Harmonic Analysis and Hausdorff Dimension

a Brief Survey

Presenter: Ruijun Lin
Advisor: Bochen Liu

Southern University of Science and Technology (SUSTech)

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Department of Mathematics

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Hausdorff Content

Let (X, d) be a metric space and $A \subset X$ be a subset.
Let $s \geq 0$ and $0 < \delta \leq \infty$. s-dimensional, δ-limited Hausdorff content:

$$\mathcal{H}_\delta^s(A) = \inf \left\{ \sum_{j=1}^{\infty} \alpha(s) 2^{-s} d\left(E_j\right)^s \middle| A \subset \bigcup_{j=1}^{\infty} E_j, \text{ and } d\left(E_j\right) < \delta, j = 1, 2, \cdots \right\}$$

►$d(E)$: diameter of the set E, $d(E) = \sup\{d(x,y) | x, y \in E\}$.

►$\alpha(s) > 0$ is a scaling factor:

- $s = n$ an integer: set $\alpha(n)$ to be the volume of the n-dimensional unit ball ($\alpha(0) = 1$);
- s a non-integer: set $\alpha(s)$ such that $\alpha(s) 2^{-s} = 1$.

Observation: $\mathcal{H}_\delta^s(A)$ increases as δ decreases, so the limit as $\delta \to 0$ exists.

Hausdorff Measure

Definition (Hausdorff measure)

Let $0 \leq s \leq n$. The s-dimensional Hausdorff measure of a set $A \subset X$ is defined to be

$$\mathcal{H}^s(A) = \lim_{\delta \to 0} \mathcal{H}^s_\delta(A)$$

Hausdorff Dimension

We have the following lemma about Hausdorff measures.

Lemma

For $A \subset X$, there is a unique non-negative real number s_0, such that $\mathcal{H}^s(A) = \infty$ if $s < s_0$ and $\mathcal{H}^s(A) = 0$ if $s > s_0$.

Then the Hausdorff dimension of a set is defined to be this critical value s_0.

Definition (Hausdorff dimension)

The Hausdorff dimension of a set $A \subset X$ is defined as

$$\dim A = \inf \{s : \mathcal{H}^s(A) = 0\} = \sup \{s : \mathcal{H}^s(A) = \infty\}$$

In the following, we use **dim$A$** to denote the **Hausdorff dimension** of a set $A \subset X$ rather than the classical integer-valued dimension.

Frostman's Lemma

For $A \subset X$, denote by $\mathcal{M}(A)$ the set of all Borel measures μ on X with $0 < \mu(A) < \infty$ and with compact support $\text{supp}\mu \subset A$ (the set of positive finite measures with compact support in A).

Theorem (Frostman's lemma)

Let $0 \leq s \leq n$. For a Borel set $A \subset \mathbb{R}^n, \mathcal{H}^s(A) > 0$ **if and only if** there is a $\mu \in \mathcal{M}(A)$ such that

$$\mu(B(x,r)) \leq r^s \text{ for all } x \in \mathbb{R}^n,\ r > 0. \qquad (\bigstar)$$

In particular,

$$\dim A = \sup\{s : \text{ there is a } \mu \in \mathcal{M}(A) \text{ such that } (\bigstar) \text{ holds.}\}.$$

Energy integral

Let $f * g$ be the convolution function of two functions f and g, defined by

$$(f * g)(x) = \int_{\mathbb{R}^n} f(x-y)g(y)dy.$$

Definition (Energy integral)

The s-dimensional energy integral of a Borel measure μ on $\mathbb{R}^n$ is defined as the integral

$$I_s(\mu) = \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} |x-y|^s d\mu(x)d\mu(y) = \int_{\mathbb{R}^n} (k_s * \mu)(x)d\mu(x),$$

where $k_s(x) = |x|^{-s}$ is known as the s-dimensional Riesz kernel.

Energy integral

Theorem

Let $0 \leq s \leq n$. For a Borel set $A \subset \mathbb{R}^n$, $\mathcal{H}^s(A) > 0$ **if and only if** there is a $\mu \in \mathcal{M}(A)$ such that the s-energy integral $I_s(\mu) < \infty$.

Equivalently,

$$\dim A = \sup\{s : \text{there is } \mu \in \mathcal{M}(A) \text{ such that } I_s(\mu) < \infty\}.$$

Outline

Fourier Transform in $L^1(\mathbb{R}^n)$

For $f \in L^1(\mathbb{R}^n)$, we have the well defined **Fourier transform**:

$$\mathcal{F}(f)(\xi) = \widehat{f}(\xi) = \int_{\mathbb{R}^n} f(x) e^{-2\pi i x \cdot \xi} dx, \ \xi \in \mathbb{R}^n.$$

If further $\widehat{f} \in L^1(\mathbb{R}^n)$, then we also have the **Fourier inversion formula**:

$$f(x) = \int_{\mathbb{R}^n} \widehat{f}(\xi) e^{2\pi i \xi \cdot x} d\xi$$

for almost every $x \in \mathbb{R}^n$.

Fourier Transform of Finite Borel Measures

Definition (Fourier transform of finite Borel measures)

Given a finite Borel measure μ on $\mathbb{R}^n$, the Fourier transform of μ on $\mathbb{R}^n$ is a function defined by

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The following theorem will be used later.

Theorem

Let $\mu \in \mathcal{M}(\mathbb{R}^n)$. If $\widehat{\mu} \in L^1(\mathbb{R}^n)$, then μ is almost a continuous function, in the sense that there is a function f_μ which is continuous almost everywhere such that $d\mu = f_\mu d\mathcal{L}^n$.

(Here $\mathcal{L}^n$ denotes the Lebesgue measure in $\mathbb{R}^n$).

Expression of Energy Integrals by Fourier Transforms

The energy integrals of a finite Borel measure **can be precisely expressed by its Fourier transforms**, as indicated in the following theorem.

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Theorem

Let $\mu \in \mathcal{M}(\mathbb{R}^n)$ and $0 < s < n$. Then

$$I_s(\mu) = \gamma(n, s) \int_{\mathbb{R}^n} |\widehat{\mu}(x)|^2 |x|^{s-n} dx,$$

where $\gamma(n, s)$ is a constant depending on n and s.

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Projection in a Direction

Given a direction $e \in S^{n-1}$, the projection of a point in $\mathbb{R}^n$ onto this direction $P_e : \mathbb{R}^n \to \mathbb{R}$ is defined by

$$P_e(x) = x \cdot e,$$

where "$\cdot$" is the standard dot product in $\mathbb{R}^n$.

Observation: $\dim P_e(A) \leq \dim A$ because P_e is a Lipschitz map ($|P_e(x) - P_e(y)| \leq c|x - y|$ for some $c > 0$) which does not increase Hausdorff dimensions.

Hausdorff Dimension of Projections

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Theorem 1

Let $A \subset \mathbb{R}^n$ be a Borel set and $s = \dim A$.

1. If $s \leq 1$, then $\dim P_e(A) = s$ for σ^{n-1}-almost all $e \in S^{n-1}$;
2. If $s > 1$, then $\mathcal{L}^1(P_e(A)) > 0$ for σ^{n-1}-almost all $e \in S^{n-1}$.

Theorem 2

Let $A \subset \mathbb{R}^n$ be a Borel set and $\dim A > 2$. Then the projection $P_e(A)$ has nonempty interior for σ^{n-1}-almost all $e \in S^{n-1}$, where σ^{n-1} denotes the spherical measure on the sphere S^{n-1}.

Hausdorff Dimension of Projections

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Sketch of proof of **Theorem 2 using Fourier analysis**:

By Frostman's lemma, we choose a measure $\mu \in \mathcal{M}(A)$ such that the energy integral $I_s(\mu) < \infty$.

Define $\mu_e(B) = \mu(P_e^{-1}(B))$ for $B \subset \mathbb{R}$.

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Define $\mu_e(B) = \mu(P_e^{-1}(B))$ for $B \subset \mathbb{R}$.
Consider the integral

$$\int_{S^{n-1}} \left(\int_{-\infty}^{\infty} |\widehat{\mu_e}(r)| dr \right) d\sigma^{n-1}(e).$$

Hausdorff Dimension of Projections

We have (by a series of calculations),

$$\begin{aligned} &\int_{S^{n-1}} \left(\int_{-\infty}^{\infty} |\widehat{\mu_e}(r)| dr \right) d\sigma^{n-1}(e) = \cdots \\ =& C_1(n, s) \cdot \left(\int_{\mathbb{R}^n} |\widehat{\mu}(x)|^2 |x|^{s-n} dx \right)^{\frac{1}{2}} + C_2(\mu) \\ =& C_1(n, s) \cdot I_s(\mu)^{1/2} + C_2(\mu) < \infty \end{aligned}$$

where C_1 is a constant depends on n and s and C_2 depends on μ.

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=& C_1(n,s) \cdot I_s(\mu)^{1/2} + C_2(\mu) < \infty
\end{aligned}$$

where C_1 is a constant depends on n and s and C_2 depends on μ.

Hence $\widehat{\mu_e} \in L^1(\mathbb{R})$ for σ^{n-1}-almost all $e \in S^{n-1}$. Thus there is a **continuous function** $g_{\mu_e} \in L^1(\mathbb{R})$ such that $d\mu_e = g_{\mu_e} d\mathcal{L}^1$ by the Radon-Nikodym theorem.

As $g_{\mu_e} \in \mathcal{M}(P_e(A))$, we conclude that the interior of $P_e(A)$ is **nonempty for σ^{n-1}-almost all** $e \in S^{n-1}$. □

Application: Borel Rings on the Real Line
Statement of the Main Theorem

Definition (Borel ring)

A Borel ring is a Borel set equipped with an (algebraic) ring structure.

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The main theorem goes as follows:

Theorem

Let $E \subset \mathbb{R}$ be a Borel set which is also an (algebraic) subring of $\mathbb{R}$. Then there are only two possibilities:

1. $\dim E = 0$;
2. $E = \mathbb{R}$.

Application: Borel Rings on the Real Line
Sketch of the Proof

Just show that if $E \subset \mathbb{R}$ has Hausdorff dimension strictly larger than zero, then $\dim E = 1$.

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We need the following lemma.

Lemma

Let A and B be nonempty Borel sets in $\mathbb{R}^n$. Then

$$\dim(A \times B) \geq \dim A + \dim B.$$

Proof. This is a direct application of the Frostman's lemma.
If $0 \leq s < \dim A$ and $0 \leq t < \dim B$, we can choose a $\mu \in \mathcal{M}(A)$ with $\mu(B(x,r)) \leq r^s$ and $\nu \in \mathcal{M}(B)$ with $\nu(B(x,r)) \leq r^t$.
Then the product measure $\mu \times \nu \in \mathcal{M}(A \times B)$ with $(\mu \times \nu)(B((x,y),r)) \leq r^{s+t}$ from which the theorem follows. □

Application: Borel Rings on the Real Line
Sketch of the Proof

Suppose $\dim E > 0$. From the above lemma, we have $\dim E^k \geq k \dim E$ for any $k \in \mathbb{N}$, where E^k is the k-fold Cartesian product of E. We can choose a sufficiently large k so that $\dim E^k > 2$.
Consider the projection operator $\varphi = P_e : \mathbb{R}^k \to \mathbb{R}$. Theorem 2 shows that $\varphi(E^k)$ has nonempty interior, and since the image $\varphi(E^k)$ is a subspace of $\mathbb{R}$, then $\varphi(E^k) = \mathbb{R}$.

The following two lemmas conclude the proof of the theorem. □

Application: Borel Rings on the Real Line
Two Lemmas

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The first lemma is a purely an algebraic proposition.

Lemma 1

Let $E \subset \mathbb{R}$ be an algebraic subring. Assume that there is a $k \in \mathbb{N}^*$ and a linear functional $\varphi : \mathbb{R}^k \to \mathbb{R}$ such that $\varphi(E^k) = \mathbb{R}$, then **such a** k **can be chosen** so that φ **maps** E^k **bijectively onto** $\mathbb{R}$.

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The second lemma forces a Borel subring with positive Hausdorff dimension to be $\mathbb{R}$.

Lemma 2

Let $E \subset \mathbb{R}$ be a Borel subring. Let k be a positive integer and $\varphi : \mathbb{R}^k \to \mathbb{R}$ a linear functional **mapping** E^k **bijectively onto** $\mathbb{R}$. Then $k = 1$ and $E = \mathbb{R}$.

Main References

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Thank you!

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