenation of paths, in the sense that the length of the concatenation of two paths is the sum of their lengths.

The set of all finite paths $E^\bullet$ becomes a small category. Objects are vertices E^0, morphisms are paths $E^\bullet$ and E^1 can be regarded as a subset of $E^\bullet$. Compositions are given by concatenations of paths. Moreover, the length function $l : E^\bullet \to \mathbb{N}$ satisfies the unique factorization property: for every path $p \in E^\bullet$ and $m, n \in \mathbb{N}$ with $l(p) = n + m$, there are unique paths $f, g \in E^\bullet$ with $l(g) = n$, $l(f) = m$ and $s(g) = r(f)$, such that $p = g|f$.

Recall that for any nonnegative integer k, the set of k-tuples of nonnegative integers $\mathbb{N}^k$ is regarded as a monoid with the implicit object. The morphisms of $\mathbb{N}^k$ are nonnegative numbers, and the composition is given by the componentwise addition of numbers, which is also commutative. Let $v, w \in \mathbb{N}$, then by definition of left divisibility, $v \preceq w$ if $w = v + a$ for some $a \in \mathbb{N}^k$. This means that $v \preceq w$ if and only if each component of w is no less than that of v. We can see immediately that this is a partial order. The category $\mathbb{N}^k$ is Noetherian with respect to this order because $(0, \ldots, 0)$ is the least element in $\mathbb{N}^k$.

Definition 6.2 (Graph of rank k)**.** Let k be an nonnegative integer. A graph of rank k (also called a k-graph) is a countable small category $\mathbf{E}$ equipped with a functor $\mathrm{d} : \mathbf{E} \to \mathbb{N}^k$ satisfying the following axiom:

For all $e \in \mathbf{E}$ and $m, n \in \mathbb{N}^k$ with $\mathrm{d}(e) = m + n$, there are unique elements $u \in \mathrm{d}^{-1}(m)$ and $v \in \mathrm{d}^{-1}(n)$ such that $e = vu$.

We often call a graph of rank k a higher-rank graph when $k \geq 2$. The functor d in Definition 6.2 is known as the degree functor and the axiom mentioned is known as the unique factorization property.

It is easy to see that the category $E^\bullet$ is a 1-graph, and the length function l is actually a functor.

Lemma 6.3 (Basic properties of a higher-rank graph)**.** *Let* $\mathbf{E}$ *be a k-graph.*

(i) $\mathrm{d}^{-1}(0) = \{\mathrm{id}_\mathbf{v} : \mathbf{v} \in \mathbf{E}^0\} = \mathbf{E}^0$.

(ii) $\mathbf{E}^* = \mathbf{E}^0$.

(iii) Let $a, b \in \mathbf{E}$*. Then* $a =^* b$ *if and only if* $a = b$*.*

Proof. (i) If $\mathbf{v} \in \mathbf{E}^0$ then by functoriality of d we have

$$\mathrm{d}(\mathrm{id}_\mathbf{v}) = \mathrm{d}(\mathrm{id}_\mathbf{v}\,\mathrm{id}_\mathbf{v}) = \mathrm{d}(\mathrm{id}_\mathbf{v}) + \mathrm{d}(\mathrm{id}_\mathbf{v}) = 2\mathrm{d}(\mathrm{id}_\mathbf{v}),$$

so that $\mathrm{d}(\mathrm{id}_\mathbf{v}) = 0$.

Conversely, if $u \in \mathrm{d}^{-1}(0)$ then $\mathrm{d}(u) = 0 = 0+0$ and $\mathrm{id}_{\mathbf{t}(u)}\, u = u = u\,\mathrm{id}_{\mathbf{d}(u)}$. The unique factorization property implies that $u = \mathrm{id}_{\mathbf{t}(u)} = \mathrm{id}_{\mathbf{d}(u)}$.

(ii) We already have that $\mathbf{E}^0 \subseteq \mathbf{E}^*$. Suppose that $u \in \mathbf{E}^*$, then $0 = \mathrm{d}(\mathrm{id}_{\mathbf{d}(u)}) = \mathrm{d}(u^{-1}u) = \mathrm{d}(u^{-1}) + \mathrm{d}(u)$. Thus $\mathrm{d}(u) = \mathrm{d}(u^{-1}) = 0$ and $u = \mathrm{id}_{\mathbf{d}(u)} \in \mathbf{E}^0$ by (i).

(iii) If $a =^* b$ in $\mathbf{E}$ then $a = bc$ for some $c \in \mathbf{E}^* = \mathbf{E}^0$ by (ii). Then we must have $c = \mathrm{id}_{\mathbf{d}(b)}$ and hence $a = b\,\mathrm{id}_{\mathbf{d}(b)} = b$.

□

Lemma 6.4 (Strictly order preserving property)**.** *The functor* $\mathrm{d} : \mathbf{E} \to \mathbb{N}^k$ *is strictly order-preserving. In other words, if* $e_2 \preceq e_1$ *in* $\mathbf{E}$ *then* $\mathrm{d}(e_2) \prec \mathrm{d}(e_1)$ *in* $\mathbb{N}^k$*. Moreover, if* $e_2 \prec e_1$ *in* $\mathbf{E}$ *then* $\mathrm{d}(e_2) \prec \mathrm{d}(e_1)$ *in* $\mathbb{N}^k$*.*

Proof. This is just because of the functoriality of d. If $e_2 \preceq e_1$ in $\mathbf{E}$, then there is an $f \in \mathbf{E}$ such that $e_1 = e_2 f$. Then $\mathrm{d}(e_1) = \mathrm{d}(e_2 f) = \mathrm{d}(e_2) + \mathrm{d}(f)$. Hence $\mathrm{d}(e_2) \preceq \mathrm{d}(e_1)$. If $\mathrm{d}(e_2) = \mathrm{d}(e_1)$ then $\mathrm{d}(f)$ is forced to be 0, which means that $f = \mathrm{id}_{\mathbf{d}(e_2)}$ by Lemma 6.3, so that $e_1 = e_2$.

□

Corollary 6.5. *The k-graph* $\mathbf{E}$ *is Noetherian.*

Proof. If $\cdots \prec e_2 \prec e_1$ is an infinite strictly decreasing sequence in $\mathbf{E}$, then $\cdots \prec \mathrm{d}(e_2) \prec \mathrm{d}(e_1)$ is also an infinite strictly decreasing sequence in $\mathbb{N}^k$ which is impossible. □

Now we need to answer the following questions:

(**Q1**) Why does a higher-rank graph possess the left cancellative property?

(**Q2**) What can be its Garside family?

(**Q3**) What do the results obtained in Section 5 mean for higher-rank graphs?

The following proposition gives the answer to the first question (**Q1**).

Proposition 6.6 (Left cancellativity of higher-rank graphs)**.** *Let* $\mathbf{E}$ *be a graph or a higher-rank graph. Then* $\mathbf{E}$ *is both left and right cancellative.*

Proof. Let $v, u, w \in \mathbf{E}$ such that $vu = vw$, we set out to verify that $u = w$. Actually we have $\mathrm{d}(vu) = \mathrm{d}(v) + \mathrm{d}(u)$ and $\mathrm{d}(vw) = \mathrm{d}(v) + \mathrm{d}(w)$. Then the identity $\mathrm{d}(v) + \mathrm{d}(u) = \mathrm{d}(v) + \mathrm{d}(w)$ with unique factorization property implies that $u = w$. This means $\mathbf{E}$ is indeed left cancellative. The same argument shows that $\mathbf{E}$ is indeed right cancellative. □

We continue to answer the second question (**Q2**).

Let $S_p := \{0,1\}^k \setminus \{(0,\dots,0)\}$ be the set k-tuples whose components are only 0 or 1, without the zero tuple. Note that S_p is a finite subset of $\mathbb{N}^k$. Take $\mathbf{G} := \mathrm{d}^{-1}(S_p)$. We will soon see that $\mathbf{G}$ is indeed a Garside family of the k-graph $\mathbf{E}$. Beforehand we note that $\mathbf{G}$ has the following properties:

- $\mathbf{G}$ is $=^*$-transverse, because by Lemma 6.3 (iii), $\mathbf{E}$ itself is already $=^*$-transverse.
- $\mathbf{G}$ is locally bounded, because for any $\mathbf{v} \in \mathbf{E}^0$, if there were an infinite strictly increasing sequence $\mathrm{id}_\mathbf{v} s_1 \prec \mathrm{id}_\mathbf{v} s_2 \prec \cdots$ in $\mathrm{id}_\mathbf{v}\mathbf{G}$ then $\mathrm{d}(\mathrm{id}_\mathbf{v} s_1) \prec \mathrm{d}(\mathrm{id}_\mathbf{v} s_2) \preceq \cdots$ is an infinite strictly increasing sequence in S_p since $\mathrm{d}(\mathrm{id}_\mathbf{v} s_i) = \mathrm{d}(\mathrm{id}_\mathbf{v}) + \mathrm{d}(s_i) = \mathrm{d}(s_i)$ and $\mathrm{d}(\mathrm{id}_\mathbf{v}) = 0$. However, there cannot exist any infinitely increasing sequence in S_p because the greatest element in it is $(1,1,\dots,1)$.
- $\mathbf{G} \cap \mathbf{E}^* = \varnothing$, because by Lemma 6.3 (i) and (ii), $\mathrm{d}(\mathbf{E}^*) = \{0\}$.

Now we look at the closure $\mathbf{G}^\sharp := \mathbf{G}\mathbf{E}^* \cup \mathbf{E}^*$. Observe that $\mathbf{G}\mathbf{E}^* = \mathbf{G}$ because $\mathbf{G}\mathbf{E}^* = \mathbf{G}\mathbf{E}^0$ by Lemma 6.3 (i) and $e = e\,\mathrm{id}_{\mathbf{d}(e)}$ for every $e \in \mathbf{E}$. Hence we can write $\mathbf{G}^\sharp = \mathbf{G} \cup \mathbf{E}^* = \mathbf{G} \cup \mathbf{E}^0$ and this is a disjoint union.

Theorem 6.7 (Existence of a (nontrivial) Garside family in a higher-rank graph)**.** *Let* $\mathbf{E}$ *be a k-graph. Then* $\mathbf{G} := \mathrm{d}^{-1}(S_p)$ *is indeed a Garside family of* $\mathbf{E}$.

Proof. For the sake of convenience, we use Corollary 4.7 as an equivalent definition of greediness.

Let e be an element of $\mathbf{E}$. If $e \in \mathbf{E}^*$, then e itself forms a $\mathbf{G}$-normal path. Thus we focus on the case when e is not invertible.

Firstly, we write $\mathrm{d}(e) = (a_1,\dots,a_k)$. and we take $b_1 = (b_1^{(1)},\dots,b_k^{(1)})$, where

$$b_i^{(1)} := \begin{cases} 1, \text{ if } a_i \neq 0, \\ 0, \text{ if } a_i = 0. \end{cases} \quad i = 1,\dots,k.$$

Secondly, we write $\mathrm{d}(e) - b_1 = (a_1 - b_1^{(1)},\dots,a_k - b_k^{(1)})$ and we take $b_2 = (b_1^{(2)},\dots,b_k^{(2)})$, where

$$b_i^{(2)} := \begin{cases} 1, \text{ if } a_i - b_i^{(1)} \neq 0, \\ 0, \text{ if } a_i - b_i^{(1)} = 0. \end{cases} \quad i = 1,\dots,k.$$

Inductively, for a general m-th step, where $2 \le m \le k$, we consider the tuple $\mathrm{d}(e) - b_1 - \cdots - b_{m-1}$ and take $b_m = (b_1^{(m)},\dots,b_k^{(m)})$, where

$$b_i^{(m)} := \begin{cases} 1, \text{ if } a_i - b_i^{(1)} - \cdots - b_i^{(m)} \neq 0, \\ 0, \text{ if } a_i - b_i^{(1)} - \cdots - b_i^{(m)} = 0. \end{cases} \quad i = 1,\dots,k.$$

By construction, we can write $\mathrm{d}(e) = b_1 + \cdots + b_k$. The unique factorization property implies that e can be written uniquely as a product $e = g_1 \cdots g_k$ in $\mathbf{E}$ such that each $\mathrm{d}(g_i) = b_i$, $i = 1,\dots,k$. Now, each g_i lies in $\mathbf{G} \subseteq \mathbf{G}^\sharp$ because $\mathrm{d}(g_i) = b_i \in S_p$. It remains to show that the path $g_1|\cdots|g_p$ is $\mathbf{G}$-greedy. For any q and r with $1 \le q < r \le k$, suppose we have a relation $s \preceq g_q \cdots g_r$ with $s \in \mathbf{G}$, then $\mathrm{d}(s) \preceq \mathrm{d}(g_q) + \cdots + \mathrm{d}(g_r) = b_q + \cdots + b_r$ and we indeed have that $\mathrm{d}(s) \preceq \mathrm{d}(g_q)$ because by construction, b_q is the largest element among all the elements in S_p which is no greater than $\mathrm{d}(e) - b_1 - \cdots - b_{q-1}$ at the q-th step. The functoriality of d and the unique factorization property imply that $s \preceq g_q$, proving that $g_1|\cdots|g_p$ is indeed $\mathbf{G}$-greedy.

□

It is the right time for us to answer the third question (**Q3**). In the beginning, we give a sufficient condition for the finite alignment of a higher-rank graph.

Lemma 6.8. *Let* $\mathbf{E}$ *be a k-graph. If the set* $\mathrm{id}_{\mathbf{v}}\,\mathrm{d}^{-1}(n)$ *is a finite set for each* $\mathbf{v} \in \mathbf{E}^0$ *and each* $n \in \mathbb{N}^k$*, then* $\mathbf{E}$ *is finitely aligned. In this case, the Garside family* $\mathbf{G}$ *is also locally finite.*

Proof. Let $\mathbf{v} \in \mathbf{E}^0$ and $n \in \mathbb{N}^k$ be given. Then $\mathrm{id}_{\mathbf{v}}\,\mathrm{d}^{-1}(n) = \{u \in \mathbf{E} : \mathbf{t}(u) = v \text{ and } \mathrm{d}(u) = n\}$ is always finite.

By Lemma 1.55 and Lemma 6.3 (iii), it suffices to verify that for every two elements $a, b \in \mathbf{E}$ the set of all minimal common right multiples $\mathrm{mcm}(a,b)$ is a finite set. Let $c \in \mathrm{mcm}(a,b)$, then by Lemma 6.4 $\mathrm{d}(c)$ is a larger than $\mathrm{d}(a)$ and $\mathrm{d}(b)$ but minimal with respect to $\preceq$ in $\mathbb{N}^k$. If we write $\mathrm{d}(a) = (n_1^{(a)}, \ldots, n_k^{(a)})$ and $\mathrm{d}(b) = (n_1^{(b)}, \ldots, n_k^{(b)})$, then the minimality of c also implies the minimality of $\mathrm{d}(c)$, so we have that

$$\mathrm{d}(c) = \left(\max\{n_1^{(a)}, n_1^{(b)}\}, \ldots, \max\{n_k^{(a)}, n_k^{(b)}\}\right).$$

Moreover, $\mathbf{t}(c) = \mathbf{t}(a) = \mathbf{t}(b)$ because $c = aa' = bb'$ for some $a', b' \in \mathbf{C}$. Therefore, by assumption, such c's can only be finitely many, so $\mathrm{mcm}(a,b)$ is a finite set.

$\mathbf{G}$ is locally finite because we note that for any $\mathbf{v} \in \mathbf{E}^0$,

$$\mathrm{id}_{\mathbf{v}}\,\mathbf{G} = \mathrm{id}_{\mathbf{v}}\,\mathrm{d}^{-1}(S_p) = \mathrm{id}_{\mathbf{v}} \bigcup_{n \in S_p} \mathrm{d}^{-1}(n) = \bigcup_{n \in S_p} \mathrm{id}_{\mathbf{v}}\,\mathrm{d}^{-1}(n)$$

which is a finite union of finite sets, hence also finite. □

For a directed graph $E^\bullet$, local finiteness of $\mathbf{G}$ means that for every vertex $v \in E^0$, there are only finitely many edges (paths of length 1) with the common target v. This is also known as row finiteness of $E^\bullet$.

Proposition 6.9 (Characterization of greediness of $\mathbf{G} \subseteq \mathbf{E}$)**.** *Let* $\mathbf{G}$ *be the Garside family in a k-graph* $\mathbf{E}$ *defined above. Given* $s, t \in \mathbf{G}$*, with* $\mathbf{t}(t) = \mathbf{d}(s)$*, the length two path* $s|t$ *is* $\mathbf{G}$*-greedy (and thus* $\mathbf{G}$*-normal) if and only if* $\mathrm{d}(t) \preceq \mathrm{d}(s)$.

Proof. We still use Corollary 4.7 as a definition of greediness.

By functoriality of d, we have that $\mathrm{d}(st) = \mathrm{d}(s) + \mathrm{d}(t) = \mathrm{d}(t) + \mathrm{d}(s)$. Then from the unique factorization property, there is a unique $u \in \mathbf{E}$ with $\mathrm{d}(u) = \mathrm{d}(t)$ and a unique $v \in \mathbf{E}$ with $\mathrm{d}(v) = \mathrm{d}(s)$ such that $st = uv$. Now since the path $s|t$ is greedy, and $u \preceq uv = st$, then $u \preceq s$ so that $\mathrm{d}(t) = \mathrm{d}(u) \preceq \mathrm{d}(s)$.

Suppose conversely that for $s, t \in \mathbf{G}$ with $\mathbf{t}(t) = \mathbf{d}(s)$ we have $\mathrm{d}(t) \preceq \mathrm{d}(s)$. Consider the relation $f \preceq st$ with $f \in \mathbf{G}$. By Lemma 6.4, $\mathrm{d}(f) \preceq \mathrm{d}(st) = \mathrm{d}(s) + \mathrm{d}(t)$ with $\mathrm{d}(f), \mathrm{d}(s)$ and $\mathrm{d}(f)$ being nonzero k-tuples whose components are only 0 or 1. Since $\mathrm{d}(t) \preceq \mathrm{d}(s)$, $\mathrm{d}(s)$ has a 1 at each component where $\mathrm{d}(t)$ has a 1, and there may be more 1's in $\mathrm{d}(s)$. Since $\mathrm{d}(f) \preceq \mathrm{d}(s) + \mathrm{d}(t)$, the nonzero component of $\mathrm{d}(s) + \mathrm{d}(t)$ must come from $\mathrm{d}(s)$. The largest possible candidate for $\mathrm{d}(f)$ with respect to $\preceq$ is the tuple with 1's at which the corresponding components of $\mathrm{d}(s) + \mathrm{d}(t)$ is nonzero, which is precisely $\mathrm{d}(s)$. Thus $\mathrm{d}(f) \preceq \mathrm{d}(s)$, and by the unique factorization property we have that $f \preceq s$, proving that $s|t$ is indeed $\mathbf{G}$-greedy. □

Proposition 6.10 (Characterization of an atom in $\mathbf{G}$)**.** *Let* $\mathbf{G}$ *be the Garside family in a k-graph* $\mathbf{E}$ *defined above. Then every atom is an element of* $\mathbf{G}$*. An element* $a \in \mathbf{G}$ *is an atom if and only if* $\mathrm{d}(a)$ *is one of the standard basis elements of* $\mathbb{N}^k$.

This is because each atom is minimal with respect to $\preceq$ among $\mathbf{G}$, each standard basis element is minimal among S_p, and the functor d preserves order.

Lemma 6.11 (Characterization of admissible pairs, H-invariance and $\max_{\preceq}^{\infty}$-closeness in a higher-rank graph)**.** *Let* $\mathbf{E}$ *be a graph or a higher-rank graph with the Garside family* $\mathbf{G}$ *defined above. Let* $\mathbf{I}$ *be a subfamily of* $\mathbf{G}$ *and* $\mathbf{D}$ *be a subfamily of* $\mathbf{E}^0$.

- *The pair* $(\mathbf{I},\mathbf{D})$ *is admissible if and only if*
  (A) *for every* $t \in \mathbf{I}$ *there exists a* $t' \in \mathbf{I}$ *with* $\mathrm{d}(t') \preceq \mathrm{d}(t)$ *or* $\mathbf{d}(t) \in \mathbf{D}$.

- $(\mathbf{I},\mathbf{D})$ *is* H*-invariant if and only if*
  (I) *for every* $t \in \mathbf{I} \cup \mathbf{D}$ *and every atom* a *with* $\mathbf{d}(a) = \mathbf{t}(t)$ *if* $\mathrm{d}(a) \npreceq \mathrm{d}(t)$ *then* $at \in \mathbf{I}$, *and if* $\mathrm{d}(a) \preceq \mathrm{d}(t)$ *and* $t = rs$ *with* $\mathrm{d}(s) = \mathrm{d}(a)$ *then* $ar \in \mathbf{I}$.

- $(\mathbf{I},\mathbf{D})$ *is* $\max_{\preceq}^{\infty}$*-closed if and only if*
  (C) *for every sequence* $\{az_i\}_i$ *with a fixed* $a \in \mathbf{G} \cup \mathbf{E}^0$ *and* $\mathrm{d}(z_i) = d \in \mathbb{N}^k$ *a constant tuple, if whenever* $e \preceq d$ *is a standard basis element of* $\mathbb{N}^k$ *and* $s_i \preceq z_i$ *satisfies* $\mathrm{d}(s_i) = e$ *we must have* $s_i \neq s_j$ *for all* $i \neq j$, *then* $a \in \mathbf{I} \cup \mathbf{D}$.

Proof sketch. The first item follows directly from Proposition 6.9.

For the second item, we note that in any case, the product $at \in \mathbf{G}$, hence $H(at) = at \in \mathbf{G}$, where a is an atom in $\mathbf{G}$.

For the third item, we have $a \preceq az_i$ for all i, and if $r \preceq az_i$ for all i, then $r \preceq a$.

□

Now, we recall the following important theorem.

Theorem 6.12 (Farthing et al. [3], Theorem 6.9)**.** *The transformation groupoid of a higher-rank graph* $\mathbf{E}$ *is a groupoid model for the Toeplitz-Cuntz-Kriger algebra* $\mathscr{T}C^*(\mathbf{E})$ *of* $\mathbf{E}$.

Summing up the results we obtained, we have the following conclusion of higher-rank graphs with the specific Garside family.

Theorem 6.13 (Li [5], Corollary 7.2)**.** *Let* $\mathbf{E}$ *be a countable finitely aligned higher-rank graph, with the Garside family* $\mathbf{G} := \mathrm{d}^{-1}(S_p)$ *and let* $I_l \ltimes \Omega$ *be the corresponding transformation groupoid. Then* $I_l \ltimes \Omega$ *is the groupoid model for the Toeplitz-Cuntz-Kriger algebra of* $\mathbf{E}$, *and there is an inclusion preserving one-to-one correspondence:*

$$\begin{aligned} \{(I_l \ltimes \Omega)\text{-invariant closed subspaces of } \Omega\} &\longrightarrow \{\text{pairs satisfying conditions (A), (I) and (C)}\} \\ X &\longmapsto (\mathbf{I}(X), \mathbf{D}(X)) \\ X(\mathbf{I},\mathbf{D}) &\longleftarrow (\mathbf{I},\mathbf{D}) \end{aligned}$$

with $\mathbf{I} \subseteq \mathbf{G}$ *and* $\mathbf{D} \subseteq \mathbf{C}^0$.

If further $\mathbf{G}$ *is locally finite, then the condition* (C) *can be removed.*

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