The categorical origins of Lebesgue integration NOTES

Tom Leinster, Notes by Ruijun Lin.

2. Integration on $[0,1]$ Let $\mathbb{F} = \mathbb{R}$ or $\mathbb{C}$ the number field.

Goals: We uniquely characterize the Banach space $L^p[0,1]$, $p \geq 1$ together with two further pieces of data: $I \in L^p[0,1]$ taking the constant value 1 and the justaposition map

$$\gamma : L^p[0,1] \oplus_p L^p[0,1] \longrightarrow L^p[0,1]$$

by

$$(\gamma(f,g))(x) = \begin{cases} f(2x) & 0 \leq x < \frac{1}{2} \\ g(2x-1) & \frac{1}{2} \leq x \leq 1 \end{cases}$$

Let Ban_1 be the category of Banach spaces with linear contractions as morphisms, where linear contractions means continuous linear operators with norms no larger than 1. Thus isomorphisms in Ban_1 are <u>isometric</u> isomorphisms.

(Note: We use Ban to denote the category of Banach spaces with morphisms the usual bounded operators)

For Banach spaces V and W, Let $V \oplus_p W$ denotes the direct sum, with norm $\|(v,w)\|_p = \frac{1}{2}(\|v\|^p + \|w\|^p)^{\frac{1}{p}}$.

Let $\mathcal{A}^p$ be the category described as follows:

<u>Objects</u>: triples (V, v, δ), where

V: Banach space, $v \in V$, $\|v\| \leq 1$, i.e. in the closed unit ball of V.

$\delta : V \oplus_p V \to V$: a map satisfying $\delta(v,v) = v$.

<u>Morphisms</u>: $(V', v', \delta') \longrightarrow (V, v, \delta)$ are the maps (contractions) $\theta : V' \longrightarrow V$ in Ban_1 preserving the structure

$\theta(v') = v$, $\theta(\delta'(v_1', v_2')) = \delta(\theta(v_1'), \theta(v_2'))$

<u>Identity</u>: The identity morphisms of Banach spaces.

The map γ defined above is an isometric isomorphism $L^p[0,1] \oplus_p L^p[0,1] \to L^p[0,1]$. and in particular, a contraction. Hence, $(L^p[0,1], I, \gamma)$ is an object of A^p.

- Theorem 2.1 (Universal property of $L^p[0,1]$)
  $(L^p[0,1], I, \gamma)$ is the initial object of A^p, whenever $1 \le p < \infty$.

(Initial objects: An object Z of a category $\mathcal{C}$ is <u>initial</u> if for each object C of $\mathcal{C}$, there is exactly one map $Z \to C$ in $\mathcal{C}$.)

Proof of Theorem 2.1. For $n \geq 0$, let E_n be the subspace of $L^p[0,1]$ consisting of (the equivalence class of) step functions on each of the intervals $\left(\frac{i-1}{2^n}, \frac{i}{2^n}\right)$ $(1 \le i \le 2^n)$. Write
$E = \bigcup_{n=0}^{\infty} E_n$, which is the space of step functions whose points of discontinuity are dyadic rationals.

(Dyadic rationals: A dyadic rational is a number that can be expressed as a fraction whose denominator is a power of 2.)

The assumption that $p < \infty$ implies that <u>E is dense</u> in the set of all step functions on $[0,1]$, which in turn is dense in $L^p[0,1]$; so E is dense in $L^p[0,1]$.

It follows that $L^p[0,1]$ is the colimit (direct limit) of the diagram $E_0 \hookrightarrow E_1 \hookrightarrow \cdots$ in Ban_1. Also note that γ restricts to an isomorphism $E_n \oplus_p E_n \longrightarrow E_{n+1}$ for each $n \geq 0$

(Cocone: Let $D : \mathbf{I} \to \mathcal{A}$ be a diagram in $\mathcal{A}$.
A cocone on D is an object $A \in \mathcal{A}$ (the vertex of the cocone) together with a family

$$\left(D(I) \xrightarrow{f_I} A \right)_{I \in \mathbf{I}}$$

of maps in $\mathcal{A}$ such that for all maps $I \xrightarrow{u} J$ in $\mathbf{I}$, the diagram

$$\begin{tikzcd}
D(I) \arrow[dr, "f_I"] \arrow[dd, "D(u)"'] & \\
& A \\
D(J) \arrow[ur, "f_J"'] &
\end{tikzcd}$$

commutes

Colimit: A colimit of D is a cocone

$$\left(D(I) \xrightarrow{p_I} C \right)_{I \in \mathbf{I}}$$

with the property that for any cocone (defined above) on D, there is a unique map $\bar{f} : C \to A$ such that $\bar{f} \circ p_I = f_I$ for all $I \in \mathbf{I}$. i.e. the following diagram commutes.

$$\begin{tikzcd}
& C \arrow[d] & \\
& A & \\
D(I) \arrow[uur, "p_I"] \arrow[ur, "f_I"'] \arrow[rr, "D(u)"'] & & D(J) \arrow[uul, "p_J"'] \arrow[ul, "f_J"]
\end{tikzcd}$$

)

- We give a explanation of how $L^p[0,1]$ is the colimit of the diagram $E_0 \hookrightarrow E_1 \hookrightarrow E_2 \hookrightarrow \cdots$ in $\mathbf{Ban}_1$.
The index small category is the category $\mathbf{I} = (\mathbb{N}, \leq)$ whose objects are natural numbers and morphisms are total orders. For any pair m, n $(m \leq n)$ there is a unique morphism $m \to n$ which is the total order relation $m \leq n$. i.e. $m \longrightarrow n \Leftrightarrow m \leq n$.

Let $\mathbf{Nor}_1$ be the category of normed spaces with linear contractions as morphisms. By definition $\mathbf{Ban}_1$ is a sub-category of $\mathbf{Nor}_1$ because the objects are only Banach spaces. Now $(E_n, \|\cdot\|_p)$ is a Banach space for each $n = 0, 1, 2, \cdots$ hence an object in $\mathbf{Nor}_1$. The colimit of the diagram $E_0 \hookrightarrow E_1 \hookrightarrow E_2 \hookrightarrow \cdots$ with inclusions as morphisms in $\mathbf{Nor}_1$ is $\bigcup_{n=0}^{\infty} E_n$. That is to say that the colimit cocone of the diagram is $D : \mathbf{I} \longrightarrow \mathbf{Nor}_1$ $D(n) = E_n$, is the cocone $\left(E_i \xrightarrow{\subseteq_i} \bigcup_{n=0}^{\infty} E_n\right)_{i \in \mathbf{I}}$ where E_n is the space of step functions that is constant on each of the intervals $\left(\frac{i-1}{2^n}, \frac{i}{2^n}\right)$ $1 \leq i \leq 2^n$, and the boundary points of the intervals may be points of discontinuity.
Verification
For any cocone $\left(E_i \xrightarrow{f_i} A\right)_{i \in \mathbf{I}}$, the unique map $\bar{f}\ E = \bigcup_{n=0}^{\infty} E_n \longrightarrow A$ such that the diagram

$$\begin{tikzcd}
 & & E_i \arrow[dll, "\subseteq_i"'] \arrow[dl, "f_i"'] \arrow[dd, "\subseteq_{ij}"] \\
E = \bigcup_{n=0}^{\infty} E_n \arrow[r, "\bar{f}"] & A & \\
 & & E_j \arrow[ull, "\subseteq_j"] \arrow[ul, "f_j"]
\end{tikzcd}$$

commutes is defined by

$$\bar{f}(\varphi) = \begin{cases} f_0(\varphi), & \varphi \in E_0 \\ f_1(\varphi), & \varphi \in E_1 \\ f_2(\varphi), & \varphi \in E_2 \\ \vdots & \\ f_i(\varphi), & \varphi \in E_i \\ \vdots & \end{cases}$$

Clearly each f_i $(i=0,1,2,\cdots)$ is of norm no larger than 1, so is $\bar{f}$. $\bar{f}$ is unique since it is determined by f_is for $i=0,1,2,\cdots$

But unfortunately $\left(\bigcup_{n=0}^{\infty} E_n, \|\cdot\|_p\right)$ is not the colimit of the diagram $E_0 \hookrightarrow E_1 \hookrightarrow E_2 \hookrightarrow \cdots$ in Ban_1 since $E = \bigcup_{n=0}^{\infty} E_n$ itself is not complete. However, since the space E is dense in $L^p[0,1]$ while $(L^p[0,1], \|\cdot\|_p)$ is indeed a Banach space, we infer that $L^p[0,1]$ is the colimit of $E_0 \hookrightarrow E_1 \hookrightarrow E_2 \hookrightarrow \cdots$ in Ban_1. This is because if $\varphi \in L^p[0,1]$ then there is a sequence say, $\{\varphi_k\}_{k=1}^{\infty}$ such that $\varphi_k \longrightarrow \varphi$ in norm $\|\cdot\|_p$ as $k \to \infty$. so the unique map $\bar{g} : L^p[0,1] \longrightarrow A$ in Ban_1 appeared in the definition of the colimit is $\bar{g}(\varphi) = \lim_{k\to\infty} \bar{f}(\varphi_k)$, where the limit exists since $\{\bar{f}(\varphi_k)\}_{k=1}^{\infty}$, which can be checked, is a Cauchy sequence, and the limit exists by completeness of $L^p[0,1]$.

-(Continue the proof) γ restrict to $E_n \oplus_p E_n \to E_{n+1}$ is an isomorphism. This is because if restricted to $E_n \oplus_p E_n$, then for $f, g \in E_n$, f and g has points of discontinuity at $i/2^n$, $i = 0, 1, 2, \cdots, 2^n$. Thus, $\gamma(f, g)$ has points of discontinuity precisely at $\frac{i}{2^{n+1}}$, $i = 0, 1, \cdots, 2^{n+1}$. The isomorphism (bijection) is directly from the previous discussion.

Remark: We also have the category Nor of normed vectorspaces with continuous (equivalently, bounded) linear maps as morphisms. However, the countable union of $\{E_n\}_{n=1}^{\infty}$, i.e. $E: \bigcup_{n=1}^{\infty} E_n$ is no longer a colimit of $E_0 \hookrightarrow E_1 \hookrightarrow E_2 \hookrightarrow \cdots$, because the unique map $\bar{f}: E \to A$ in Nor_1 is no longer bounded.

An interesting thing is that: we can see that E is not isomorphic to $L^p[0,1]$ from both analytical and categorical perspective:

From analytical perspective: If two spaces are isomorphic, they should be both complete or not complete. But $L^p[0,1]$ is complete while E is not.

From categorical perspective: Colimits are universal objects. This means that if C is a colimit of a diagram, then C' is isomorphic (in the same category) to C if and only if C' is also a colimit of the same diagram. However, in Nor (or Ban_1), $L^p[0,1]$ is a colimit of $E_0 \hookrightarrow E_1 \hookrightarrow E_2 \hookrightarrow \cdots$ while E is not.

The above discussion shows that $L^p[0,1]$ and E are both colimits of the increasing sequence $E_0 \hookrightarrow E_1 \hookrightarrow E_2 \hookrightarrow \cdots$ in Nor, but only $L^p[0,1]$ is a colimit in Ban$_1$.
Now, let $(V, v, \delta) \in A^p$. We must show that there exists a unique morphism $(L^p[0,1], I, \gamma) \to (V, v, \delta)$ in A^p.

Uniqueness: Let θ be a map $(L^p[0,1], I, \gamma) \to (V, v, \delta)$.
Then $\theta(I) = v$, which by linearity determines $\theta|_{E_0}$ uniquely.
To be rigorous, consider the object (E_0, I, γ), where E_0 is the Banach space of step functions with (possible) points of discontinuity on 0 and 1, which is

$$E_0 = \{ c\chi_{(0,1)} \mid c \in \mathbb{F} \}$$

Since $\theta(I) = v$, then $\theta|_{E_0}(c\chi_{(0,1)}) = \theta(cI) = c\theta(I) = cv$.
Hence $\theta|_{E_0}$ is uniquely determined. It is easy to check that $\theta|_{E_0}(\gamma(c_1\chi_{(0,1)}, c_2\chi_{(0,1)})) = \delta(\theta|_{E_0}(c_1\chi_{(0,1)}, c_2\chi_{(0,1)}))$
(structure preserving) $\theta|_{E_0}(\gamma(cI, cI)) = \delta(\theta|_{E_0}(cI, cI)) = cI$.
Induction: Base case: The propery that (E_0, I, γ) is initial is satisfied by the above discussion.
Induction hypothesis: Suppose inductively that $\theta|_{E_n}$ is determined uniquely. Since θ is a map in A^p, the square

$$\begin{tikzcd}
E_n \oplus_p E_n \arrow[r, "\gamma|_{E_n \oplus E_n}"] \arrow[d, "\theta|_{E_n} \oplus \theta|_{E_n}"'] & E_{n+1} \arrow[d, "\theta|_{E_{n+1}}"] \\
V \oplus_p V \arrow[r, "\delta"'] & V
\end{tikzcd}$$

(in Ban$_1$) commutes.

But $\gamma|_{E_n \oplus E_n} : E_n \oplus_p E_n \to E_{n+1}$ is invertible, we can recover two functions in E_n from E_{n+1}, so $\theta|_{E_{n+1}}$ is uniquely determined by $\theta|_{E_n}$, that is

$$\theta|_{E_{n+1}} = \delta \circ (\theta|_{E_n} \oplus \theta|_{E_n}) \circ (\gamma|_{E_n \oplus_p E_n})^{-1}.$$

This completes the induction.

Hence, (recalling $E = \bigcup_{n=0}^{\infty} E_n$), $\theta|_E$ is uniquely determined on the dense subspace E of $L^p[0,1]$. (This is because if $f \in E$, then $f \in E_n$ for some n, and hence

$\theta|_E(f) = \theta|_{E_n}(f)$ for this n, which is uniquely determined.

As $\theta|_E$ is bounded, θ is bounded on $L^p[0,1]$ itself.

(Here $\theta : L^p[0,1] \to V$ is defined as follows:

Let $f \in L^p[0,1]$, choose $\{f_k\} \subset E$ such that $f_k \to f$ in L^p. Then define

$\theta(f) =$ the limit of $\theta|_E(f_k)$ $(V, \| \|_V)$ as $k \to \infty$)

(the limit exists since f_k is Cauchy)

Existence. For each $n \geq 0$, define a map

$\theta_n : E_n \to V$ in Ban_1 as follows:

$\theta_0 : E_0 \cong \mathbb{F} \to V$ given by $\theta_0(I) = v$.

(and is a contraction because $\|v\| \leq 1$) and inductively,

$$\theta_{n+1} = \left(E_{n+1} \xrightarrow{\gamma|^{-1}_{E_n \oplus E_n}} E_n \oplus_p E_n \xrightarrow{\theta_n \oplus \theta_n} V \oplus_p V \xrightarrow{\delta} V \right)$$

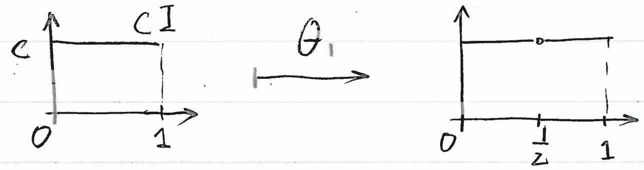
($\gamma|^{-1}_{E_n \oplus E_n}$ is well defined by the structure of $\gamma|_{E_n \oplus E_n}$)

$\theta_{n+1} : E_{n+1} \to V$, $\theta_{n+1} = \delta \circ (\theta_n \oplus \theta_n) \circ \gamma^{-1}$

One can check that θ_{n+1} extends θ_n for each $n \geq 0$ using the axiom $\delta(v, v) = v$.

(Explanation: $\theta_0 : E_0 \cong \mathbb{R}(\text{or } \mathbb{C}) \longrightarrow V$, $\theta_0(I) = v$.

$\theta_1 : E_1 \to V$ $\quad \theta_1 = \delta \circ (\theta_0 \oplus \theta_0) \circ \gamma^{-1}$



We can see that $\theta_1|_{E_0} = \theta_0$

$\theta_2 : E_2 \to V$ $\quad \theta_2 = \delta \circ (\theta_1 \oplus \theta_1) \circ \gamma^{-1}$

can also check that $\theta_2|_{E_1} = \theta_1$.

Inductively θ_{n+1} extends θ_n for each $n \geq 0$)

Since $L^p[0,1]$ is the colimit of $E_0 \hookrightarrow E_1 \hookrightarrow \cdots$, there is a unique map $\theta : L^p[0,1] \longrightarrow V$ such that $\theta|_{E_n} = \theta_n$ for each n. (θ can be defined as in the discussion of uniqueness)

(We can see this from the following commutative diagram:

$$\begin{tikzcd}
 & L^p[0,1] \arrow[d, "(\exists!)\,\theta"] & \\
 & V & \\
E_i \arrow[uur, hook, "\text{inclusion}"] \arrow[ur, "\theta_i"] \arrow[rr, hook, "\text{inclusion}"'] & & E_j \arrow[uul, hook', "\text{inclusion}"'] \arrow[ul, "\theta_j"']
\end{tikzcd}$$

$i < j$)

It remains to prove that θ is a map $(L^p[0,1], I, \gamma) \longrightarrow (V, v, \delta)$ i.e. θ preserves the structure of A^p.

First: $\theta(I) = \theta_0(I) = v$.

Second, we must show that $\theta \circ \gamma = \delta \circ (\theta \oplus \theta)$, that is, the lower square of the diagram:

$$\begin{tikzcd}
E_n \oplus_p E_n \arrow[r, "\gamma|_{E_n \oplus E_n}"] \arrow[d, hook] \arrow[dd, bend right=60, "\theta_n \oplus \theta_n"'] & E_{n+1} \arrow[d, hook] \arrow[dd, bend left=60, "\theta_{n+1}"] \\
L^p[0,1] \oplus_p L^p[0,1] \arrow[r, "\gamma"] \arrow[d, "\theta \oplus \theta"] & L^p[0,1] \arrow[d, "\theta"] \\
V \oplus_p V \arrow[r, "\delta"'] & V
\end{tikzcd}$$

commutes. The upper square commutes trivially, the outer square commutes by the definition of $\theta_{n+1} = \delta \cdot (\theta_n \oplus \theta_n) \cdot \gamma^{-1}_{E_n \oplus E_n}$.

Thus, the lower square commutes on the subspace $E_n \oplus_p E_n$ of $L^p[0,1] \oplus L^p[0,1]$ for each n.

(since $\theta|_{E_n} = \theta_n$ obtained from the previous discussion)

But
$$\bigcup_{n=0}^{\infty} E_n \oplus_p E_n = E \oplus_p E,$$

and $E \oplus_p E$ is dense in $L^p[0,1] \oplus_p L^p[0,1]$ because E is dense in $L^p[0,1]$ and $\oplus_p$ induces the product topology, so the lower square commutes

(θ is unique and the square commutes because of the colimit property, so do $\theta \oplus \theta$ and $L^p[0,1] \oplus_p L^p[0,1]$. This is because $E \oplus E$ is dense in $L^p[0,1] \oplus L^p[0,1]$ and

they are isomorphic, and they are both colimits of
$E_0 \oplus E_0 \hookrightarrow E_1 \oplus E_1 \hookrightarrow E_2 \oplus E_2 \hookrightarrow \cdots$)
The proof is complete $\square$

Taking $p = 1$, we immediately obtain a characterization of integration. Let $m : \mathbb{F} \oplus \mathbb{F} \to \mathbb{F}$ denote the arithmetic mean, $m(x, y) = \frac{1}{2}(x+y)$. Then $(\mathbb{F}, 1, m)$ is an object of A^1

Proposition 2.2. (Uniqueness of integration)
The unique map $(L^1[0,1], I, \gamma) \to (\mathbb{F}, 1, m)$

in A^1 is the integration operator $\int_0^1$.

Proof: By Theorem 2.1, it suffices to prove that $\int_0^1$ is a map in A^1. This statement amounts to the linearity of $\int_0^1$ together with the properties

$$\left|\int_0^1 f\right| \leq \int_0^1 |f|$$

$$\left.\begin{aligned} \int_0^1 I &= 1 \\ \int_0^1 \gamma(f, g) &= \frac{1}{2}\left(\int_0^1 f + \int_0^1 g\right) \end{aligned}\right\} \text{(preserving structure)}$$

$(f, g \in L^1[0,1])$

$\square$

Comment: Proposition 2.2 actually uses only the properties to define the integral: triangle inequality, linearity, dilation and translation, $\int_0^1 I = 1$

Corollary 2.3. $\int_0^1$ is the unique bounded linear functional on $L^1[0,1]$ such that $\int_0^1 1 = 1$ and

$$\int_0^1 f(x)\,dx = \frac{1}{2}\left(\int_0^1 f\left(\frac{x}{2}\right)dx + \int_0^1 f\left(\frac{x+1}{2}\right)dx\right)$$

for all $f \in L^1[0,1]$

Proof: This is just a restatement of Proposition 2.2, except that the hypothesis that $\int_0^1$ is a contraction has been weaked to boundedness. That sufficies because the uniqueness part of the proof of Theorem 2.1 uses only boundedness, not contractivity of the maps in A^P.

The above gives us the integration on $[0,1]$. The universal property of $L^1[0,1]$ also produces integration on subintervals of $[0,1]$. (The fullowing result is due to Mark Meckes)

Write

$$C_*[0,1] = \{\text{continuous function } F:[0,1]\to\mathbb{F} \mid F(0)=0\}$$

and give it the supremum norm. Then $C_*[0,1]$ is a Banach space. Define $\iota \in C_*[0,1]$, $\iota(x) = x$ (the identity function on $[0,1]$) and (the juxtaposition map)

$$\kappa : C_*[0,1] \oplus C_*[0,1] \longrightarrow C_*[0,1]$$

$$\kappa(F,G)(x) = \begin{cases} \frac{1}{2}F(2x) & \text{if } 0 \le x \le \frac{1}{2} \\ \frac{1}{2}(F(1)+G(2x-1)) & \text{if } \frac{1}{2} \le x \le 1 \end{cases}$$

$(F, G \in C_*[0,1])$

Then $(C_*[0,1], i, \kappa)$ is an object of $\mathcal{A}^1$.

Given $f \in L^1[0,1]$, define $\int_0^- f \in C_*[0,1]$ by

$$x \mapsto \int_0^x f$$

Proposition 2.4 (Meckes) The unique map

$$(L^1[0,1], I, \gamma) \longrightarrow (C_*[0,1], i, K)$$

in $\mathcal{A}^1$ is the definite integration operator

$$\int_0^- : L^1[0,1] \longrightarrow C_*[0,1]$$

Proof: By Theorem 2.1, it suffices to show that $\int_0^-$ is a map in $\mathcal{A}^1$. This amounts to the linearity of integration together with the elementary fact that

$$\left| \int_0^x f \right| \leq \int_0^x |f|$$

$$\int_0^x 1 = x$$

$$\int_0^x \gamma(f,g) = \left(K\left(\int_0^- f, \int_0^- g \right) \right)(x)$$

for all $f, g \in L^1[0,1]$ and $x \in [0,1]$. $\square$

Theorem 2.1 uniquely characterizes $L^1[0,1]$ as a Banach space, but Proposition 2.4 allows us to realize its elements as equivalence class of functions. Given an element $\alpha \in L^1[0,1]$, first apply the map of Proposition 2.4 to obtain an element of $C_*[0,1]$, then differentiate to obtain a function defined almost everywhere on $[0,1]$. This function is a representative of α, since every integrable function f satisfies

$$f(x) = \frac{d}{dx}\int_0^x f \quad \text{for almost all } x.$$

Remark 2.5. (i) The obvious analogue of Theorem 2.1 for $p=\infty$ is False.

Let $\oplus_\infty$ denote the direct sum with norm

$$\|(v,w)\| = \max\{\|v\|, \|w\|\}.$$

Define the category $\mathcal{A}^\infty$ analogously to $\mathcal{A}^p$. Then, by the same argument as for $p<\infty$, the initial object of $\mathcal{A}^\infty$ is the closure of E in $L^\infty[0,1]$, together with I and γ. This is NOT $L^\infty[0,1]$; for example, $\bar{E}$ does not contain the indicator function $\chi_{[0,\frac{1}{3}]}$ (actually any (step) function with points of continuity not at dyadic points cannot be approximated uniformly).

(ii) $[0,1]$ can be replaced by the Cantor space $\{0,1\}^{\mathbb{N}}$, i.e. the set of infinite $0,1$ sequences, with the probability measure in which the set of sequences beginning with n prescribed bits has measure 2^{-n}. The measure preserving surjection

$$s : \{0,1\}^{\mathbb{N}} \to [0,1]$$

(binary representations) $$(x_0, x_1, \cdots) \mapsto \sum_{n=0}^{\infty} x_n 2^{-(n+1)}$$

induces an isomorphism $L^p[0,1] \cong L^p(\{0,1\}^{\mathbb{N}})$ for each $p \in [1,\infty]$ (in $\mathbf{Ban}_1$).

We give a rigorous discussion.

The Cantor space endowed with the measure defined as follows: The set fixing the beginning $n+1$ bits

$$\{(y_0, y_1, \cdots, y_n, x_{n+1}, x_{n+2}, \cdots) \mid y_0, \cdots, y_n \text{ fixed}, x_{n+1}, x_{n+2}, \cdots \in \{0,1\}\}$$

has measure $2^{-(n+1)}$. We see that this measure is a probability measure, because $\{0,1\}^{\mathbb{N}}$ has measure 1. The map s defined above between $\{0,1\}^{\mathbb{N}}$ and $[0,1]$ preserves measures.

(Under the construction, a function $f: \{0,1\}^{\mathbb{N}} \to \mathbb{F}$ can be identified with some $f^*: [0,1] \to \mathbb{F}$, because S is "nearly" a bijection since it converts the binary representation of real numbers in $[0,1]$ back to the real number itself, and the image of $(x_0, x_1, \cdots x_m, 0, 1, 1, \cdots)$ under the map s is the same as that of $(x_0, x_1, \cdots, x_m, 1, 0, 0, \cdots)$)

Let $\sim$ denotes the equivalence relation:

$$(x_0, x_1, \cdots, x_m, 0, 1, 1, \cdots) \sim (x_0, x_1, \cdots, x_m, 1, 0, 0, \cdots)$$

Then
$$f': \{0,1\}^{\mathbb{N}}/\sim \;\longrightarrow \mathbb{F} \quad \text{is well-defined.}$$
$$\vec{x} \longmapsto f(\vec{x})$$

and can be identified canonically with f^*.

Also, $\{0,1\}^{\mathbb{N}}/\sim$ is of the same measure as $\{0,1\}^{\mathbb{N}}$, since only countably many elements are moded out, which does not change measures.

$$\bigcup_{n=0}^{\infty} \{(x_0, \cdots, x_n, 1, 0, 0, \cdots) \mid x_i \in \{0,1\}, i = 0, 1, 2, \cdots, n\}$$

is countable.

Thus the space of functions

$$f' : \{0,1\}^{\mathbb{N}}/\sim \;\longrightarrow \mathbb{F}$$

is canonically isomorphic to the space of functions

$$f^* : [0,1] \rightarrow \mathbb{F}.$$

The isometric isomorphism $L^p(\{0,1\}^{\mathbb{N}}) \cong L^p[0,1]$ is also canonical since the additional restrictions:
$(\|f'\| < \infty$ and $\|f^*\| < \infty)$ are the same, with the map
s measure preserving.
Under this isomorphism, γ corresponds to the map

$$\gamma : L^p(\{0,1\}^{\mathbb{N}}) \oplus L^p(\{0,1\}^{\mathbb{N}}) \longrightarrow L^p(\{0,1\}^{\mathbb{N}})$$

defined by

$$(\gamma(f,g))(x_0, x_1, \cdots) = \begin{cases} f(x_1, x_2, \cdots) & \text{if } x_0 = 0 \\ g(x_1, x_2, \cdots) & \text{if } x_1 = 1 \end{cases} \qquad (1)$$

$(f, g \in L^p(\{0,1\}^{\mathbb{N}}) \quad x_i \in \{0,1\})$

Thus, $(L^p(\{0,1\}^{\mathbb{N}}), I, \gamma)$ is initial in A^p, whenever $p < \infty$, where I is the constant function 1 on $\{0,1\}^{\mathbb{N}}$ (slightly abuse of notation).

We now show that the initial object of A^∞ consists of not the bounded functions on the Cantor space (or equivalently the interval) but the continuous functions on it.

Let's recall the definition of A^∞ analogous to $A^p (1 \le p < \infty)$

A^∞ is the category with

Objects: triples (V, v, δ) where

V is a Banach space, $v \in V$, $\|v\| \le 1$,

$\delta : V \oplus_\infty V \longrightarrow V$ satisfying $\delta(v, v) = v$.

Morphisms: $(V', v', \delta') \longrightarrow (V, v, \delta)$ are the maps

$\theta : V' \to V$ in Ban_1 preserving the structure

$\theta(v') = v$, $\theta(\delta(v_1', v_2'))= \delta(\theta(v_1'), \theta(v_2'))$

Identity: The identity morphisms of Banach spaces.

Here $V \oplus_\infty V$ is the direct sum of two copies of V,

with the norm $\|(v, w)\|_\infty = \max\{\|v\|, \|w\|\}$

Given $\{0,1\}^{\mathbb{N}}$ the product topology and $C(\{0,1\}^{\mathbb{N}})$ the supremum norm. The map

$$\gamma : C(\{0,1\}^{\mathbb{N}}) \oplus_\infty C(\{0,1\}^{\mathbb{N}}) \longrightarrow C(\{0,1\}^{\mathbb{N}})$$

defined by (1) is an isometric isomorphism, so $(C(\{0,1\}^{\mathbb{N}}), I, \gamma)$ is an object of A^∞

Proposition 2.6 (Universal property of functons on Cantor space)

$(C(\{0,1\}^{\mathbb{N}}), I, \gamma)$ is the initial object of A^∞.

Proof: For $n \ge 0$, let E_n be the subspace of $C(\{0,1\}^{\mathbb{N}})$ consisting of the functions f such that for all $x = (x_0, x_1, \dots)$ and $y = (y_0, y_1, \dots)$ in $\{0,1\}^{\mathbb{N}}$,

$$x_i = y_i \text{ for all } i < n \Rightarrow f(x) = f(y)$$

(i.e. only look at the first $n-1$ components)

Then E_0 is the linear span of I and $E_{n+1} = \gamma(E_n \oplus E_n)$ for each $n \geq 0$.

The same argument as used to prove Theorem 2.1 shows that $(C(\{0,1\}^{\mathbb{N}}), I, \gamma)$ is initial in $\mathcal{A}^\infty$, provided that $E = \bigcup_{n \geq 0} E_n$ is dense in $C(\{0,1\}^{\mathbb{N}})$. We show this now.

Metrize $\{0,1\}^{\mathbb{N}}$ by putting $d(x,y) = 2^{-\min\{n,\ x_n \neq y_n\}}$.
For $n \geq 0$, define $\pi_n \{0,1\}^{\mathbb{N}} \longrightarrow \{0,1\}^{\mathbb{N}}$ by

$$\pi_n(x) = (x_0, \dots, x_{n-1}, 0, 0, \dots)$$

Thus $d(x, \pi_n(x)) \leq 2^{-n}$ for all x.
Now let $f \in C(\{0,1\}^{\mathbb{N}})$; we prove that $f \in \bar{E}$.
For each $n \geq 0$, we have $f \circ \pi_n \in E_n$ and

$$\| f - f \circ \pi_n \|_\infty \leq \sup_{x,y:\, d(x,y) \leq 2^{-n}} |f(x) - f(y)|$$

But since $\{0,1\}^{\mathbb{N}}$ is compact, f is uniformly continuous, so the right-hand side converges to 0 as $n \to \infty$.
Hence $f = \lim_{n \to \infty} f \circ \pi_n \in E$ as required. $\square$

Now, let $1 \leq p \leq r < \infty$. For any Banach spaces V and W, the identity map

$$V \oplus_r W \longrightarrow V \oplus_p W$$

is a contraction, by the elementary fact that power means are increasing in their order. (To be rigorous,

$$\|(v,w)\|_p = \left[\tfrac{1}{2}\left(\|v\|^p + \|w\|^p\right)\right]^{\frac{1}{p}}$$

$$\leq \left[\tfrac{1}{2}\left(\|v\|^r + \|w\|^r\right)\right]^{\frac{1}{r}}$$

$$= \|(v,w)\|_r \qquad 1 \leq p \leq r < \infty$$

This is obtained by Jensen's inequality)

Hence A^p is a subcategory of A^r, and in particular, $(L^p[0,1], I, \gamma)$ can be regarded as an object of A^r.

Proposition 2.7. The unique map $(L^r[0,1], I, \gamma) \to (L^p[0,1], I, \gamma)$ in A^r is the inclusion $L^r[0,1] \hookrightarrow L^p[0,1]$

Proof: Clearly the inclusion is one such map, and by Theorem 2.1, it is the only one. $\square$

Similarly A^p is a subcategory of A^∞ for every p, so $(L^p[0,1], I, \gamma)$ is an object of A^∞. Hence by Proposition 2.6, there is a unique map

$$(C\{0,1\}^{\mathbb{N}}, I, \gamma) \longrightarrow (L^p[0,1], I, \gamma) \qquad (2)$$

in A^∞. Let $i: [0,1] \longrightarrow \{0,1\}^{\mathbb{N}}$ be any <u>section</u> of the surjection of Remark 2.5 (ii) (a choice of binary expansion of each element of $[0,1]$) One easily checks that $f \mapsto f \circ i$ is a map of the form (2) in A^∞, so it is the unique such map.

Moreover $s : \{0,1\}^{\mathbb{N}} \to [0,1]$ induces a map

$$C[0,1] \to C(\{0,1\}^{\mathbb{N}}) \quad \text{by}$$
$$g \mapsto g \circ s.$$

Its composite with (2) is the map $C[0,1] \to L^p[0,1]$ given by $g \mapsto g \circ s \circ i = g$. In otherwords,

We have derived the inclusion $C[0,1] \hookrightarrow L^p[0,1]$ from the universal property of $C(\{0,1\}^{\mathbb{N}})$.

This inclusion relates the abstractly characterized space $L^p[0,1]$ to genuine function space $C[0,1]$

Next, we use universal properties to construct the multiplication map : $L^p[0,1] \times L^q[0,1] \xrightarrow{\cdot} L^1[0,1]$ (3)

for each $p, q > 1$ such that $\frac{1}{p} + \frac{1}{q} = 1$.

For the rest of this section, write L^p as a shorthand for $L^p[0,1]$. For Banach spaces V and W, we write $\mathrm{Hom}(V, W)$ for the Banach space of all bounded linear maps from V to W with the operator norm.

To construct the multiplication map (3), we will give $\mathrm{Hom}(\underline{L^q}, L^1)$ the structure of an object $\underline{A^p}$. By Theorem 2.1, this structure will induce a map $L^p \to \mathrm{Hom}(L^q, L^1)$ or equivalently a map $L^p \times L^q \to L^1$. We will show that this is the multiplication.

First recall that we have already constructed the inclusion $j : L^q \hookrightarrow L^1$ (Proposition 2.7). This j is a map in Ban_1,

that is, an element of a closed unit ball of $\mathrm{Hom}(L^q, L^1)$.
Now define a linear map
$$\Gamma : \mathrm{Hom}(L^q, L^1) \oplus_p \mathrm{Hom}(L^q, L^1) \longrightarrow \mathrm{Hom}(L^q, L^1)$$
as follows: for $\varphi_1, \varphi_2 \in \mathrm{Hom}(L^q, L^1)$, let $\Gamma(\varphi_1, \varphi_2)$ be the composite:
$$L^q \xrightarrow{\gamma^{-1}} L^q \oplus L^q \xrightarrow{\varphi_1 \oplus \varphi_2} L^1 \oplus L^1 \xrightarrow{\gamma} L^1$$

Lemma 2.8 $(\mathrm{Hom}(L^q, L^1), j, \Gamma)$ is an object of A^p.

Proof: It is immediate from the definitions that $\Gamma(j, j) = j$, and $\|j\| = 1$, so we only have to show that Γ is a contraction.
Let $g \in L^q$. Writing $\gamma^{-1}(g) = (g_1, g_2)$
$$\begin{aligned}
\|\Gamma(\varphi_1, \varphi_2)(g)\| &= \|\gamma(\varphi_1(g_1), \varphi_2 g_2)\| \\
&= \tfrac{1}{2}(\|\varphi_1(g_1)\| + \|\varphi_2(g_2)\|) \quad (4) \\
&\leq \tfrac{1}{2}(\|\varphi_1\|\,\|g_1\| + \|\varphi_2\|\,\|g_2\|) \\
&\leq \tfrac{1}{2}(\|\varphi_1\|^p + \|\varphi_2\|^p)^{\frac{1}{p}}(\|g_1\|^q + \|g_2\|^q)^{\frac{1}{q}} \quad (5) \\
&= \left(\tfrac{1}{2}(\|\varphi_1\|^p + \|\varphi_2\|^p)\right)^{\frac{1}{p}}\left(\tfrac{1}{2}(\|g_1\|^q + \|g_2\|^q)\right)^{\frac{1}{q}} \\
&= \|(\varphi_1, \varphi_2)\|\,\|g\| \quad (6)
\end{aligned}$$
where (4) holds because $\gamma : L^1 \oplus_1 L^1 \longrightarrow L^1$ is an isometry. (5) is by Hölder's inequality for vectors in $\mathbb{R}^2$, and (6) holds because $\gamma : L^q \oplus_q L^q \longrightarrow L^q$ is an isometry.
Hence Γ is a contraction, as claimed. □

By Theorem 2.1, then, there is a unique map
$$(L^p, I, \gamma) \longrightarrow (\mathrm{Hom}(L^q, L^1), j, \Gamma) \quad (7)$$
in A^p. As a linear map $L^p \longrightarrow \mathrm{Hom}(L^q, L^1)$ it corresponds to

a <u>bilinear</u> map:

$$\square : L^p \times L^q \longrightarrow L^1$$

Proposition 2.9. The map $\square : L^p \times L^q \longrightarrow L^1$ is the product

$$(f, g) \longmapsto f \cdot g$$

Proof: By Hölder's inequality for functions on $[0,1]$, there is a linear contraction (pointwise)

$$\theta : L^p \longrightarrow \mathrm{Hom}(L^q, L^1)$$
$$f \longmapsto f \cdot -$$

By uniqueness part of Theorem 2.1, it suffices to show that θ is a map of the form (7) in $\mathcal{A}^p$. (prescrving the structure)
Evidently $\theta(\mathbb{I}) = j$. It remains to show that the square

$$\begin{tikzcd}
L^p \oplus_p L^p \arrow[r, "\gamma"] \arrow[d, "\theta\oplus\theta"'] & L^p \arrow[d, "\theta"] \\
\mathrm{Hom}(L^q, L^1) \oplus_p \mathrm{Hom}(L^q, L^1) \arrow[r, "\Gamma"'] & \mathrm{Hom}(L^q, L^1)
\end{tikzcd}$$

commutes, or equivalently that for all $f_1, f_2 \in L^p$,

$$\Gamma(\theta(f_1), \theta(f_2)) = \theta(\gamma(f_1, f_2))$$

This in turn is equivalent to

$$\gamma(f_1 \cdot g_1, f_2 \cdot g_2) = \gamma(f_1, f_2) \cdot \gamma(g_1, g_2)$$

for all $f_1, f_2 \in L^p$ and $g_1, g_2 \in L^q$, which follows from the definition of γ. $\square$

Endofunctor: A functor $F : C \to C$ from a category to itself is called an endofunctor.

Algebra for an endofunctor

Let $\mathcal{D}$ be a category, and $T : \mathcal{D} \to \mathcal{D}$ is an endofunctor. A T-algebra (or algebra for T) is an object D of $\mathcal{D}$ together with a morphism $\delta : T(D) \to D$, written as a pair (D, δ). A map of T-algebras $(D', \delta') \to (D, \delta)$ is a morphism $\theta : D' \to D$ in $\mathcal{D}$ such that the following diagram commutes:

$$\begin{tikzcd} T(D') \arrow[r, "\delta'"] \arrow[d, "T(\theta)"'] & D' \arrow[d, "\theta"] \\ T(D) \arrow[r, "\delta"] & D \end{tikzcd}$$

i.e. $\theta \circ \delta' = \delta \circ T(\theta)$

Denote by $\mathcal{D}(T)$ the category of T-algebras.

Objects (D, δ), $\delta : T(D) \to D$

Morphisms $(D', \delta') \xrightarrow{\theta} (D, \delta)$ such that the above diagram commutes.

Identity: the identity map. 1_D.

Coalgebra for an endofunctor

Let $\mathcal{C}$ be a category and $F : \mathcal{C} \to \mathcal{C}$ is an endofunctor. An F-coalgebra (or coalgebra for F) is an object C of $\mathcal{C}$ together with a morphism $\alpha : C \to F(C)$ written as a pair (C, α). A map of F-coalgebras $(C', \alpha') \to (C, \alpha)$ is a morphism $\varphi : C' \to C$ in $\mathcal{C}$ such that the following diagram commutes

$$\begin{tikzcd} C' \arrow[r, "\alpha'"] \arrow[d, "\varphi"'] & F(C') \arrow[d, "F(\varphi)"] \\ C \arrow[r, "\alpha"] & F(C) \end{tikzcd}$$

i.e. $\alpha \circ \varphi = F(\varphi) \circ \alpha'$

In other words: A coalgebra for an endofunctor $T: \mathcal{D} \to \mathcal{D}$ an algebra for $T^{op}: \mathcal{D}^{op} \to \mathcal{D}^{op}$.

Examples: Coalgebras for endofunctors on $\mathcal{S}et$

- Let $\mathcal{P}: \mathcal{S}et \to \mathcal{S}et$ be the power set functor.
  where $\mathcal{P}$ takes $C' \in \mathcal{S}et$ to its power set $\mathcal{P}(C')$
  $\mathcal{P}$ takes $\varphi: C' \to C$ to $\mathcal{P}(\varphi): \mathcal{P}(C') \to \mathcal{P}(C)$
  by sending $S' \subset C'$ to $\varphi(S')$.
  Then a coalgebra for $\mathcal{P}$ is a pair (C, α) with
  $\alpha: C \to \mathcal{P}(C)$, such that the diagram

$$\begin{tikzcd} C' \arrow[r, "\alpha'"] \arrow[d, "\varphi"'] & \mathcal{P}(C') \arrow[d, "\mathcal{P}(\varphi)"] \\ C \arrow[r, "\alpha"] & \mathcal{P}(C) \end{tikzcd}$$

  commutes for any other pair (C', α').
  What can α be? α is the map sending an element to the singleton.
  $\alpha: C \to \mathcal{P}(C)$ $\quad x \mapsto \{x\}$
  It is easy to check that this diagram commutes.

- An initial algebra of an endofunctor T on a category $\mathcal{D}$ is an initial object in the category of algebras of T.

Lemma 1 (Lambek's theorem)

If an endofunctor $T : \mathcal{D} \to \mathcal{D}$ has an initial algebra (D, δ), then $\delta : T(D) \to D$ is an isomorphism.

Proof: Suppose (D, δ), where $\delta : T(D) \to D$, is an initial algebra, then we define an algebra structure on $T(D)$ to be $T(\delta) : T(T(D)) \to T(D)$, i.e. the pair $(T(D), T(\delta))$.

Now, (D, δ) is initial, by definition there is a unique T-algebra morphism $i : D \to T(D)$, so that the diagram

$$\begin{tikzcd} T(D) \arrow[r, "T(i)"] \arrow[d, "\delta"'] & T(T(D)) \arrow[d, "T(\delta)"] \\ D \arrow[r, "i"] & T(D) \end{tikzcd}$$

commutes.

Note that $\delta \circ i : D \to D$ is the identity 1_D since D is initial and $\delta \circ i$ is the unique morphism. Then

$T(\delta) \circ T(i) = T(\delta \circ i) = T(1_D) = 1_{T(D)}$

As a result, $i \circ \delta : T(D) \to T(D)$ is the identity $1_{T(D)}$ because $i \circ \delta = T(\delta) \circ T(i) = 1_{T(D)}$. Hence δ is an isomorphism. $\square$

Well-ordered sets and ordinals.

- A well-ordered set is a totally ordered set $X = (X, \leq)$ such that every non-empty subset A of X has a minimal element $\min(A) \in A$.

Let $A \sqcup B$ denote the disjoint union of the sets A and B.

-Given a subset A of a non-empty well-ordered set $(X, \leq)$, we define the supremum $\sup(A) \in X \sqcup \{+\infty\}$ of A to be the least upper bound of A:

$$\sup(A) = \min(\{y \in X \sqcup \{+\infty\} : x \leq y \text{ for all } x \in A\})$$

(thus for instance the supremum of the empty set is $\min(X)$)

If $x \in X$, we define the successor

$$\mathrm{succ}(x) = \min((x, +\infty]).$$

Proposition (Peano-type axioms)

If x is an element of a non-empty well-ordered set X, then exactly one of the following statements holds

I (Limit case) $x = \sup([\min(X), x))$

II (Successor case) $x = \mathrm{succ}(y)$ for some $y \in X$.

In particular, $\min(X)$ is not the successor of any element in X.

Proof: Suppose Case I and Case II simultaneously hold. we have a $y \in X$ such that $x = \mathrm{succ}(y) = \min((x, +\infty])$. Hence $y < x$. Because $x = \sup([\min(X), x))$, the Least upper bound, then there exists a $z \in X$ such that $y < z < x$. This contradicts the minimality of Y.

Suppose neither of the cases holds. Then x is not the successor of any element. In other words, there is no $y \in X$ such that $x = \min((y, +\infty])$ i.e. for any $y \in X$, $x \neq \min((y, +\infty])$. We have $x > \min((y, +\infty])$ for all $y \in X$ and $x < \min((y, +\infty])$ for all $y \in X$. In the first case, $x = +\infty$ which is not in X, and in the second case,

$x \le y$ for all $y \in X$, which means $x = \min(X)$. But this is when Case I holds. Both lead to contradiction. $\square$

Corollary. If x and y are elements of a well-ordered set such that $\mathrm{succ}(x) = \mathrm{succ}(y)$, then $x = y$. Therefore

$$x = y \iff \mathrm{succ}(x) = \mathrm{succ}(y).$$

Theorem: (Transfinite induction for well ordered sets)
Let X be a non-empty well-ordered set, and let

$$P : X \longrightarrow \{\text{True}, \text{False}\}$$

be a property of elements of X.
Suppose that:

1. (Base case): $P(\min(X))$ is true.
2. (Successor case): If $x \in X$ and $P(x)$ is true, then $P(\mathrm{succ}(x))$ is true.
3. (Limit case): If $x = \sup([\min(X), x))$ and $P(y)$ is true for all $y < x$, then $P(x)$ is true. [Note this subsumes the base case].

Then P is true for all $x \in X$.

Proof: This is equivalent to say:

$$\{x \in X : P(x) = \text{True for all the three cases}\} = X$$

but Peano-type axioms just said that the three cases above involve all the elements of X $\square$

- Ordinals.

An ordinal is a well-ordered set α with the property that $x = \{y \in \alpha : y < x\}$ for all $x \in \alpha$.
(In particular, each element of α is also a subset of α, and strict order relation "<" on α is identical to the set member relation "$\in$")

Example For each natural number $n = 0, 1, 2, \cdots$, define the ordinal number n^{th} $(1^{st}, 2^{nd}, 3^{rd}, 4^{th}, \ldots)$ recursively by setting

$$0^{th} := \emptyset, \text{ and } n^{th} := \{0^{th}, 1^{st}, \ldots, (n-1)^{th}\}$$

for all $n \geq 1$. thus for instance

$$0^{th} = \emptyset,$$
$$1^{st} = \{0^{th}\} = \{\emptyset\}$$
$$2^{nd} = \{0^{th}, 1^{st}\} = \{\emptyset, \{\emptyset\}\}.$$
$$3^{rd} = \{0^{th}, 1^{st}, 2^{nd}\} = \{\emptyset, \{\emptyset\}, \{\emptyset, \{\emptyset\}\}\}$$

and so forth. One can easily check that n^{th} is an ordinal for every n. (and $(n-1)^{th}$ is known as the immediate predecessor of n^{th})
Furthermore, if we define

$$\omega = \{n^{th} : n \in \mathbb{N}\},$$

then ω is also an ordinal. ω is called the first transfinite ordinal.

The (partially) ordered sets form the category <u>Ord</u> of ordered sets, with

Objects: ordered sets $(X, \leq_X), (Y, \leq_Y), \cdots$

Morphisms: $\varphi : X \to Y$ such that $\varphi(x) \leq_Y \varphi(x')$ whenever $x \leq_X x'$.

Identity: identity morphisms $X \xrightarrow{id_X} X$.

- A representation ρ of the well-ordered sets is an assignment of a well-ordered set $\rho(X)$ to every well-ordered set X, such that

1. $\rho(X)$ is isomorphic to X for every well-ordered set X. (In particular, if $\rho(X)$ and $\rho(Y)$ are equal, then X and Y are isomorphic, in the sense of isomorphisms in the category Ord)
2. If there exists a morphism from X to Y, then $\rho(X)$ is a subset of $\rho(Y)$ (and the order structure on $\rho(X)$ is induced from that on $\rho(Y)$). (In particular, if X and Y are isomorphic, then $\rho(X)$ and $\rho(Y)$ are equal).

In the language of category theory, a representation is a covariant functor from the category of well-ordered sets to itself which turns all morphisms into inclusions, and which is naturally isomorphic to the identity functor.

Remark: Because the collection of well-ordered set is a class rather than a set, ρ is not actually a function (it is sometimes referred as a class function).

There is a standard representation of ordinals: (continuing the example before)

$0 = 0^{th} = \emptyset$

$1 = 1^{st} = \{\emptyset\} = \{0\}$

$2 = 2^{nd} = \{\emptyset, \{\emptyset\}\} = \{0, 1\}$

$\vdots$

$\omega = \{n^{th} : n \in \mathbb{N}\} = \{1, 2, 3, 4, \cdots\}$

(the first infinite ordinals)

$\omega + 1 = \{n^{th} : n \in \mathbb{N}\} \cup \{\omega\} = \{1, 2, 3, 4, \cdots, \omega\}$

$\vdots$

- An ordinal β is called the immediate predecessor of α if $\beta + 1 = \alpha$, in the sense that (in the form of sets)

$$\alpha = \beta \cup \{\emptyset\}. \quad \text{or} \quad \alpha = \beta \cup \{1\}.$$

- An ordinal $\alpha > 0$ is called a limit ordinal if α has no immediate predecessor.

Theorem:

(i) Given any two ordinals α and β, one is a subset of the other (and the order structure is induced from that on β).

(ii) Every well-ordered set X is isomorphic to exactly one ordinal $\mathrm{ord}(X)$.

In particular, ord is a representation of the well-orded sets.

(We omit the proof here)

From the above theorem, we have the following definition of comparison of ordinals: (they have order relations)
— Given two ordinals α and β, we say that $\underline{\alpha \leq \beta}$ if α is a subset of β. Thus, the ordinal numbers are totally ordered by the order relations, in fact they are well-ordered.

Transfinite composition.

Let $\mathcal{C}$ be a category, and let $I \subseteq \mathrm{Mor}(\mathcal{C})$ be a class of its morphisms. For α an ordinal, (regarded as a category with total orders as morphisms), an α-indexed transfinite sequence of elements in I is the diagram

$$X_\bullet : \alpha \longrightarrow \mathcal{C}$$

such that

— $X_\bullet$ takes all the successor morphisms $\beta \xrightarrow{\leq} \beta+1$ in α to the elements $X_\beta \xrightarrow{x_{\beta,\beta+1}} X_{\beta+1}$ in I, and
— $X_\bullet$ is "continuous" in the sense that for every non-zero limit ordinal $\beta < \alpha$, $X_\bullet$ restricted to the full subdiagram $\{\gamma \mid \gamma < \beta\}$ is a colimiting cocone in $\mathcal{C}$ for $X_\bullet$ restricted to $\{\gamma \mid \gamma < \beta\}$:

$$X_\beta \cong \varinjlim_{\gamma < \beta} X_\gamma$$

The corresponding transfinite composition is the induced morphism $\quad X_0 \longrightarrow X_\alpha := \varinjlim_\alpha X_\bullet$ (universal cocone)

into the colimit of the diagram. Schematically,

$$\begin{tikzcd}
X_0 \arrow[r, "x_{0,1}"] \arrow[dr] & X_1 \arrow[r, "x_{1,2}"] \arrow[d] & X_2 \arrow[r, "x_{2,3}"] \arrow[dl] & X_3 \arrow[r] \arrow[dll] & \cdots \\
& X_\alpha & & &
\end{tikzcd}$$

Lemma 2. (Adámek's theorem)

Let $\mathcal{C}$ be a category with an initial object 0 and a transfinite composition of length ω (the first infinite ordinal) (hence colimits of sequences $\omega \to \mathcal{C}$). Suppose $F: \mathcal{C} \to \mathcal{C}$ preserves colimit of ω-chains. Then the colimit $F^{(\omega)}(0)$ of the chain

$$0 \xrightarrow{i} F(0) \xrightarrow{F(i)} \cdots \longrightarrow F^{(n)}(0) \xrightarrow{F^n(i)} F^{(n+1)}(0) \longrightarrow \cdots$$

(where i is the unique morphism $0 \to F(0)$) carries a structure of initial F-algebra.

(Proof omitted here)

Remark 2.11 Theorem 2.1 on $L^p[0,1]$, Proposition 2.6 on $C(\{0,1\}^{\mathbb{N}})$ and Proposition 2.10 on ℓ^r and c_0 are all instances of general categorical theorem that not only proves the existence of these objects but also construct them.

Let D be a category, and $T: D \to D$ be an endofunctor. Suppose that D has an initial object Z such that the diagram

$$Z \xrightarrow{!} T(Z) \xrightarrow{T(!)} T^2(Z) \xrightarrow{T^2(!)} \cdots$$

has a colimit (where $!$ is the unique map $Z \to T(Z)$), and that this colimit is preserved by T; in other words, the canonical map: $\eta: \mathrm{colim}_n\, T(T^n(Z)) \to T\, \mathrm{colim}_n(T^n(Z))$ is an isomorphism. The Adámek's theorem states that $(\mathrm{colim}_n\, T^n(Z), \eta^{-1})$ is the initial T-algebra.

$$(\mathrm{colim}_n = \varinjlim_{\mathbb{N}} = \varinjlim_{\omega})$$

Let D be the category Ban_* of pairs (V, v) where $V \in \mathrm{Ban}_1$ and $v \in V$ with $\|v\| \leq 1$; maps in Ban_* are maps in Ban_1 preserving the chosen points. (This is the coslice category $\mathbb{F}/\mathrm{Ban}_1$) Define $T: D \to D$ by

$$T(V, v) = (V \oplus_p V, (v, v)), \quad T(\theta) = \theta \oplus \theta$$

The category of T-algebras is $\mathcal{A}^r$.

(Verification Should verify the commutativity of diagrams given in the definition of T-algebras. Recall that $\mathcal{A}^r$ has

Objects: (V, v, δ) where $(V, v) \in \mathrm{Ban}_*$, $\delta: V \oplus_p V \to V$ is a map of Banach spaces satisfying $\delta(v, v) = v$.

Morphisms: θ preserving structure $\theta(v') = v$, $\theta \circ \delta = \delta \circ (\theta \oplus \theta)$ which precisely satisfying the commutative diagram:

$$\begin{tikzcd}
(V' \oplus_p V', (v', v')) \arrow[r, "\delta'"] \arrow[d, "T(\theta)"'] & (V', v') \arrow[d, "\theta"] \\
(V \oplus_p V, (v, v)) \arrow[r, "\delta"] & (V, v)
\end{tikzcd}$$

Moreover $(\mathbb{F}, 1)$ is initial in Ban_*, and the hypothesis of Adámek's theorem hold, so the initial object of A^r is constructed as the colimit over $n \geq 0$ of the objects $T^n(\mathbb{F}, 1)$ of Ban_*, i.e. $\operatorname{colim}_n T^n(\mathbb{F}, 1)$.

Concretely, $T^n(\mathbb{F}, 1)$ is a Banach space for all $n \in \mathbb{N}$ by the definition of T, where

- the norm of $\mathbb{F}$ is given by the absolute value $\|\cdot\| = |\cdot|$
- the norm of $\mathbb{F} \oplus_p \mathbb{F}$ is given by

$$\|(a, b)\| = \left(\tfrac{1}{2}(|a|^p + |b|^p)\right)^{\frac{1}{p}}$$

Recall that in Theorem 2.1, E_n is defined as

$E_n = \{$ step functions defined on $[0,1]$ that is constant on each of the intervals $\left(\frac{i-1}{2^n}, \frac{i}{2^n}\right)$ $1 \leq i \leq 2^n \}$

In order to give a norm for each E_n and make it into a Banach space as a subspace of $L^p[0,1]$ with our familiar L^p-norm, we let

$$\|I_{[0,1]}\| = 1$$ (I is defined to be the constant function on $[0,1]$ as in Theorem 2.1)

so that

$$(\mathbb{F}, 1) \cong (E_0, I) \quad \text{in } Ban_*$$

Hence $T^n(\mathbb{F}, 1) \cong (E_2, I)$ for all $n = 0, 1, 2, \cdots$

The categorical construction of $L^p[0,1]$ is the colimit

$E_0 \hookrightarrow E_1 \hookrightarrow \cdots$ in Ban_1.

From Adámek's theorem we see that

$(L^p[0,1], I, \gamma)$ is an initial in $\mathcal{A}^p$.

This is already proved in Theorem 2.1.

Lambek's theorem tells us that γ is an isomorphism.

This can also be seen in the definition of γ in the front parts of Section 2.

The definition of $L^p[0,1]$ with its norm is summarized below: ① Define $E_0 = \{$ step functions constant on $(0,1)\}$

$= \{ cI \mid c \in \mathbb{F}\}$, with $\|I\| = 1$

so that every element is given a norm, and

$(E_0, I) \cong (\mathbb{F}, 1)$ in Ban_*

② Define $T: \mathrm{Ban}_* \to \mathrm{Ban}_*$ by

$T(V, v) = (V \oplus_p V, (v, v))$ with the norm on

$V \oplus_p V$ to be $\|(v, w)\| = \left(\frac{1}{2}\left(\|v\|^p + \|w\|^p\right)\right)^{\frac{1}{p}}$.

$T(\theta) = \theta \oplus \theta$

Hence if we let E_n defined as

$E_n = \{$step functions defined on $[0,1]$ that is continuous on $\left(\frac{i-1}{2^n}, \frac{i}{2^n}\right)$, $1 \le i \le 2^n\}$

then $(E_n, I) = T^n(E_0, I) \cong T^n(\mathbb{F}, 1)$

and the norm on E_n is given for each n.

③ $(L^p[0,1], I)$ is defined to be the colimit of

$(E_0, I) \hookrightarrow (E_1, I) \hookrightarrow (E_2, I) \hookrightarrow \cdots$ in Ban_*. and

$(L^p[0,1], I, \gamma)$ is the initial object in $\mathcal{A}^p$. ◀

3. Integration on an arbitrary measure space.

Goals: To characterize the functor L^p from measure spaces to Banach spaces, again by universal property.

Throughout, the measure on a measure space X is written as μ_X, and all measure spaces are understood to be finite ($\mu_X(X) < \infty$).

- An embedding $i: Y \to X$ of measure spaces is an injection such that $B \subseteq Y$ is measurable if and only if $iB \subseteq X$ is measurable, and in that case, $\mu_Y(B) = \mu_X(iB)$.
Measure spaces and embeddings form a category $\mathbf{Meas}_{emb}$.

- A measure-preserving partial map

$$(A, s): X \to Y$$

is a measurable subset $A \subset X$ together with a measure-preserving map $s: A \to Y$. Here A is given the unique measure space structure such that the inclusion $A \hookrightarrow X$ is an embedding. The composite of measure-preserving partial maps

$$X \xrightarrow{(A,s)} Y \xrightarrow{(B,t)} Z$$

is $(s^{-1}B, u)$, where $u: s^{-1}B \longrightarrow Z$ is defined by

$$u(x) = t(s(x)).$$

Measure spaces and measure-preserving partial maps form a category $\mathbf{Meas}$.
(Recall that a map $f: X \to Y$ between two measure spaces X and Y is called measure preserving if $\mu_X(f^{-1}(A)) = \mu_Y(A)$ for all $A \subseteq Y$ measurable)

We describe them again:

- The category of measures and embeddings Meas_{emb} consists of

Objects: measure spaces (X, μ_X).

Morphisms: embeddings $i: Y \to X$ (injections) satisfying

$$B \subseteq Y \text{ is measurable} \iff i(B) \subseteq X \text{ is measurable}$$

with $\mu_Y(B) = \mu_X(i(B))$

Identity: identity map.

- The category of measure spaces and measure-preserving partial maps Meas consists of

Objects: measure spaces (X, μ_X).

Morphisms: $(A, s): X \to Y$, where $A \subseteq X$ is measurable, and $s: A \to Y$ is a measure-preserving map: $\mu_Y(B) = \mu_A(s^{-1}(B))$ for all measurable $B \subseteq Y$. Here A is given a unique measure space structure such that the inclusion $i: A \hookrightarrow X$ is an embedding.

(The structure is intuitive: For every measurable set $B \subseteq Y$, $B \cap A$ is measurable in A.)

Composition: If $X \xrightarrow{(A,s)} Y$ and $Y \xrightarrow{(B,t)} Z$, then the composition

$$(B, t) \circ (A, s) = (s^{-1}(B), t \circ s|_{s^{-1}(B)}).$$

(Should note that the domain changes to be smaller)

For each $p \in [1, \infty]$ there is a functor

$$L^p : \mathrm{Meas}^{op} \longrightarrow \mathrm{Ban}_1$$

defined on objects in the usual way and on maps as follows: Given a measure-preserving map $(A, s): X \to Y$, the induced map $L^p(Y) \to L^p(X)$ is $g \mapsto (g \circ s)^X$, where $(g \circ s)^X$ denotes the composite $g \circ s: A \to \mathbb{F}$ extend by zero to X. In other words,

$$L^p(s) : L^p(Y) \longrightarrow L^p(X)$$
$$g \longmapsto (g\circ s)^X$$

where $(g\circ s)^X(x) = \begin{cases} g\circ s\,(x) & x\in A \\ 0 & x\in X\setminus A \end{cases}$

Any embedding $i : Y \to X$ determines a map

$$(iY, i^{-1}) : X \to Y$$ in Meas ; note the change of direction. (In particular, every measurable subset A of X gives a map $(A, id_A) : X \to A$) Also, every measure-preserving map $s : X \to Y$ determines a map

$$(X, s) : X \to Y \text{ in Meas}.$$

Now, the category $\text{Meas}_{\text{pres}}$ of measure spaces and measure preserving maps is defined as follows

Objects : measure spaces

Morphisms : pairs $(X, s) : X \to Y$, where

$s : X \to Y$ is a measure preserving map.

$\mu_Y(B) = \mu_X(s^{-1}(B))$ for every $B \subseteq Y$ measurable.

Thus both of the categories $\text{Meas}^{op}_{\text{emb}}$ and $\text{Meas}_{\text{pres}}$ can be viewed as subcategories of Meas

$$\begin{tikzcd} \text{Meas}^{op}_{\text{emb}} \arrow[dr, hook] & \\ & \text{Meas}. \\ \text{Meas}_{\text{pres}} \arrow[ur, hook] & \end{tikzcd}$$

(Should note that the direction of arrows in Meas_{emb} and $\text{Meas}_{\text{pres}}$ are opposite)

Furthermore, every measure-preserving partial map factors canonically into maps of these two special types:

$$X \xrightarrow{(A,s)} Y = (X \xrightarrow{(A, id_A)} A \xrightarrow{(A,s)} Y).$$

This has two consequences. (contravariant functor)
First, any functor $F : \mathrm{Meas}^{op} \longrightarrow \mathcal{C}$ to another category $\mathcal{C}$ is determined by its effect on objects, embeddings and measure-preserving maps.
For simplicity, we will write $F(i)$ for the map

$$F(i\,Y, i^{-1}) : F(Y) \to F(X)$$

induced by an embedding $i : Y \to X$, and $F(s)$ for the map $F(X,s) : F(Y) \longrightarrow F(X)$ for the map

$$F(X, s) : F(Y) \longrightarrow F(X)$$

induced by a measure preserving map $s : X \to Y$.

Second, a transformation between two functors

$$\mathrm{Meas}^{op} \longrightarrow \mathcal{C}$$

is natural <u>if and only if</u> it is natural with respect to both embeddings and measure-preserving maps.
We explain this concisely below:
On the one hand, it is obvious from the definition that the "only if" part holds.
On the other hand, by the decomposition

$$X \xrightarrow{(A,s)} Y = (X \xrightarrow{(A, id_A)} A \xrightarrow{(A,s)} Y)$$

The naturality of the embeddings gives $A \xrightarrow{(id_A(A),\, id_A^{-1})} X$ and hence the first component of the decomposition. The naturality of measure preserving gives $A \xrightarrow{(A,s)} Y$. Thus the "if" part holds.

Lemma 3.1 Let $F : \mathrm{Meas}^{op} \to \mathcal{C}$ be a functor into a category $\mathcal{C}$. Let $s : X \to Y$ be a measure preserving map and B a measurable subset of Y. Write

$$\begin{tikzcd} s^{-1}(B) \arrow[r, "s'"] \arrow[d, hook, "i"'] & B \arrow[d, hook, "j"] \\ X \arrow[r, "s"] & Y \end{tikzcd}$$

for the inclusions and the restriction of s. (Here $s' = s|_{s^{-1}(B)}$)

Then the square

$$\begin{tikzcd} F(s^{-1}(B)) \arrow[d, "F(i)"'] & F(B) \arrow[l, "F(s')"'] \arrow[d, "F(j)"] \\ F(X) & F(Y) \arrow[l, "F(s)"] \end{tikzcd}$$

in $\mathcal{C}$ commutes.

(Result of this type are known as Beck-Chevalley conditions)

Proof: The square

$$\begin{tikzcd} s^{-1}(B) \arrow[r, "{(s^{-1}(B), s')}"] & B \\ X \arrow[u, "{(s^{-1}(B), id_{s^{-1}(B)})}"] \arrow[r, "{(X, s)}"] & Y \arrow[u, "{(B, id_B)}"'] \end{tikzcd}$$

in Meas commutes, and the result follows by functoriality of F. $\square$

The proof of the main theorem is decomposed into three steps. First: the universal vector space obtained from a measure space consists of the simple functions.

Second, the universal normed vector space from a measure space consists of the a.e. equivalence classes of simple functions.

Third, the universal Banach space from a measure space consists of integrable functions.

Each step has two versions, according to whether one considers functoriality with respect to all measure preserving partial maps or only the embeddings.

We now begin the first step

- Two embeddings

$$Y \xrightarrow{i} X \xleftarrow{j} Z$$

of measure spaces are complementary if $i(Y) \cap j(Z) = \emptyset$ and $i(Y) \cup j(Z) = X$.

- Write Vect for the category of vector spaces (over $\mathbb{F}$).
- Let $\mathcal{V}_{emb}$ be the category of pairs (F, v) consisting of
  - a functor $F : \mathrm{Meas}_{emb} \longrightarrow \mathrm{Vect}$ together with
  - an element $v_X \in F(X)$ for each measure space X,

satisfying the following axiom:

(I) $(Fi)(v_Y) + (Fj)(v_Z) = v_X$ for all pairs of complementary embeddings $Y \xrightarrow{i} X \xleftarrow{j} Z$.

A map $(F', v') \longrightarrow (F, v)$ in $\mathcal{V}_{emb}$ is a natural transformation $\psi : F' \longrightarrow F$ such that $\psi_X(v'_X) = v_X$ for all $X \in \mathrm{Meas}_{emb}$.

When $i : Y \to X$ is an embedding and i is understood, we write $(F(i))(v_Y) \in F(X)$ as v_Y^X. In this notation, axiom (I) states that $v_Y^X + v_Z^X = v_X$.

Example: There is an object $(\mathcal{S}, I)$ of $\mathcal{V}_{emb}$ defined as follows: The functor $\mathcal{S}: \mathrm{Meas}_{emb} \longrightarrow \mathrm{Vect}$ assigns each measure space X the space of simple functions $X \to \mathbb{F}$ and is defined on embeddings by extending simple functions by zero. The element $I_X \in \mathcal{S}(X)$ is the function with constant value 1. Given an embedding $i: Y \to X$, the simple function $I_Y^X = (\mathcal{S}(i))(I_Y)$ on X is the indicator function (characteristic function) of $iY \subseteq X$. Axiom (I) is the elementary fact that $I_Y^X + I_Z^X = I_X$ for any measurable partition $X = Y \sqcup Z$. (This will be used later)

Let $\mathcal{V}$ be the category of pairs (F, v) consisting of a functor $F: \mathrm{Meas}^{op} \longrightarrow \mathrm{Vect}$ together with an element $v_X \in F(X)$ for each measure space X, satisfying axiom (I) and

(II) $(F(s))(v_Y) = v_X$ for all measure-preserving maps $X \xrightarrow{s} Y$.

The maps in $\mathcal{V}$ are defined as in $\mathcal{V}_{emb}$, which is a natural transformation (preserving structures similar in $\mathcal{V}_{emb}$.

The functor $\mathcal{S}$ on Meas_{emb} extends naturally to a functor $\mathcal{S}: \mathrm{Meas}^{op} \longrightarrow \mathrm{Vect}$, since the composite of a simple function $Y \longrightarrow \mathbb{F}$ with a measure-preserving map $X \to Y$ is again simple. The pair $(\mathcal{S}: \mathrm{Meas}^{op} \longrightarrow \mathrm{Vect}, I)$ is an object of $\mathcal{V}$. By abuse of notation, we write $(\mathcal{S}, I)$ for both this object of $\mathcal{V}$ and the object $\mathcal{V}_{emb}$ defined previously, which is its image under the evident forgetful functor $\mathcal{V} \longrightarrow \mathcal{V}_{emb}$.

Lemma 3.2. Let $(F, \nu) \in \mathcal{V}_{emb}$. Let $n \geq 0$ and let $(i_r : Y_r \to X)_{1 \leq r \leq n}$ be a family of embeddings with pairwise disjoint images. Then

$$\nu^X_{i_1 Y_1 \cup \cdots \cup i_n Y_n} = \nu^X_{Y_1} + \cdots + \nu^X_{Y_n}$$

In particular, $\nu^X_{\emptyset} = 0$ for any measure space X.

Proof. Put $Y = Y_1 \cup \cdots \cup Y_n$, and $i : Y \to X$ is defined as
$i|_{Y_r} = i_r : Y_r \longrightarrow X$. $1 \leq r \leq n$.

Remember

ν^X_Y i embedding

Y is not necessarily a subset of X

First we show that

$$\nu_Y = \nu^Y_{Y_1} + \nu^Y_{Y_2} + \cdots + \nu^Y_{Y_n} \qquad (9)$$

where $\nu^Y_{Y_r}$ is understood as $F(inc_r)(\nu_Y) \in F(Y)$, where inc_r is the inclusion $Y_r \xrightarrow{\subseteq} Y$. which is a special kind of embedding. $\nu^Y_{Y_r}$ inc_r

Axiom (I) applied to the embeddings
$$\begin{array}{ccccc} Y_1 & \longrightarrow & Y & \longleftarrow & Y_2 \\ \| & & \| & & \| \\ \emptyset & \longrightarrow & \emptyset & \longleftarrow & \emptyset \end{array}$$
gives $\nu_\emptyset + \nu_\emptyset = \nu_\emptyset$ and so $\nu_\emptyset = 0$. The general case follows by induction, again using (I), which proves (9).
The functoriality of $F : \mathrm{Meas}_{emb} \longrightarrow \mathrm{Vect}$ gives:

$$\nu^X_Y = (Fi)(\nu_Y) = (Fi)(\nu^Y_{Y_1} + \cdots + \nu^Y_{Y_n}) = (\nu^Y_{Y_1})^X + \cdots + (\nu^Y_{Y_n})^X$$
$$= \nu^X_{Y_1} + \cdots + \nu^X_{Y_n}$$ as required.

Here: $(\nu^Y_{Y_r})^X = \nu^X_{Y_r}$ is because

$$Y_r \xhookrightarrow{incr} Y \xrightarrow{i} X \qquad \text{and}$$

$$\begin{aligned}\nu^X_{Y_r} = (Fi)(\nu_{Y_r}) &= (F(i\circ incr))(\nu_{Y_r}) \\ &= (Fi)[F(incr)(\nu_{Y_r})] \\ &= (Fi)(\nu^Y_{Y_r}) \\ &= (\nu^Y_{Y_r})^X\end{aligned}$$

The last part of the lemma is the case $n=0$ $\square$

(Comment: We should write $(\nu^Y_{Y_r})^X_Y = \nu^X_{Y_r}$ to express the relations)

$$\underbrace{\overline{Y_r \xhookrightarrow{incr} Y}}_{Y \xrightarrow{i} X} \qquad \underbrace{\qquad}_{Y_r \hookrightarrow X}$$

Lemma 3.3. Let $(F,\nu)\in\mathcal{V}$. Let $s: X\to Y$ be a measure-preserving map and B a measurable subset of Y. Then
$(Fs)(\nu^Y_B) = \nu^X_{s^{-1}B}$

Proof: Define i, j, s' as in Lemma 3.1, those are

$$\begin{aligned} s' &= s|_{s^{-1}B} : s^{-1}B \to B \\ i &: s^{-1}B \hookrightarrow X \quad \text{the inclusion} \\ j &: B \hookrightarrow Y \quad \text{the inclusion}\end{aligned}$$

We have: $(Fs)(\nu^Y_B) = (Fs)(Fj)(\nu_B)$
by definition of ν^Y_B and
$\nu^X_{s^{-1}B} = (Fi)(\nu_{s^{-1}B}) = (Fi)(Fs')(\nu_B)$

by definition of $\nu^{X}_{s^{-1}B}$ and axiom (II). The results follows from Lemma 3.1 $(Fi)(Fs') = (Fs)(Fj)$

Now establish the <u>universal properties of simple functions</u>:
The proof implicitly uses the fact that: A function on a measure space is simple if and only if its image is <u>finite</u> and each fibre is measurable.

Proposition 3.4 (Universal properties of spaces of simple functions)
(i) (S, I) is the initial object of $\mathcal{V}_{emb}$.
(ii) (S, I) is the initial object of $\mathcal{V}$.

Proof: For (i) let $(F, \nu) \in \mathcal{V}_{emb}$. We show that there exists a unique map $\psi : (S, I) \longrightarrow (F, \nu)$ in $\mathcal{V}_{emb}$.
<u>Uniqueness</u>. Let $\psi : (S, I) \longrightarrow (F, \nu)$ in $\mathcal{V}_{emb}$.
For each measure space X and measurable $Y \subseteq X$, the naturality of ψ with respect to the inclusion $i : Y \hookrightarrow X$ gives a commutative square

$$\begin{tikzcd} S(Y) \arrow[r, "S(i)"'] \arrow[d, "\psi_Y"'] & S(X) \arrow[d, "\psi_X"] \\ F(Y) \arrow[r, "Fi"'] & F(X) \end{tikzcd}$$

Evaluating the square at $I_Y \in S(Y)$ gives:

$$\psi_X(I^X_Y) = (Fi)(\psi_Y(I_Y)) = \nu^X_Y$$

Hence ψ_X is uniquely determined on indicator functions of measurable subsets of X, and therefore, by linearity, on all of $S(X)$.

DATE

<u>Existence.</u> For each measure space X and $f \in S(X)$, define

$$\psi_X(f) = \sum_{c \in \mathbb{F}} c\, \nu^X_{f^{-1}(c)} \in F(X).$$

This sum has only finitely many nonzero summands, since if c is not in the image of f then $\nu^X_{f^{-1}(c)} = \nu^X_{\emptyset} = 0$ by Lemma 3.2.

To show that $\psi_X : S(X) \longrightarrow F(X)$ is linear, let $f, g \in S(X)$.
Then $\psi_X(f+g) = \sum_c c\, \nu^X_{(f+g)^{-1}(c)}$ (f, g are simple functions $X \to \mathbb{F}$)

But $(f+g)^{-1}(c) = \bigsqcup_{a,b:\, a+b=c} f^{-1}(a) \cap g^{-1}(b)$, so by Lemma 3.2

$$\nu^X_{(f+g)^{-1}(c)} = \sum_{a,b:\, a+b=c} \nu_{f^{-1}(a) \cap g^{-1}(b)}. \qquad (11)$$

Hence by (11)

$$\psi_X(f+g) = \sum_a a \sum_b \nu^X_{f^{-1}(a) \cap g^{-1}(b)}$$
$$+ \sum_b b \sum_a \nu^X_{f^{-1}(a) \cap g^{-1}(b)}$$
$$= \sum_a a\, \nu^X_{f^{-1}(a)} + \sum_b b\, \nu^X_{g^{-1}(b)} \qquad (12)$$
$$= \psi_X(f) + \psi_X(g).$$

where (12) is obtained by applying Lemma 3.2 to the disjoint unions

$$f^{-1}(a) = \bigsqcup_b f^{-1}(a) \cap g^{-1}(b), \quad g^{-1}(b) = \bigsqcup_a f^{-1}(a) \cap g^{-1}(b)$$

We have now shown that $\psi_X(f+g) = \psi_X(f)+\psi_X(g)$ for all $f, g \in S(X)$. A similar argument shows that ψ_X preserves scalar multiplication. Thus, ψ_X is a linear map $S(X) \longrightarrow F(X)$.

Next we show that ψ_X is natural in $X \in \mathrm{Meas}_{\mathrm{emb}}$. That is, given an embedding $i: Y \longrightarrow X$, we show that the square

$$\begin{array}{ccc} S(Y) & \xhookrightarrow{i} & S(X) \\ {\scriptstyle \psi_Y}\big\downarrow & & \big\downarrow{\scriptstyle \psi_X} \\ F(Y) & \xrightarrow{Fi} & F(X) \end{array}$$

commutes.

By linearity, it suffices to check this on the indicator function I_B^Y of a measurable subset $B \subseteq Y$.

(Comment: I_B^Y is again abuse of notation: $B \overset{inc}{\underset{\subseteq}{\hookrightarrow}} Y$
$I_B^Y = F(inc_B^Y)(I_B)$)

On the one hand, $\psi_Y(I_B^Y) = \nu_B^Y$ by definition of ψ_Y, so $(Fi)(\psi_Y(I_B^Y)) = (\nu_B^Y)^X = \nu_B^X$. On the other hand, the extension by zero of I_B^Y to X is I_B^X, and $\psi_X(I_B^X) = \nu_B^X$. The square therefore commutes.

We have now shown that $\psi = (\psi_X)_{X \in \mathrm{Meas}_{\mathrm{emb}}}$ defines a natural transformation on $S \longrightarrow F$. Moreover, $\psi_X(I_X) = \nu_X$ for each measure space X, by definition of ψ_X. Hence ψ is a morphism $(S, I) \longrightarrow (F, \nu)$ in $\mathcal{V}_{\mathrm{emb}}$, completing the proof of (i).

To prove (ii), let $(F, \nu) \in \mathcal{V}$. By (i), there is a unique map $\psi : (S, I) \longrightarrow (F, \nu)$ in $\mathcal{V}_{emb}$, and it suffices to prove that ψ is a map in $\mathcal{V}$. This reduces to showing that for every measure preserving map $s : X \to Y$, the naturality square

$$\begin{tikzcd} S(Y) \arrow[r, "- \circ s"] \arrow[d, "\psi_Y"'] & S(X) \arrow[d, "\psi_X"] \\ F(Y) \arrow[r, "Fs"] & F(X) \end{tikzcd}$$

commutes. (Actually, the maps in Meas are measure-preserving partial maps $X \xrightarrow{(A,s)} Y$, where $A \subseteq X$ is measurable and $s : A \to Y$ is a measure-preserving map, but we can take $X \xrightarrow{s} Y$ without loss of generality). By linearity it suffices to check this on the indicator function I_B^Y of a measurable set $B \subseteq Y$. On the one hand,

$$\psi_X(I_B^Y \circ s) = \psi_X(I^X_{s^{-1}B}) = \nu^X_{s^{-1}B}.$$

On the other hand, $Fs(\psi_Y(I_B^Y)) = (Fs)(\nu_B^Y)$.
Lemma 3.3 concludes the proof. $\square$
This completes the first step towards the "main theorem". We now embark on the second.

Let Nor_1 be the category of normed vector spaces with contractions. Let $p \in [1, \infty]$. Let $\mathcal{N}^p_{emb}$ be the category of pairs (F, ν) consisting of
- a functor $F : \mathrm{Meas}_{emb} \longrightarrow \mathrm{Nor}_1$ together with
- an element $\nu_X \in F(X)$ for each measure space X

satisfying the axiom (I): $(Fi)(\nu_Y)+(Fj)(\nu_Z)=\nu_X$ for all pairs of complementary embeddings $Y \xrightarrow{i} X \xleftarrow{j} Z$ and the following axioms:

(III) $\|\nu_X\| \le \mu_X(X)^{1/p}$ for all measure spaces X;

(IV) $\|(Fi)(u)+(Fj)(w)\|_{F(X)} \le (\|u\|^p_{F(Y)}+\|w\|^p_{F(Z)})^{\frac{1}{p}}$ for all pairs $Y \xrightarrow{i} X \xleftarrow{j} Z$ of complementary $u \in F(Y)$ and $w \in F(Z)$.

The maps in $\mathcal{N}^p_{emb}$ are defined as in $\mathcal{V}_{emb}$: A map $(F',\nu') \longrightarrow (F,\nu)$ in $\mathcal{N}^p_{emb}$ is a natural transformation $\psi : F' \longrightarrow F$ such that $\psi_X(\nu'_X)=\nu_X$ for all X.

Remark: When $p=1$, axiom (IV) is redundant, since the maps in ~~Ban~~ Norm_1 are contractions. When $p=\infty$, the expression $\mu_X(X)^{1/p}$ in (III) is interpreted as 0 if $\mu_X(X)=0$ and 1 otherwise, and the right-hand side of the inequality in (IV) as $\max\{\|u\|,\|w\|\}$.

Let $\mathcal{N}^p$ be the category defined in the same way, but replacing Meas_{emb} by Meas^{op}, and requiring that each object (F,ν) satisfies axioms (I)-(IV)

(Recall Axiom (II). $F(s)(\nu_Y)=\nu_X$ for all measure-preserving maps $X \xrightarrow{s} Y$.)

<u>Example</u>. An object $(\mathcal{S}^p, I)$ of $\mathcal{N}^p_{emb}$ and an object $(\mathcal{S}^p, I)$ for $\mathcal{N}^p$ defined as follows (abusively using the same name for both):

For a measure space X, write $\mathcal{S}^p(X)$ for the vector space of equivalence classes of simple functions on X under equality almost everywhere, with the L^p-norm

$$\|f\|_p = \sum_{c \in \mathbb{F}} |c|^p \mu_X(f^{-1}(c)) \qquad (3)$$

if $p < \infty$, and

$$\|f\|_\infty = \max\{|c| : c \in \mathbb{F},\ \mu_X(f^{-1}(c)) > 0\}$$

i.e. $S^p(X) = S(X) / (f \sim g \Leftrightarrow \|f-g\|_p = 0)$ for $1 \le p \le \infty$.

This construction defines functors:

$$S^p : \mathrm{Meas}_{\mathrm{emb}} \longrightarrow \mathrm{Nor}_1$$
$$S^p : \mathrm{Meas}^{op} \longrightarrow \mathrm{Nor}_1.$$

with the same functorial action as S (in the category V_{emb} and V respectively). They determine an object (S^p, I) of N^p_{emb} and an object (S^p, I) of N^p. Axioms (I) - (IV) are elementary properties of indicator functions and the p-norm. Equality holds in the inequalities (III) and (IV). Axiom (IV) is a consequence of the fact that the p-norm of a function on a disjoint union $Y \sqcup Z$ is determined in the evident way by p-norms of its restrictions Y and Z.

Lemma 3.5. Let $(F, \nu) \in N^p_{\mathrm{emb}}$. Then $\|\nu^X_Y\| \le \mu_X(iY)^{1/p}$ for any embedding $i : Y \to X$ of measure spaces.
In particular, $\nu^X_Y = 0$, whenever Y is a measure-zero subspace of a measure space X.

Proof: $\|\nu^X_Y\| = \|F_i(\nu_Y)\| \le \|\nu_Y\|$ since F_i is a contraction, and $\|\nu_Y\| \le \mu_Y(Y)^{1/p} = \mu_X(iY)^{1/p}$ by axiom (III). $\square$

Proposition 3.6 (Universal properties of spaces of simple functions). Let $1 \leq p \leq \infty$.

(i) (S^p, I) is the <u>initial object</u> of $\mathcal{N}^p_{emb}$.

(ii) (S^p, I) is the <u>initial object</u> of $\mathcal{N}^p$.

Proof: For (i), let $(F, \nu) \in \mathcal{N}^p_{emb}$. We show that there is a unique map $(S^p, I) \to (F, \nu)$ in $\mathcal{N}^p_{emb}$. Write ψ for the unique map $(S, I) \to (F, \nu)$ in $\mathcal{V}_{emb}$.

<u>Uniqueness</u>. Let φ be a map $(S^p, I) \to (F, \nu)$ in $\mathcal{N}^p_{emb}$. Applying the forgetful functor $\mathcal{N}^p_{emb} \to \mathcal{V}_{emb}$ gives a map $(S^p, I) \to (F, \nu)$ in $\mathcal{V}_{emb}$, which we also denote by φ. Its composite with the quotient map $(S, I) \to (S^p, I)$ in $\mathcal{V}_{emb}$ can only be ψ, so φ is uniquely determined.

To make it clear:

$(S^p, I) \xrightarrow{\varphi} (F, \nu)$ in $\mathcal{N}^p_{emb}$ with $S^p, F : \mathrm{Meas}_{emb} \to \mathrm{Nor}_1$ and $\nu_X \in F(X)$ satisfying axioms related to $1 \leq p < \infty$

$\Big\downarrow$ forgetful functor $\mathcal{N}^p_{emb} \to \mathcal{V}^p_{emb}$

$(S^p, I) \xrightarrow{\varphi} (F, \nu)$ in $\mathcal{V}^p_{emb}$ with $S^p, F : \mathrm{Meas}_{emb} \to \mathrm{Vect}$

In the category $\mathcal{V}^p_{emb}$, the map known as the quotient map $(S, I) \xrightarrow{\pi} (S^p, I)$ is defined as the natural transformation $(\pi_X : S(X) \to S^p(X))_{X \in \mathrm{Meas}_{emb}}$ satisfying

$$\pi_X(I_X) = I_X \text{ (equivalence class of the indicator function on } X)$$

The composite

$$(S, I) \xrightarrow{\pi} (S^p, I) \xrightarrow{\varphi} (F, \nu)$$

is a map $(S, I) \to (F, \nu)$ in $\mathcal{V}_{emb}$. But (S, I) is initial by Proposition 3.4, so $\varphi \circ \pi$ can only be ψ. Thus φ is uniquely determined.

<u>Existence</u>. First we prove that $\|\psi_X(f)\| \leq \|f\|_p$ for each measure space X and $f \in S(X)$. If $p < \infty$, then

$$\|\psi_X(f)\| = \Big\| \sum_{c \in \mathbb{F}} c\, \nu^X_{f^{-1}(c)} \Big\| \qquad (14)$$

$$\leq \Big(\sum_{c \in \mathbb{F}} \| c\, \nu^X_{f^{-1}(c)} \|_p \Big)^{1/p} \qquad (15)$$

$$\leq \Big(\sum_{c \in \mathbb{F}} |c|\, \mu_X (f^{-1}(c)) \Big)^{1/p} \qquad (16)$$

$$= \|f\|_p \qquad (17)$$

where (14) follows from (10) for ψ (the definition of ψ), inequality (15) follows by induction from axiom (IV), (16) follows from Lemma 3.5 (with inclusions $f^{-1}(c) \subseteq X$ as embeddings) and (17) follows from (13) (the definition of p-norm). The proof for $p = \infty$ is similar. Hence $\|\psi_X(f)\| \leq \|f\|_p$, as claimed.

It follows that for each measure space X, there is a unique linear map φ_X such that the triangle

$$\begin{tikzcd}
S(X) \arrow[r, "\pi"] \arrow[dr, "\psi_X"'] & S^p(X) \arrow[d, dashed, "\exists!\, \varphi_X"] \\
& F(X)
\end{tikzcd}$$

commutes (where the horizontal arrow is the quotient map), and that φ_X is a contraction. Moreover, φ_X is natural in X because ψ_X is, and $\varphi_X(I_X) = \psi_X(I_X) = \nu_X$. Hence φ is a map $(S^p, I) \longrightarrow (F, \nu)$ in $\mathcal{N}^p_{emb}$, completing the proof of (i).

Part (ii) follows from the fact that ψ is natural with respect to measure-preserving maps, by Proposition 3.4(ii). $\square$

Now come to the main theorem.

Let $\mathcal{B}^p_{emb}$ denote the full subcategory of $\mathcal{N}^p_{emb}$ consisting of pairs (F, ν) such that F takes values in Banach spaces, and similarly $\mathcal{B}^p \subseteq \mathcal{N}^p$. Equivalently, $\mathcal{B}^p_{emb}$ and $\mathcal{B}^p$ are defined in the same way as $\mathcal{N}^p_{emb}$ and $\mathcal{N}^p$, respectively, but replacing Nor_1 by Ban_1. We have already defined the functor $L^p : \mathrm{Meas}^{op} \longrightarrow \mathrm{Ban}_1$ and we also write L^p for the restricted functor $\mathrm{Meas}_{emb} \longrightarrow \mathrm{Ban}_1$. These functors define an object (L^p, I) of $\mathcal{B}^p_{emb}$ and an object (L^p, I) of $\mathcal{B}^p$.

<u>Theorem 3.7</u> (Universal Properties of the L^p functors)

Let $1 \leq p \leq \infty$.

(i). (L^p, I) is the initial object of $\mathcal{B}^p_{emb}$.

(ii) (L^p, I) is the initial object of $\mathcal{B}^p$.

<u>Proof.</u> The inclusion $\mathrm{Ban}_1 \hookrightarrow \mathrm{Nor}_1$ has a left adjoint, the completion functor $V \mapsto \overline{V}$. Given $(F, \nu) \in \mathcal{B}^p_{emb}$, define

$$\begin{aligned} \overline{F} : \mathrm{Meas} &\longrightarrow \mathrm{Ban}_1 \quad \text{by} \\ X &\longmapsto \overline{F(X)}, \quad \text{i.e. } \overline{F}(X) = \overline{F(X)} \end{aligned}$$

and regard $\nu_X \in F(X)$ as an element of $\overline{F}(X)$. It is routine to verify that $(\overline{F}, \nu) \in \mathcal{B}^p_{emb}$ and that forgetful functor $\mathcal{B}^p_{emb} \longrightarrow \mathcal{N}^p_{emb}$ has left adjoint $(F, \nu) \mapsto (\overline{F}, \nu)$.

Left adjoints preserve initial objects, so by Proposition 3.6, the initial objects of $\mathcal{B}^p_{emb}$ is $(\overline{S^p}, I)$. But $L^p(X)$ is the

completion of $S^P(X)$ for each X, so the initial object of $\mathcal{B}^P_{emb}$ is (L^P, I). This proves (i).
The same argument proves (ii).

$\square$

Theorem 3.7 characterizes the space $L^P(X)$ up to isometric isomorphism since the maps in Ban_1 are contractions.

Completion functor $\mathrm{Nor}_1 \longrightarrow \mathrm{Ban}_1$.
Recall that: A metric space is complete when every Cauchy sequence in it converges.
Fact: The number fields $\underset{\mathbb{R} \text{ or } \mathbb{C}}{\overset{\shortparallel}{\mathbb{F}}}$ (with metric $d(x,y) = |x-y|$) is complete.
Theorem: Every metric space can be completed, that is, there is a complete metric space $\bar{X}$, containing (a copy of) X and extending its distance function.
Construction of such $\bar{X}$: Let $C(X)$ be the set of Cauchy sequences of X. For any two Cauchy sequences $a = (x_n)$, $b = (y_n)$, it is easy to check that the real sequence $(d(x_n, y_n))$ is also a Cauchy sequence. Since $\mathbb{R}$ is complete, it converges to a real number $D(a,b) = \lim_{n\to\infty} d(x_n, y_n)$.

We say that $\overset{a=}{(x_n)}$ and $\overset{b=}{(y_n)}$ are asymptotic when $d(x_n, y_n) \to 0$ as $n \to \infty$. The relation $(x_n) \sim (y_n) \iff d(x_n, y_n) \to 0$ as $n \to \infty$ is an equivalence relation. Therefore, $C(X)$ partitions into equivalence classes.

Define: $\bar{X} = C(X) / [(x_n) \sim (y_n) \iff d(x_n, y_n) \to 0,\ n \to \infty]$

to be the space of equivalence class of Cauchy sequences, and define the metric: $\bar{d}: \bar{X} \times \bar{X} \longrightarrow \mathbb{R}$ as

$$\bar{d}([a],[b]) = D(a,b)$$

were $[a]$ and $[b]$ denote the equivalence class of a and b, respectively.

The function $\bar{d}$ is well-defined, since for any other representative sequences $a' \in [a]$, $b' \in [b]$, we have

$$D(a',b') \leq D(a',a) + D(a,b) + D(b,b') = D(a,b)$$

(because $d(x_n, y_n) \leq d(x_n, z_n) + d(z_n, y_n)$ by the definition of a metric, and taking the limit as $n \to \infty$ does not change the direction of inequality) Similarly $D(a,b) \leq D(a',b')$ so

$$\bar{d}([a],[b]) = D(a,b) = D(a',b') = \bar{d}([a'],[b'])$$

It remains to verify that D is indeed a metric:

Let $[a], [b] \in \bar{X}$ with a and b representatives respectively.

- $\bar{d}([a],[b]) = 0 \iff D(a,b) = 0$
$$\iff \lim_{n\to\infty} d(x_n, y_n) = 0$$
$$\iff a \sim b \iff [a] = [b].$$
- $\bar{d}([b],[a]) = \bar{d}([a],[b])$ because $d(y_n, x_n) = d(x_n, y_n)$ so that $D(b,a) = D(a,b)$ by definition of a metric.

- $\bar{d}([a],[b]) \leq \bar{d}([a],[c]) + \bar{d}([c],[b])$ because $D(a,b) \leq D(a,c) + D(c,b)$ mentioned above.

Thus, $(\bar{X}, \bar{d})$ is a metric space, which is complete and contains (a dense copy of) X, extending its distance function, with a further argument. i.e.,

$$\bar{d}([x],[y]) = d(x,y) \quad \text{when } x,y \in X$$

and $[x]$ $[y]$ are viewed as $[(x,x,\cdots)]$ and $[(y,y,\cdots)]$ respectively. We can simply identify X as a subset of the completion $\bar{X}$.

For a normed vector space $(X, \|\cdot\|)$, the function $d(x,y) = \|x-y\|$ is a metric. Completion makes a normed space into a Banach space, so we have a completion functor

$$\overline{(\cdot)} : \mathrm{Nor}_1 \longrightarrow \mathrm{Ban}_1$$

from the category of normed spaces with linear contractions to the full subcategory of Banach spaces with linear contractions:

$$\overline{(\cdot)} : X \mapsto \bar{X}, \quad \text{the completion,}$$

For any contraction $T: X \to Y$, $\bar{T}([x]) = [T(x)]$ for all Cauchy sequences $x \in C(X)$, with $[x] \in \bar{X}$.

Indeed, the completion functor is just the left adjoint to the inclusion functor

$$i : \mathrm{Ban}_1 \longrightarrow \mathrm{Nor}_1$$

$$X \longmapsto X$$

$$[T : X \to Y] \longmapsto [T : X \to Y].$$

This can be checked directly by definition

We now focus on the case $p=1$. As in the case of the interval, the universal characterization of the integrable functions yields a unique characterization of integration as follows.

Write $\mathbb{F}: \mathrm{Meas}_{\mathrm{emb}} \to \mathrm{Ban}_1$ for the functor that sends all measure spaces to the ground field $\mathbb{F} \in \mathrm{Ban}_1$ ($\mathbb{R}$ or $\mathbb{C}$, maybe) and all maps in $\mathrm{Meas}_{\mathrm{emb}}$ to $\mathrm{id}_{\mathbb{F}}$. For each measure spaces X, put $t_X = \mu_X(X) \in \mathbb{F}$. Then $(\mathbb{F}, t) \in \mathcal{B}^1_{\mathrm{emb}}$. (I think there is an abuse of notation about $\mathbb{F}$)

Proposition 3.8. (Uniqueness of integration) The unique map $(L^1, I) \longrightarrow (\mathbb{F}, t)$ is the family of operators

$$\int_{-} = \left(\int_X : L^1(X) \longrightarrow \mathbb{F} \right)_{\substack{(X \in \mathrm{Meas}_{\mathrm{emb}}) \\ \text{measure spaces } X}}.$$

Proof: By Theorem 3.7(i), it suffices to show that $\int_-$ is indeed a map $(L^1, I) \to (\mathbb{F}, t)$ in $\mathcal{B}^1_{\mathrm{emb}}$. This reduces to the following standard properties of integration.

First, whenever X is a measure space, $\int_X$ is a map in $\mathcal{B}^1_{\mathrm{emb}}$ by linearity of integration and the triangle inequality

$$\left| \int_X f \right| \leq \int_X |f|$$

for $f \in L^1(X)$. Second, $\int_-$ defines a natural transformation $L^1 \to \mathbb{F}$ because

$$\int_X g^X = \int_Y g$$

for any embedding of measure spaces $Y \to X$ and $g \in L^1(Y)$ where g^X denotes the extension by zero to X. Finally,

$$\int_X I_X = \mu_X(X)$$

for every measure space X.

$\square$

(Comment: Similar to Proposition 2.2, here we define the integral operator $\int_{-}$ by required properties.)

Remark 3.9. ~~Here we have treated $(\mathbb{F}, t)$ as an object of $\mathcal{B}^1_{emb}$. But the constant functor $\mathbb{F} : \mathrm{Meas}^{op} \to \mathrm{Ban}$ together~~ with the element $t_X \in \mathbb{F}$ also define an object $(\mathbb{F}, t)$ of $\mathcal{B}^1$. Theorem 3.7 (ii) implies that the unique map $(L^p, I) \to (\mathbb{F}, t)$ in $\mathcal{B}^1_{emb}$ is in fact a map in $\mathcal{B}^1$. In concrete terms, this statement is the formula for integration under a change of variables

$$\int_Y g = \int_X g \circ s \left(= \int_X (g \circ s)^X \right)$$

whenever $s : X \to Y$ is a measure-preserving map and $g \in L^1(Y)$.

The next result relates abstractly characterized spaces $L^1(X)$ to actual spaces of functions. (We cannot hope to evaluate an element in $L^1(X)$ at a point of X, since it is only an equivalence class of integrable functions. The best we can hope for is to be able to integrate it over any measurable subset of X.) That is, we would like to construct for each $f \in L^1(X)$ the signed or complex measure $f\mu_X$. $f\mu_X(-) = \int_{-} f\,d\mu_X$, defined by

$$(f\mu_X)(A) = \int_A f\,d\mu_X$$

We now show that this construction arises naturally from the universal property of L^1.

Write $M(X)$ for the <u>Banach space</u> (indeed it is) of finite signed measures if $\mathbb{F} = \mathbb{R}$ or complex measures if $\mathbb{F} = \mathbb{C}$. on the underlying σ-algebra of a measure space X, with total variation norm $\nu \mapsto |\nu|(X)$.

[Recall that: For signed measure, the total variation of a measurable set E is defined by $|\mu|(E) = \mu^+(E) + \mu^-(E)$, where $\mu = \mu_+ - \mu_-$ is the unique decomposition of μ into mutually singular unsigned measures, according to the Jordan decomposition theorem. For complex measure, the total variation is $|\mu|(E) = \sup_{\pi:\text{ partition of } E \text{ into countable number of subsets}} \sum_{A \in \pi} |\mu(A)|$.]

Any embedding $i: Y \to X$ induces an isometry $M(Y) \to M(X)$ that extends measures by zero; thus, M defines a functor $\mathrm{Meas}_{\mathrm{emb}} \to \mathrm{Ban}_1$. Together with the elements $\mu_X \in M(X)$, it gives an object (M, μ) of $\mathcal{B}^1_{\mathrm{emb}}$.

Proposition 3.10. The unique map $(L^1, I) \to (M, \mu)$ in $\mathcal{B}^1_{\mathrm{emb}}$ has X-component

$$L^1(X) \to M(X)$$
$$f \mapsto f\mu_X.$$

for each measure space X.

Proof. For each X, define $\theta_X : L^1(X) \to M(X)$ by
$$\theta_X(f) = f\mu_X.$$
By Theorem 3.7(i), it suffices to show that θ defines a map $(L^1, I) \to (M, \mu)$ in $\mathcal{B}^1_{ams}$. It is "elementary" that θ_X is an isometry: (We show in the complex case)

For $f \in L^1(X)$, $\|\theta_X(f)\| = \|f\mu_X\| = |f\mu_X|(X)$. On the one hand, for every countable partition π of X,

$$\sum_{A \in \pi} |(f\mu_X)(A)| = \sum_{A \in \pi} \left|\int_A f \, d\mu_X\right|$$
$$(\text{Triangle inequality}) \leq \sum_{A \in \pi} \int_A |f| \, d\mu_X$$
$$= \int_X |f| \, d\mu_X$$
$$= \|f\|_1$$

so that $$\|f\mu_X\| = |f\mu_X|(X) = \sup_{\pi \text{ countable partition of } X} \sum_{A \in \pi} |(f\mu_X)(A)|$$
$$\leq \|f\|_1.$$

On the other hand, let $\varepsilon > 0$ be any positive number. Since the space of simple functions on X is dense in $L^1(X)$, We choose a simple function g such that
$$\|g - f\|_1 < \varepsilon.$$

Write $g = \sum_{k=1}^{n} c_k \chi_{A_k}$, where χ_{A_k} is the indicator function of A_k, $\{A_1, \ldots, A_n\}$ is a countable, measurable partition of X, and $c_1, \ldots, c_k \in \mathbb{C}$

For this partition, $\sum_{k=1}^{n} (f\mu_X)(A_k) = \sum_{k=1}^{n} \left| \int_{A_k} f \, d\mu_X \right|$

$$= \sum_{k=1}^{n} \left| \int_{A_k} [g-(g-f)] \, d\mu_X \right|$$

$$\geq \sum_{k=1}^{n} \left| \int_{A_k} g \, d\mu_X \right| - \sum_{k=1}^{n} \left| \int_{A_k} (g-f) \, d\mu_X \right|$$

$$= \sum_{k=1}^{n} |c_k| \mu_X(A_k) - \sum_{k=1}^{n} \left| \int_{A_k} (g-f) \, d\mu_X \right|$$

$$= \int_X |g| \, d\mu_X - \sum_{k=1}^{n} \left| \int_{A_k} (g-f) \, d\mu_X \right|$$

$$\geq \int_X |g| \, d\mu_X - \sum_{k=1}^{n} \int_{A_k} |g-f| \, d\mu_X$$

$$\geq \int_X |f| \, d\mu_X - 2 \int_X |g-f| \, d\mu_X$$

$$= \|f\|_1 - 2\varepsilon$$

Since ε is arbitrary, $\|f\mu_X\| \geq \|f\|_1$. Hence $\|f\mu_X\| = \|f\|_1$.

Also, $\theta_X(I_X) = \mu_X$ for each X. So it only remains to prove that θ_X is natural in $X \in \text{MeasEmb}$. Let $i: Y \to X$ be an embedding. We may assume i is an inclusion. We must show that the square

$$\begin{array}{ccc} L^1(Y) & \longrightarrow & L^1(X) \\ \theta_Y \downarrow & & \downarrow \theta_X \\ M(Y) & \longrightarrow & M(X) \end{array}$$

commutes.

where both horizontal maps are extensions by zero. Equivalently, writing $(\ )^X$ for the extension by zero on X of a function or measure on Y, we must show that

$$g^X \mu_X = (g\mu_Y)^X \quad \text{(measures)}$$

for all $g \in L^1(Y)$. But this just states that

$$\int_A g^X \, d\mu_X = \int_{A\cap Y} g \, d\mu_Y$$

for all measurable $A \subseteq X$, which is true. $\square$

Remark: There is a similar theorem in which $M(X)$ is replaced by the subspace $AC(X)$ of signed or complex measures absolutely continuous with respect to μ_X: the unique map $(L^1, I) \to (AC, \mu)$ in $\mathcal{B}^1_{\text{emb}}$ is $f \mapsto f\mu_X$. This map is an isomorphism (the Radon–Nikodym theorem), but that does not seem to be an easy consequence of the universal property of (L^1, I).

Finally, consider the case $p=2$.
Theorem 3.7 characterizes (L^2, I) as the initial object of $\mathcal{B}^2$, but there is an alternative characterization.
Let Hilb_1 be the category of Hilbert spaces with linear contractions. Let $\mathcal{H}$ be the category of pairs (F, ν) consisting of a functor $F: \mathrm{Meas}^{op} \longrightarrow \mathrm{Hilb}_1$ and an element $\nu_X \in F(X)$ for each measure space X, subject to axioms (I) – (III) (with $p=2$ in (III)) and

(IV_H) $\langle (Fi)(v_Y), (Fj)(v_Z) \rangle = 0$ whenever $Y \xrightarrow{i} X \xleftarrow{j} Z$ are embeddings with disjoint images. (that is $(Fi)(v_Y)$ is orthogonal to $(Fj)(v_Z)$)

Thus, the difference between the categories $\mathcal{B}^2$ and $\mathcal{H}$ is that: Ban_1 has been replaced by Hilb_1 and (IV) by (IV_H).

The functor $L^2 : \mathrm{Meas}^{op} \longrightarrow \mathrm{Hilb}_1$ together with the constant functions $I_X \in L^2(X)$, defines an object (L^2, I) of $\mathcal{H}$. It is universal as such:

Proposition 3.11 (Universal property of the L^2 functor) (L^2, I) is the initial object of $\mathcal{H}$.

Proof. In the proof of Theorem 3.6, the only point where axiom (IV) was used to prove the inequality (15), which in the case $p=2$, states that

$$\left\| \sum_{c \in \mathbb{F}} c\, v^X_{f^{-1}(c)} \right\|^2 \leq \sum_{c \in \mathbb{F}} \left\| c\, v^X_{f^{-1}(c)} \right\|^2 \quad (18)$$

for any simple function f on a measure space X and any object $(F, v) \in \mathcal{B}^2$. So it suffices to prove that: (18) also holds for any $(F, v) \in \mathcal{H}$.

Indeed, recall that if we write i_c for the inclusion $f^{-1}(c) \hookrightarrow X$, then by definition, $v^X_{f^{-1}(c)} = F(i_c)(v_{f^{-1}(c)})$. Axiom (IV_H) therefore implies that the elements $v^X_{f^{-1}(c)}$ of $F(X)$ are orthogonal for distinct c, giving equality in (18).

by the Pythagora's theorem for inner products.

$\square$

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